SHALLOW WATER EQUATIONS: ACCURACY OF HIGH ORDER BIORTHOGONAL SCHEMES

Margarete O. Domingues
margaret@met.inpe.br
DCM-INPE/ IMECC-UNICAMP
CP 515 12227-970 SJC-SP

Abstract

The effects of high order schemes of biorthogonal spline spatial discretization are studied in a shallow water f plane model. A semi-implicit model is used in this temporal discretization. The accuracy of this discrete model is analyzed in phase frequency and group velocity. Thus, a significant discretization error reduction is found in the slow mode oscillations for high order schemes. On the other hand, the error of the fastest modes ("gravity modes") is not improved when order schemes higher than four are applied.

Key words: Shallow water model - Biorthogonal splines - Wavelets - Group velocity

1. INTRODUCTION

Elvius and Sundström (1972) analyzed the error introduced in time and space discretization in finite differences of a linear shallow water model. These authors used a method of semi-implicit time discretization in this model. This method separates the spatial derivative from the temporal one. In this last case, this method divides the temporal derivative in two categories: the responsible terms for severe restrictions in the time step are dealt with an implicit dicretization, while the other terms are dealt with an explicit discretization.

In 1976, Grotjam and O'Brien evaluated this type of error in phase frequency and in group velocity of a semi-implicit model, in approaches of second order. The group velocity components are calculated by the phase frequency wave number derivative, respectativelly. This physical amount is very important in the understanding of the dispersion of energy of these waves (Dias, 1996). Moreover, this amount is important for the agreement of numerical models, because even if a differential equation is non dispersive, its numerical equivalents are dispersive (Trefethen, 1982).

In this work the error introduced in the phase frequency and group velocity of a non dissipate shallow water f plane model is evaluated for the biorthogonal spline higher order space discretization in the semi-implicit time model described above.

2. PRELIMINARY

The families of biorthogonal spline functions were built by Cohen et al. (1992), and are generated by a couple of functions$\{\phi(x),\phi^*(x)\}$, where $\phi^*(x)$ is a *B-splines* of order N^* and $\phi(x)$ depends on a parameter N with the same parity of N^*. From these functions the operators of restriction and prolongation define themselves $(r^d f)_k = h^{-1} \int f(x)\, \phi_k^{*h}(x)\, dx$, $(p^d f^d)(x) = \sum_k f_k^d\, \phi_k^h(x)$, where $\phi_k^{*h}(x) = \phi^*(h^{-1}x\text{-}k)$ and $\phi_k^h(x) = \phi(h^{-1}x\text{-}k)$. Biorthogonality means that the approximation scheme $\{r^d,p^d\}$ is conservative in the sense that $r^d[p^d f^d]= f^d$.

A discretization of a differential operator D in the form $D^d f^d = r^d[Dp^d f^d]$ can be thought as a Petrov-Galerkin type scheme, where ϕ_k^{*h} are the test function and ϕ_k^h are the trial functions. For the linear case and in case that operators have constant coefficients, $D^d f^d$ it can also be interpreted as a scheme of finite differences. It considers, for instance, $Df(x)=f\,'(x)$.The discrete operator $D^d f^d$ has the following expression: $(Dp^d f^d)_p = h^{-1} \sum_k f^d_k\, \Gamma(p\text{-}k)$, where $\Gamma(s)= \int \phi'(z)\, \phi^*(z+s)\, dz$. It can be proved that these coefficients depend only on $M=N+N^*$ (Cunha and Gomes, 1995). Therefore the analysis done in this work is restricted to the case $N^{*}= 0$, for which $\phi^{*}(x) = \delta(x)$, Dirac function, and $\phi(x)$ is an interpolator function, of degree M-1, that corresponds to a collocation scheme.

3. ANALYSIS OF THE SHALLOW WATER DISCRETE MODEL

The following shallow water discrete f plane model can be obtained by coupling the biorthogonal spline space discretization, presented in Domingues (1997), with a semi-implicit temporal scheme of finite differences.

$$L^{d} u^{d} = f_0 v^{d} - D_x^{d} \varphi^{d},$$
$$L^{d} v^{d} = -f_0 u^{d} - D_y^{d} \varphi^{d},$$
$$L^{d} \varphi^{d} = - \Phi(D_x^{d} u^{d}+ D_y^{d} v^{d}),$$

where $(L^{d} g^{d})_{p,q,n} = 1/(2 \Delta t) [g^{d}(p,q,n+1) - g^{d}(p,q,n-1)] + U/h \sum_k g^{d}(k,q,n) \Gamma(p-k) + V/h \sum_l g^{d}(p,l,n) \Gamma(q-l)$,
$(D_x^{d}g^{d})_{p,q} = 1/2 \sum_k [g^{d}(k,q,n+1) + g^{d}(k,q,n-1)] \Gamma(p-k)$ e $(D_y^{d}g^{d})_{p,q} = 1/2 \sum_l [g^{d}(p,l,n+1) +g^{d}(p,l,n-1)] \Gamma (q-l)$.

As these discrete operators are expressed as convolutions, they can commute. Thus, after some manipulation an φ_d expression is derived as: $L_d [[L_d^{2}+f_0^{2}] - \Phi [D_{xd}^{(2)} + D_{yd}^{(2)}]] \varphi_d = 0$. Assuming $\varphi^{d}(ph,qh,n\Delta t) = e^{i (h\xi p+h \upsilon q - n \omega \Delta t)}$, three solutions are obtained, as in the continuous case. The first one is given by ω^{e} and the others by $\omega_{\pm}^{si}$. To follow values of ω^{e} and its respective zonal group velocity are presented:

$$\omega^{e} = \frac{1}{\Delta t \ \text{arcsen}\, A}, \qquad V_{gx}^{e} = \frac{-iU}{\sqrt{1-A^2}} \frac{d\Gamma'(h\xi)}{d\xi},$$

where $A= i/h [U \ \Gamma'(h\xi)+V\Gamma'(h\upsilon)]$ and $\Gamma'(\zeta)=\sum_s e^{-i\zeta s} \Gamma(s)$. This phase frequency is associated to the explicit discretization solution of the linear advection equation. Figure 1 presents the graphs of the error $|1-\omega /\omega_e|$ for *M=2,...,8*. A significant error reduction occurs for *M* values higher than *4*, as observed in this figure. The same kind of error analysis is made for the zonal group velocity (for $h\upsilon$~0 – Figure 2). The two other solutions are given below:

$$\omega_{\pm}^{si} = \frac{1}{\Delta t \ \text{arcsen}\, G_{\pm}},$$

where $G_{\pm} =-iA\pm R /(1-\Lambda B)$, $R^{2}= (\Delta t^{2}f_0^{2}-\Lambda B) (1-\Lambda B) - \lambda \ BA^{2}$, $B=\Gamma'^{2}(h\xi)+\Gamma'^{2}(h\upsilon)$, $\Lambda=\Phi(\Delta t/h)^{2}$ and $\lambda=U\Delta t/h$. These two phase frequencies are associate to the discrete operator $[[L_d^{2}+f_0^{2}] - \Phi [D_{xd}^{(2)} + D_{yd}^{(2)}]] \varphi^{d} =0$. In Figure 3 is presented the error $|1-\omega^{si}_{+}/\omega_{+}|$ for *M=2,.., 8*. The zonal component of the group velocity associated to the frequency ω^{si}_{+} is given by

$$V_{gx}^{si} = \frac{1}{\Delta t \sqrt{1-G^2}} \frac{\partial G_+}{\partial \xi},$$

where $\partial G_{+}/ \partial\xi = (-i \ \partial A/\partial\xi + \partial R/\partial\xi) / (1 - \Lambda B) + \Lambda\partial B/\partial\xi \ \ (-i A + R)/(1- \Lambda B)^{2}$,
$\partial A/\partial\xi = \lambda \ d \Gamma'(h \xi) /d\xi$,
$\partial B/\partial\xi = 2 \Gamma'(h \xi) \ \ d \Gamma'(h \xi)/d \xi$,
$\partial R/\partial\xi = - 1/2 \Lambda R^{-1} [\partial B/\partial\xi (-2 \Lambda B + \Delta t^{2} f_0^{2} + A^{2} + 1) + 2 A B \partial A/\partial \xi]$. In these two cases (Figures 3 and 4), a significant error reduction do not occur, in phase frequency or in group velocity, by introducing the discretization in order higher than four.

4. CONCLUSIONS

Swartz and Wendroff (1974) demonstrated that the phase frequency error, in an undimensional advection equation for usual finite differences, decreases to $h\xi<\pi/4$, when the order of the numerical schemes is incresed. This result was also numerically verified in the present work for the biorthogonal discretization. However, for values $\pi/4<h\xi, h\upsilon>\pi/2$, the best results are obtained for the family M=4. Mesinger and Arakawa(1976) and Grotjam and O'Brien(1976) emphasized the importance of the inversion of the sign of the group velocity propagation sign (with the occurrence of the group velocity zero) in the discrete finite difference model of second order. This occurs in regions $h\xi \sim \pi/2$. In these regions, the relative discretization errors are significantly high, what discards the physical usage of such regions. As the scheme order increases, the physical reliability region increases, for ω^e and for its group velocity. In this case ω^e is almost non divergent and geostrophically balanced, moreover it possess slower modes than the other two solutions (Elvius and Sundström, 1972). In many situations of meteorological interest, this frequency is of great importance.Therefore, the methods of higher order are better than the ones of low order in relation to the accuracy in the phase frequency and group velocity, in these slow modes. The other two frequencies, associated to the gravity waves, are many times assumed as noise. So, the effort to use higher order schemes is of great interest, specially in these cases.

As it was already mentioned, biorthogonal discretization method can be interpreted as a finite difference scheme. In this case, it is possible to take advantage of the multi-level basis of wavelet associates and to develop adaptive schemes in the space (Bacry et al., 1992), which is being treated in a work in progress. However, it is still necessary to evaluate the expenses of these methods in terms of flop requirements and computational efficiency implementation.

Acknowledgements

The author thanks the financing support of FAPESP (94-2016-9) and the suggestions of Dr. Sônia Maria Gomes (Imecc-Unicamp).

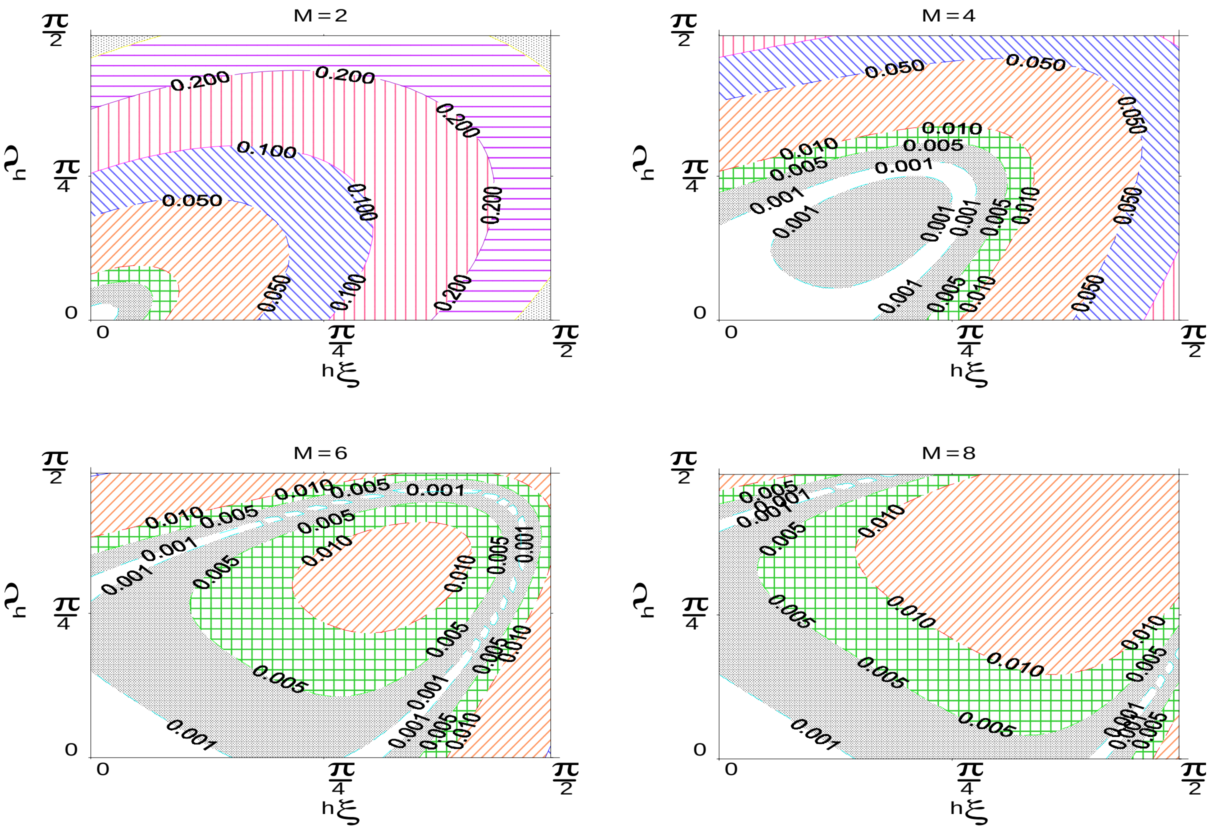


Fig.1 - Relative error in the phase frequency ω^e ($\lambda = 0.17$, U=V).

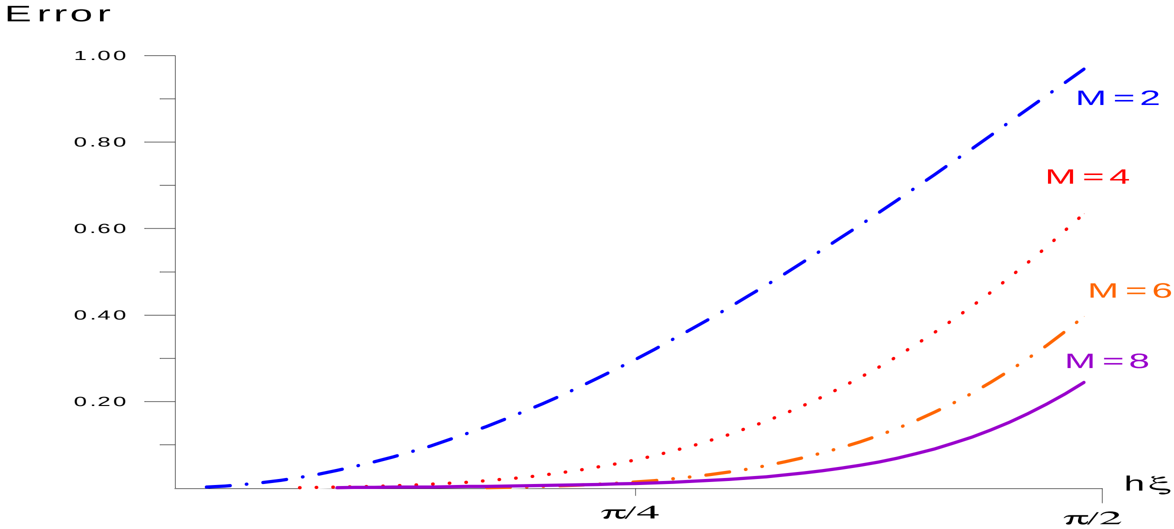


Fig. 2 - Relative error in group velocity w^e (for hυ~0, λ=0.17, U=V).

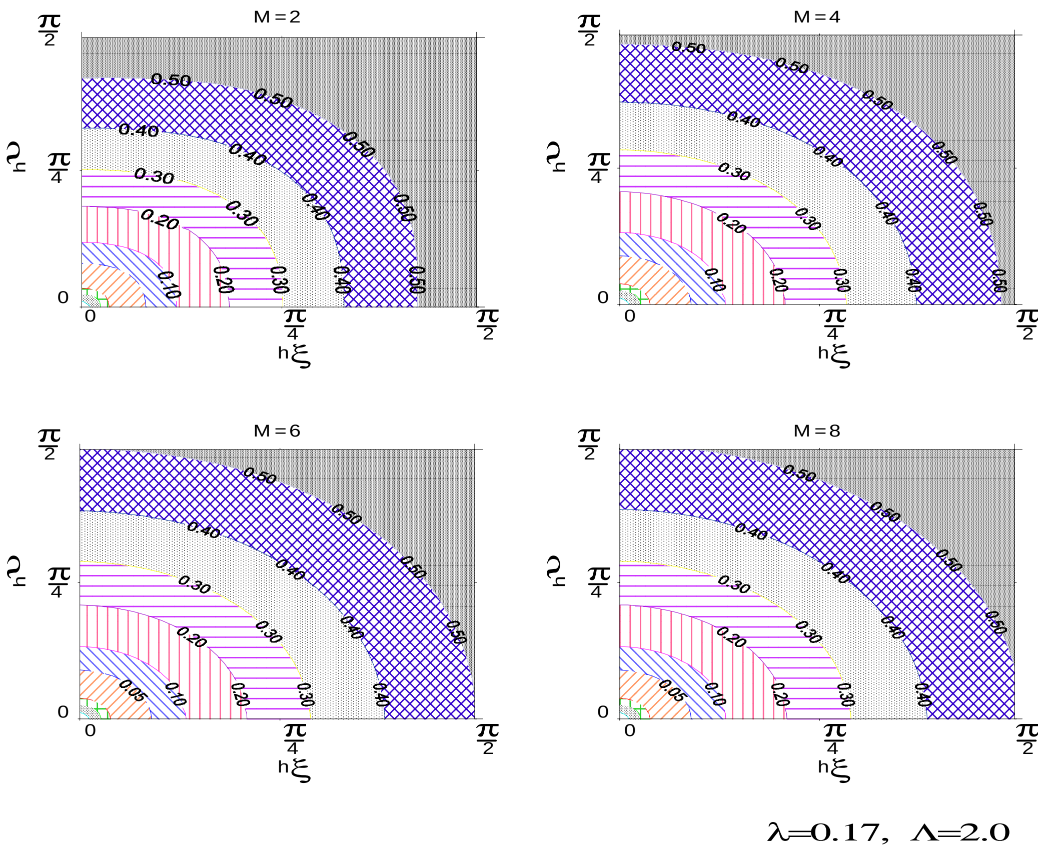
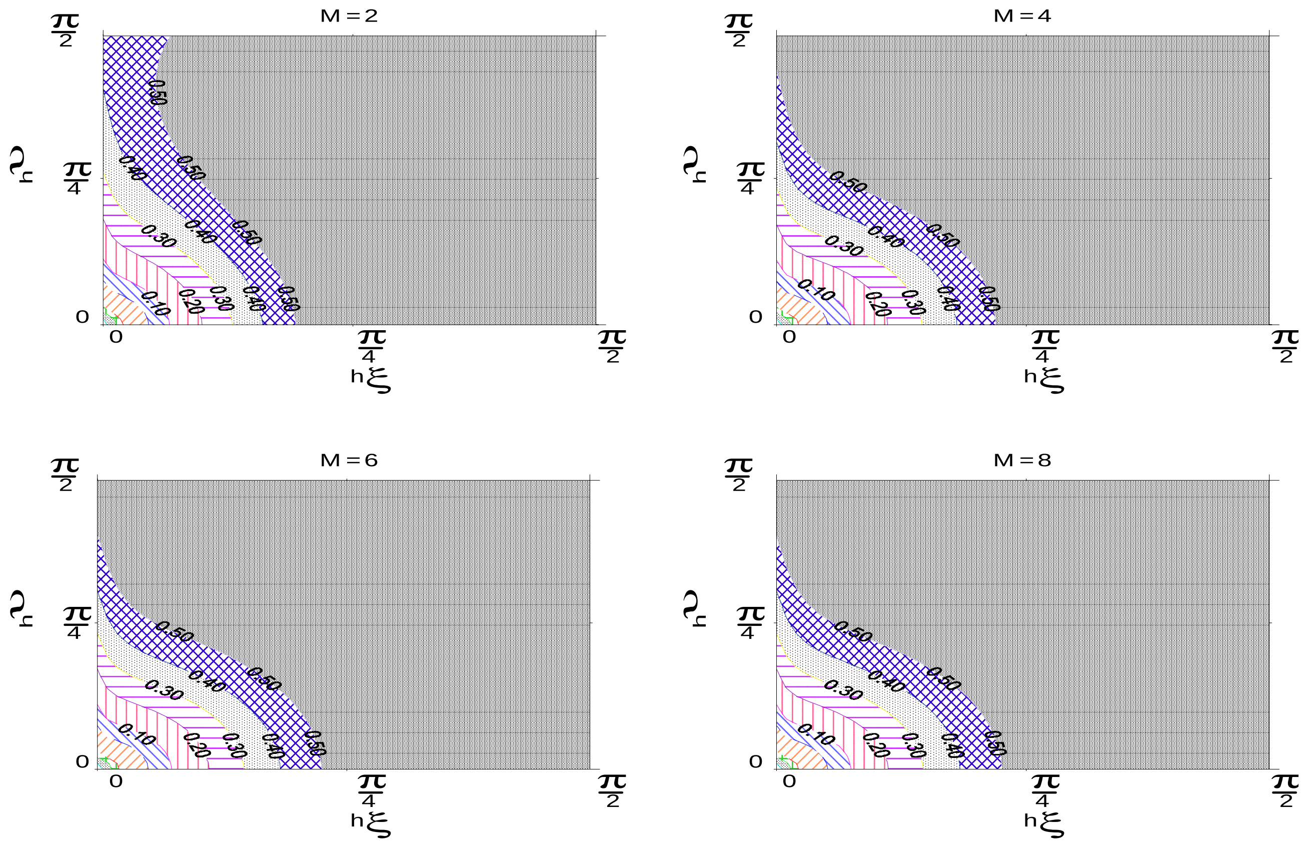


Fig. 3 - Relative error in phase frequency w^{si}_+.



λ=0.17, Λ=2.0

Fig. 4 - Relative error in group velocity w^{si}_{+}.

References

Bacry, E. ; Mallat, S.; Papanicolau. A wavelet based space-time adaptative numerical method for partial differential equations. **M^{2}AN,** 26:793-834, 1992.

Cohen, A.; Daubechies, I.; I. Feauveau, J. C., Biorthogonal bases of compactly supported wavelets. **Communications in Pure and Applied Mathematics**, 45: 485-560, 1992.

Cunha, C.; Gomes, S. M., Petrov-Galerkin and collocation methods using biorthogonal spline wavelets.**RP 26/95-Unicamp**, Campinas, Brasil, maio, 1995, 19p.

Domingues, M.O., Estudo do Erro na Freqüência de Fase e na Velocidade de Grupo em um Modelo Discreto Biorthogonal para as Equações de Águas Rasas, **Unicamp**, Campinas, SP, 1997. 27p. (http:// www.ime.unicamp.br/~margaret/meg.html)

Grotjahn, R.; O'Brien, J. J., Some Inaccuracies in Finite Differencing Hyperbolic Equations, **Monthly Weather Review**, 104, 180--194, 1976.

Elvius, T., Sundström, A. Computationally efficient schemes and boundary conditions for a fine mesh barotropic model based on the shallow water equations. **Tellus** XXV(**2**), 1973.

Mesinger, F.; Arakawa, A.,Numerical Methods Used in Atmospheric Models**. GARP Publication, WMO-ICSU Joint Organizing Committee**, New York, 17(**I**), 64, 1976.

Silva Dias, P. L., Equação da água rasa**. workshop Computational Methods for Oceanic and Atmospheric Flows - LNCC/CNPq**, LNCC - Rio de Janeiro, RJ, Set 18-29, 1995.

Swartz, B.; Wendroff, B.,The relative efficiency of finite difference and finite elements methods. I: Hyperbolic problems and splines, **SIAM Journal of Numerical Analysis**, 11, 979--993,1974.

Trefethen, L. N. Group velocity in finite differences schemes. **SIAM Review** 24(**2**), 1982.