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Digital Steiner sets and Matheron semi-groups

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- *is the dilate of a segment by itself still a segment ?*
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Or notions that admit several definitions,e.g.

- *Digital convexity is defined in five different manners in literature. Which one to take?*

Convexity

- In maths, convexity is a notion defined for ***vector spaces.***

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$\Rightarrow$ X equals the intersection of the half spaces that contain it,

$\Rightarrow$ or $\{x,y\} \in X \Rightarrow [x,y] \in X$

$\Rightarrow$ or the measure of $X \oplus B$, both compact convex sets, is a linear function of their Minkowski functionals, e.g. in R^2

$$\underline{A}(X \oplus B) = A(X) + U(X).\, U(B) / 2\pi + A(B)$$

Convexity and Scale-space Representation

- Still in space R^n, denote by λB the set similar to B by factor λ. Then the semi-group law:

$$[(A \oplus \lambda B) \oplus \mu B)] = A \oplus (\lambda + \mu) B$$

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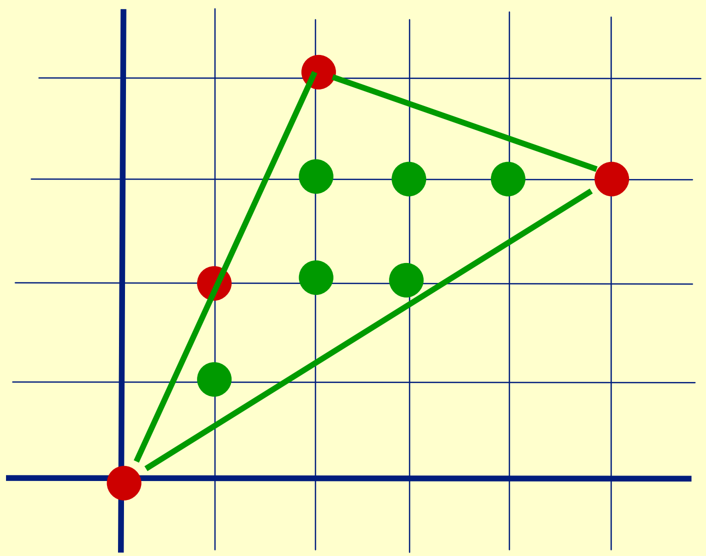
- Note that set **A** is arbitrary. In particular we have that

$$\lambda B \oplus \mu B = (\lambda + \mu) B$$

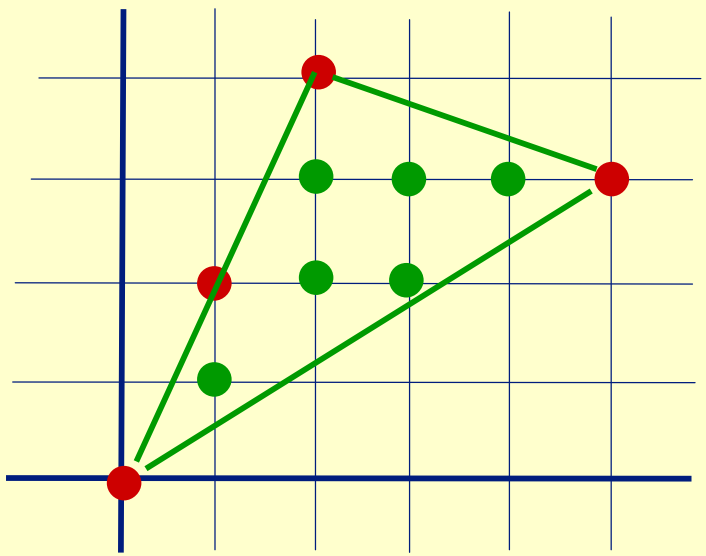
Digital Convexity

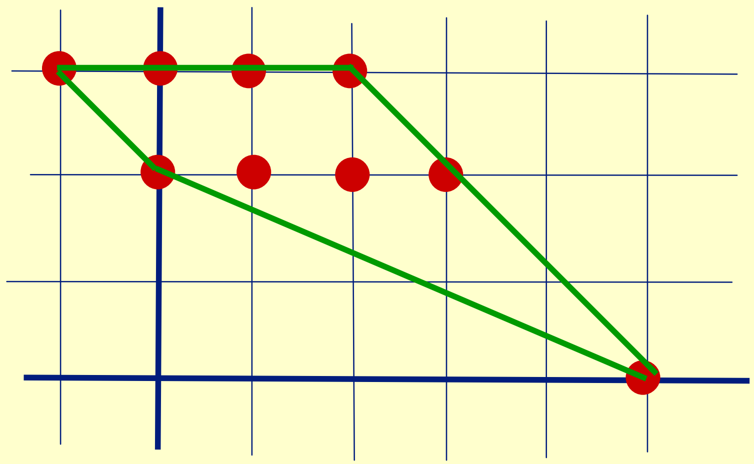
- Unfortunately, when passing from R^n to Z^n all these nice equivalences vanish…
- e.g., the three segments belong to set X, which it is not convex,



Digital Convexity

- When passing from R^n to Z^n all these nice equivalences vanish…
- e.g., the three segments belong to set X, which it is not convex,
- Also, a digital convex set may be non arcwise connected.





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- Therefore we must analyse exactly how convexity appears, so that to chose the most convenient digital convexity
- Indeed, the morph. scale-space pyramids are governed by ***Matheron semi-group*** law

$$\lambda \geq \mu > 0 \Rightarrow \psi_\mu \circ \psi_\lambda = \psi_\lambda$$

Where $\{\psi_\lambda, \lambda > 0\}$ is a family of morph. filters

- The law applies for opening, ASF and levelling.

Granulometries

- In case of opening, Matheron semi-group is called a ***granulometry***:

$$\lambda \geq \mu > 0 \Rightarrow \gamma_\mu \circ \gamma_\lambda = \gamma_\lambda \qquad (1)$$

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- For

$\Rightarrow$ $\mathcal{P}(E)$ lattices (e.g. $E = R^n$ or Z^n)

$\Rightarrow$ and $\{\delta_\lambda\}$ a family of dilations

Rel.(1) is equivalent to $\lambda \geq \mu \Rightarrow \delta_\lambda(\mathbf{x}) = \gamma_\mu \delta_\lambda(\mathbf{x})$

i.e. each structuring element ***is open by the smaller ones.***

Granulometries

- ***In the Euclidean and translation invariant case***

  $\lambda \geq \mu \Rightarrow \delta_\lambda(x) = \gamma_\mu \delta_\lambda(x)$ becomes

  $\lambda \geq \mu \Rightarrow B_\lambda = \gamma_\mu B_\lambda$ (structuring elements)

- ***Then magnification ≡ convexity***

  $\{\lambda \geq \mu \Rightarrow B_\lambda = \gamma_\mu B_\lambda\}$ + Homothetics B_λ

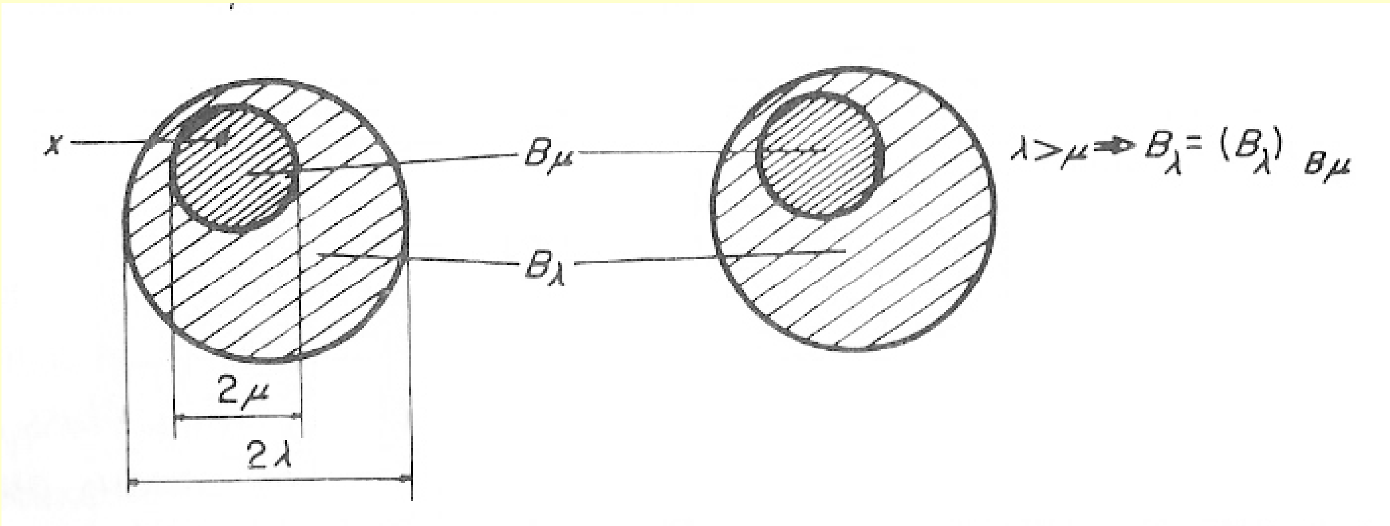
  is equivalent to

  $\{\lambda \geq \mu \Rightarrow B_\lambda = \gamma_\mu B_\lambda\}$ + convex B_λ

- For Matheron semi-groups, magnification and convexity are the ***same notion.***

Granulometries

- Conversely, we can drop convexity



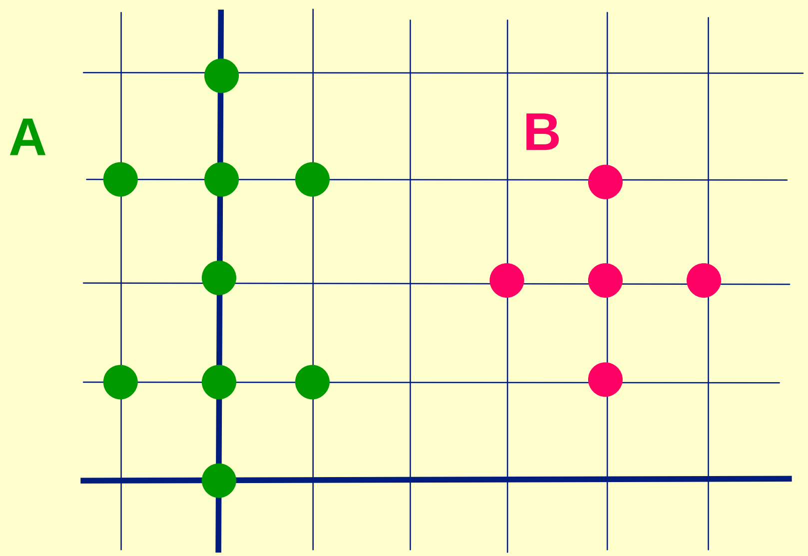
- The B's are not convex, but also not homothetic,

  …. however the semi-group is satisfied.

Granulometries

- Note also that A = A ○B is not an inclusion relation

When A is open by B,

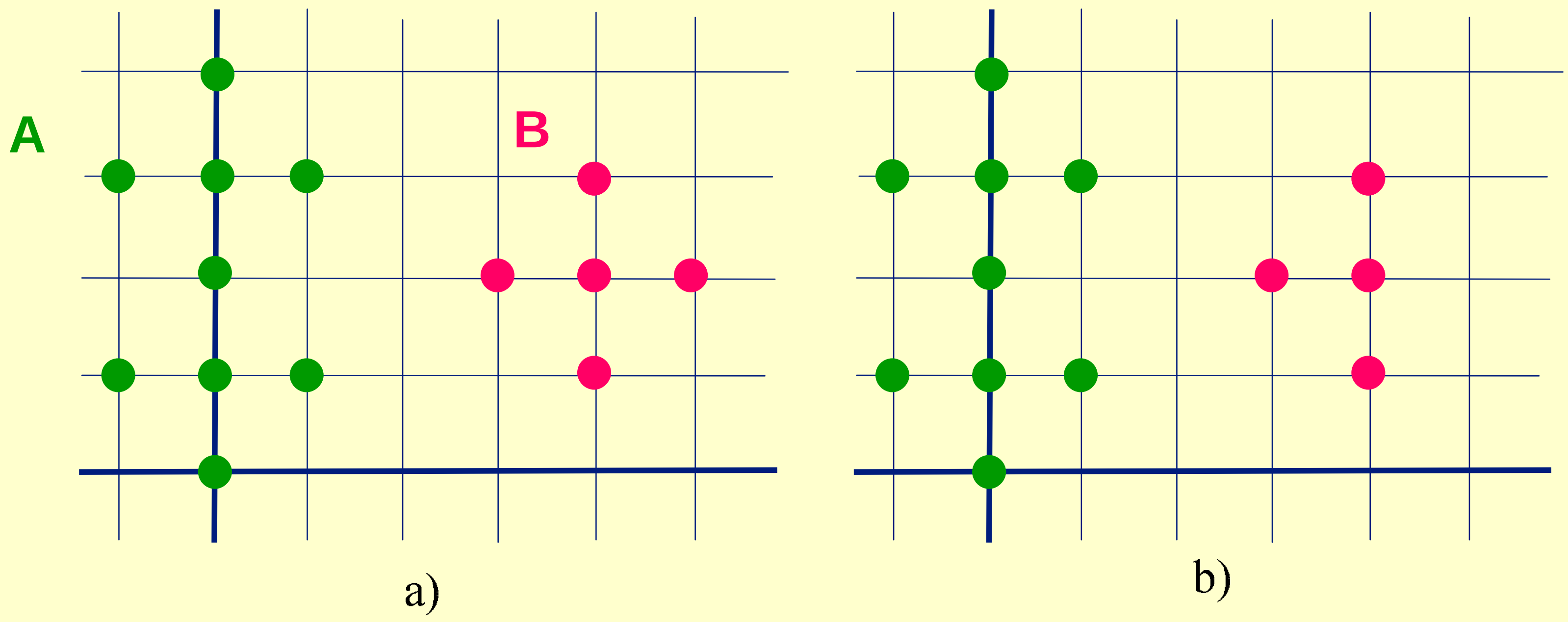


a)

Granulometries

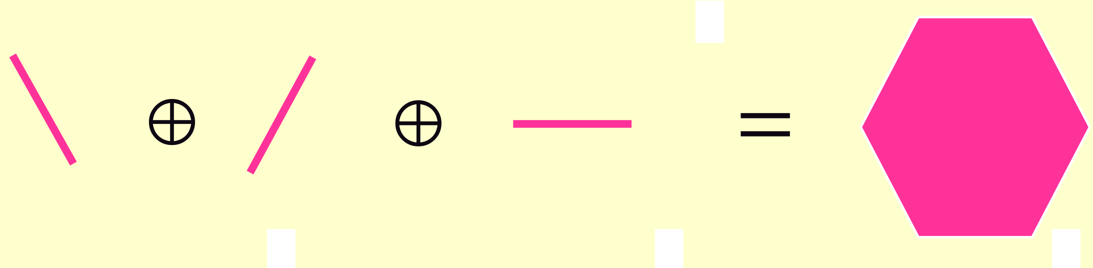
- Note also that A = A ○B is not an inclusion relation

When A is open by B, it may be not open by smaller sets

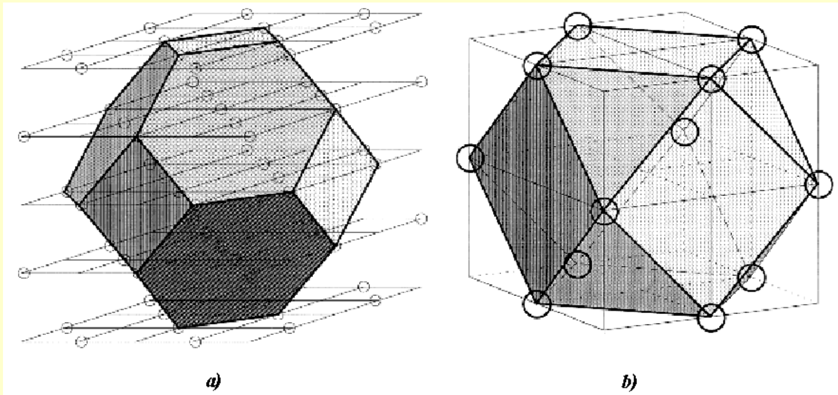


Euclidean Steiner class

- ***Steiner class*** : In R^n, the convex sets which are dilates of segments, and their limits (e.g. the disc) are ***Steiner***



- In R^2, they coincide with all convex sets with a centre of symmetry, but no longer in R^3.



Euclidean Steiner class

- ***Directional measure***: The Steiner set X is equivalent to the measure s_X ($d\alpha$), with

$$\mathbf{X = \oplus \{L[s_X\ (d\alpha)],\ \alpha \in \Omega\ \}}$$

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- This directional measure ***exchanges dilation and addition***

$$\mathbf{s_{X\oplus Y} = s_X + s_Y}$$

hence

$$\mathbf{s_X \leq s_Y \;\Rightarrow\; s_{X\ominus X} = s_X - s_Y \;\Rightarrow\; Y \text{ is open by } X}$$

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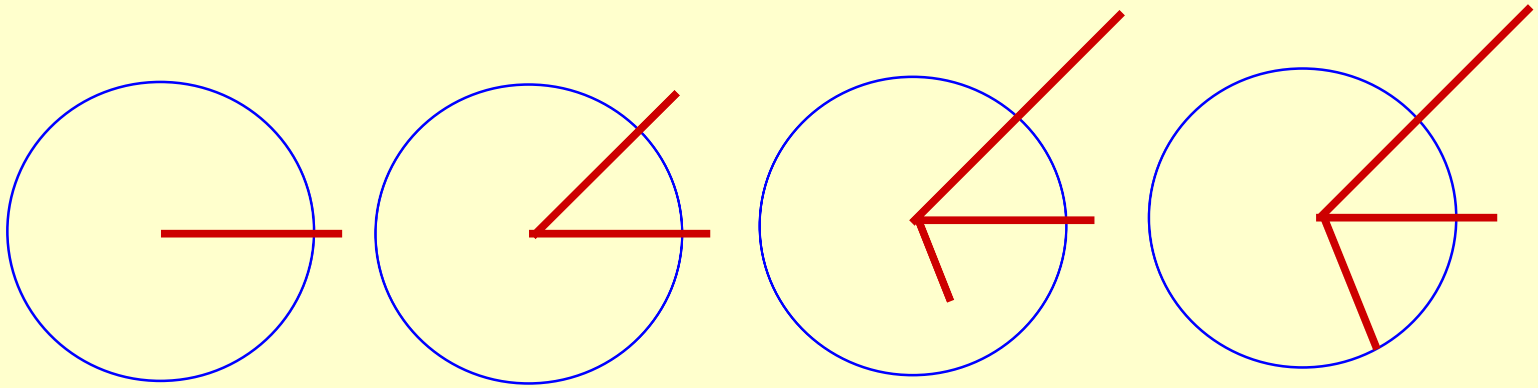
- Every family of Steiner sets with increasing measures ***generates a granulometry***.

An example



- This sequence of Steiner sets generates a granulometry

From R^n to Z^n

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- What is a digital ***Steiner set*** ?
- What is a digtal ***convex set*** ?
- Under which conditions is a digital convex set ***connected***?

Bezout planes in Z^n

- ***Bezout theorem***: The equation

$$\mathbf{a_1 u_1 + a_2 u_2 + \ldots\ a_n u_n = 1} \qquad (1)$$

has solutions in Z^n iff the a_i are relatively prime.

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- ***General solution***: One goes from the solutions of

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- ***Spanning of the space*** Therefore the hyper-planes (2) span the space Z^n , so that each point is met ***once and only once.***

N.B. in Z^2 this is also true for Bresenham lines (H.Talbot)

Bezout lines in Z^2

- When *a* and *b* are relatively prime, then

$$\exists\ \mathbf{u,v} \in \mathbf{Z} \quad \text{such that} \quad \mathbf{au + bv = 1}$$

If (x_0, y_0) is solution of $ax + by = c$, then

$$\mathbf{a\,(x_0 + u) + b\,(y_0 + v) = c + 1}$$

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- All solutions of the equation $ax+by = c+1$ derive from the solutions of $ax+by = c$ by translation of vector (u,v)

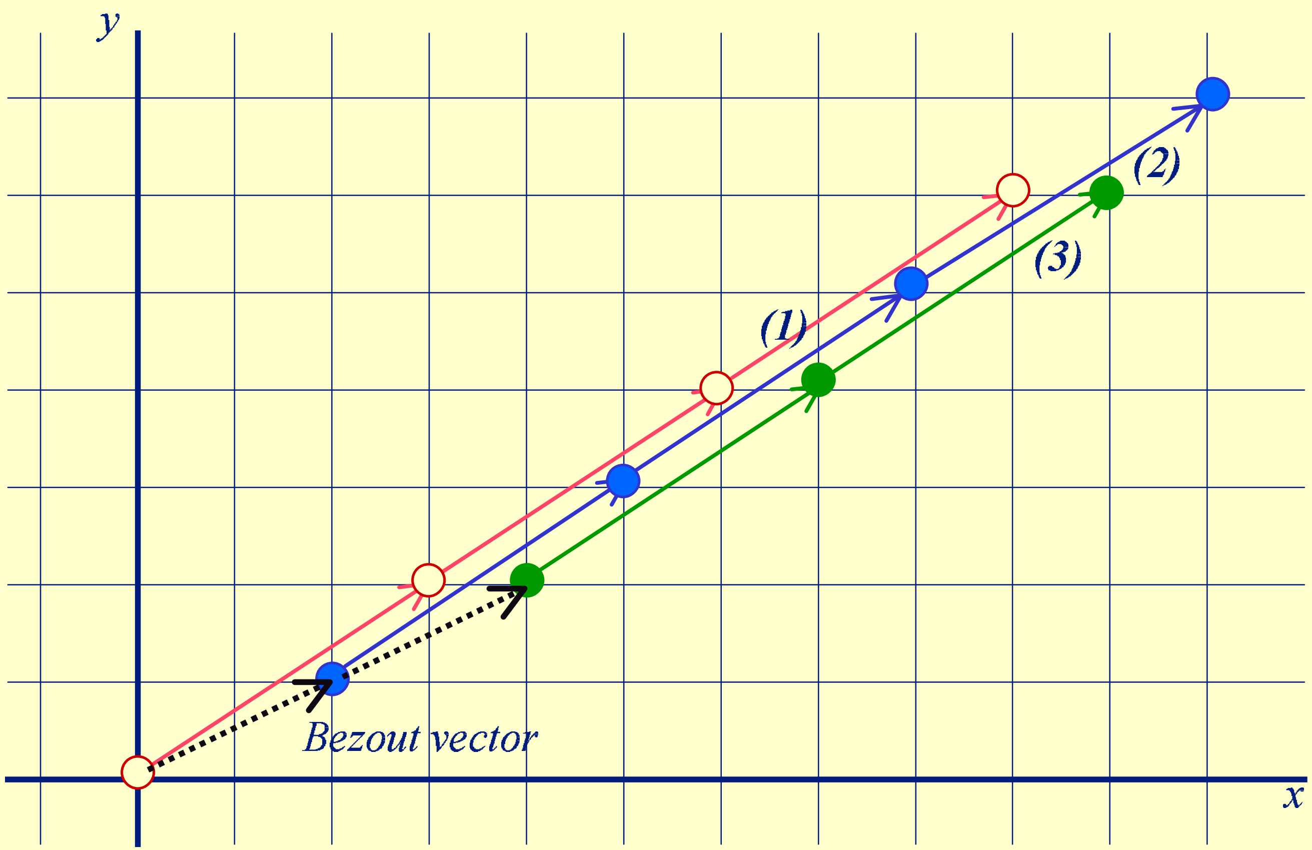
- ***An example*** : take the Bezout straight line

$$\mathbf{2\ x - 3\ y\ = 1}$$

which has vector (2,1) for solution.

Bezout lines in Z^2

The translates of the line by the Bezout vector span the digital plane



Bezout directions and segments

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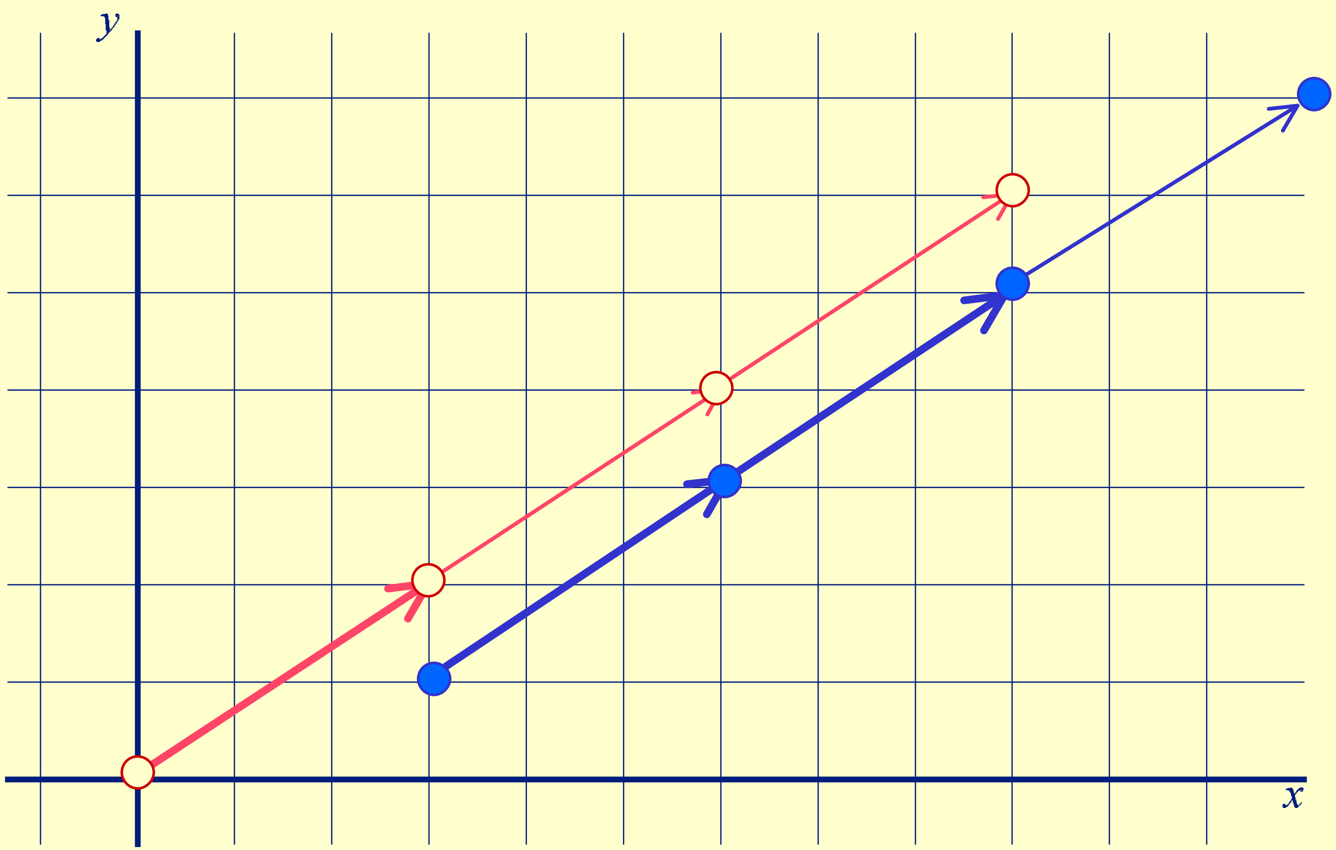
- ***Bezout segment*** : the sequence of the (k+1) points

$$L_x(k, \omega) = \{ x + p\,\omega,\ 0 \leq p \leq k \}$$

Bezout lines in Z^2

Examples of Bezout vector, lines, and segment in the digital plane



Dilation on Bezout segments

Theorem 1 :

- 1/ The Minkowski sum of the segments $L_x(k, \omega)$ and $L_y(m, \omega)$ is the segment

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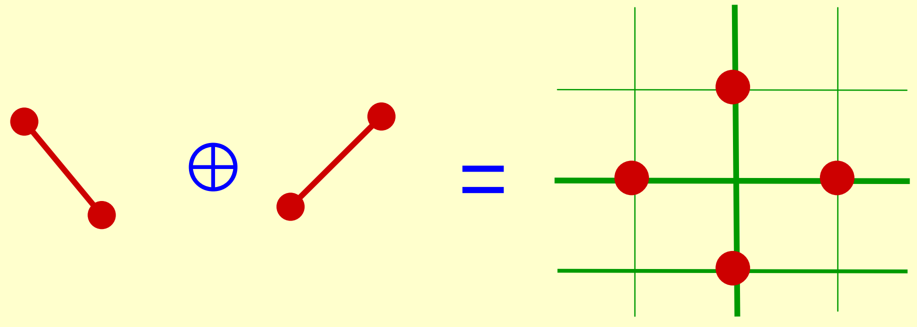
- 3/ The ***only*** digital segments that satisfy these two properties are the ***Bezout*** ones (because of their unit thickness).

Digital Steiner sets

- ***Steiner sets*** : A set in Z^n is ***Steiner*** when it can be decomposed into Minkowski sum of Bezout segments.

- A Steiner set is not always convex.



Digital Steiner sets

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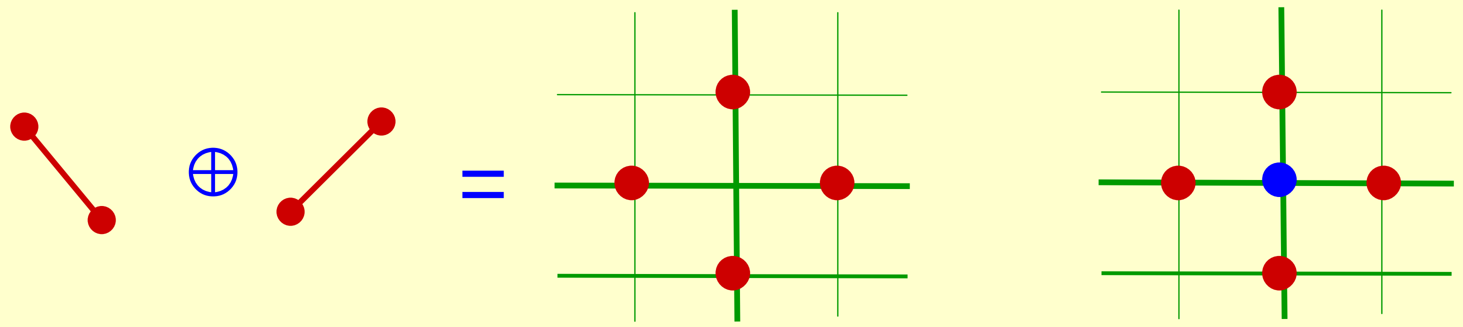
- A Steiner set is not always convex. In the figure, if we add the centre, the set becomes convex, but it is no longer Steiner (though it is symmetrical…)



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- ***Theorem 2***
  - Every segment is convex ;
  - When points x and y belong to the convex set X, then all points of the Bezout segment [x,y] belong to X

- Hence, By using Bezout' background, we can identify both approaches of convexity, by convex hull, and by barycentre

Reveillès Straight lines

affine shift

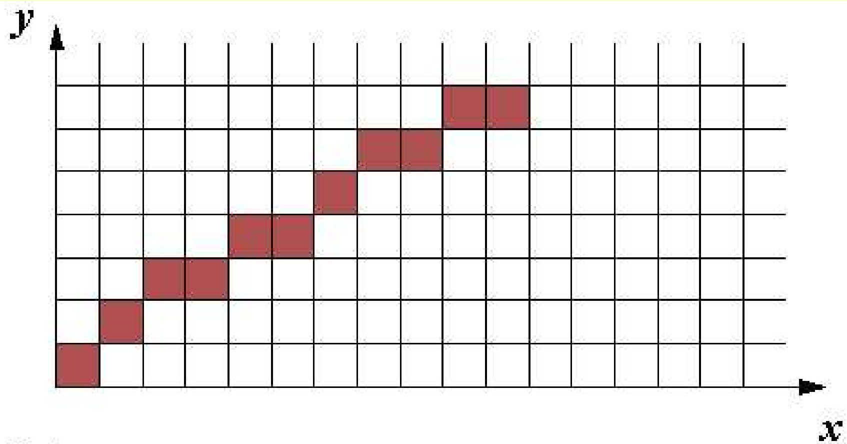
$$\gamma \leq ax + by < \gamma + \rho$$

direction

thickness

Where the directional parameters $a, b,$ are relatively prime

Reveillès Straight lines



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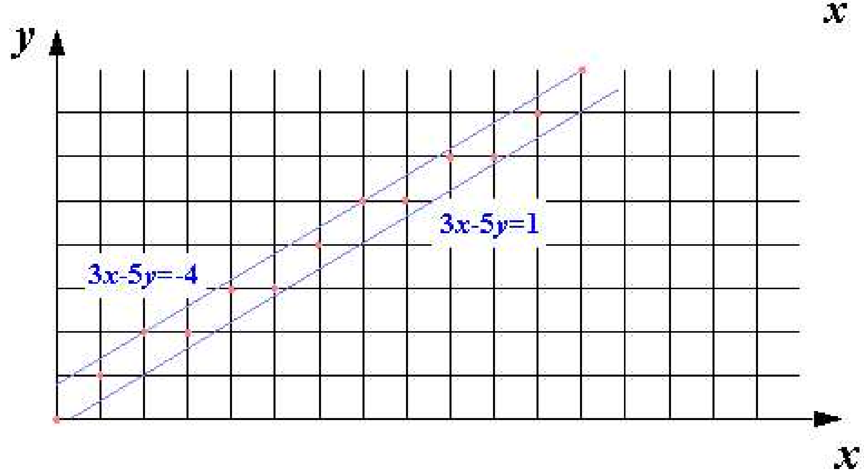
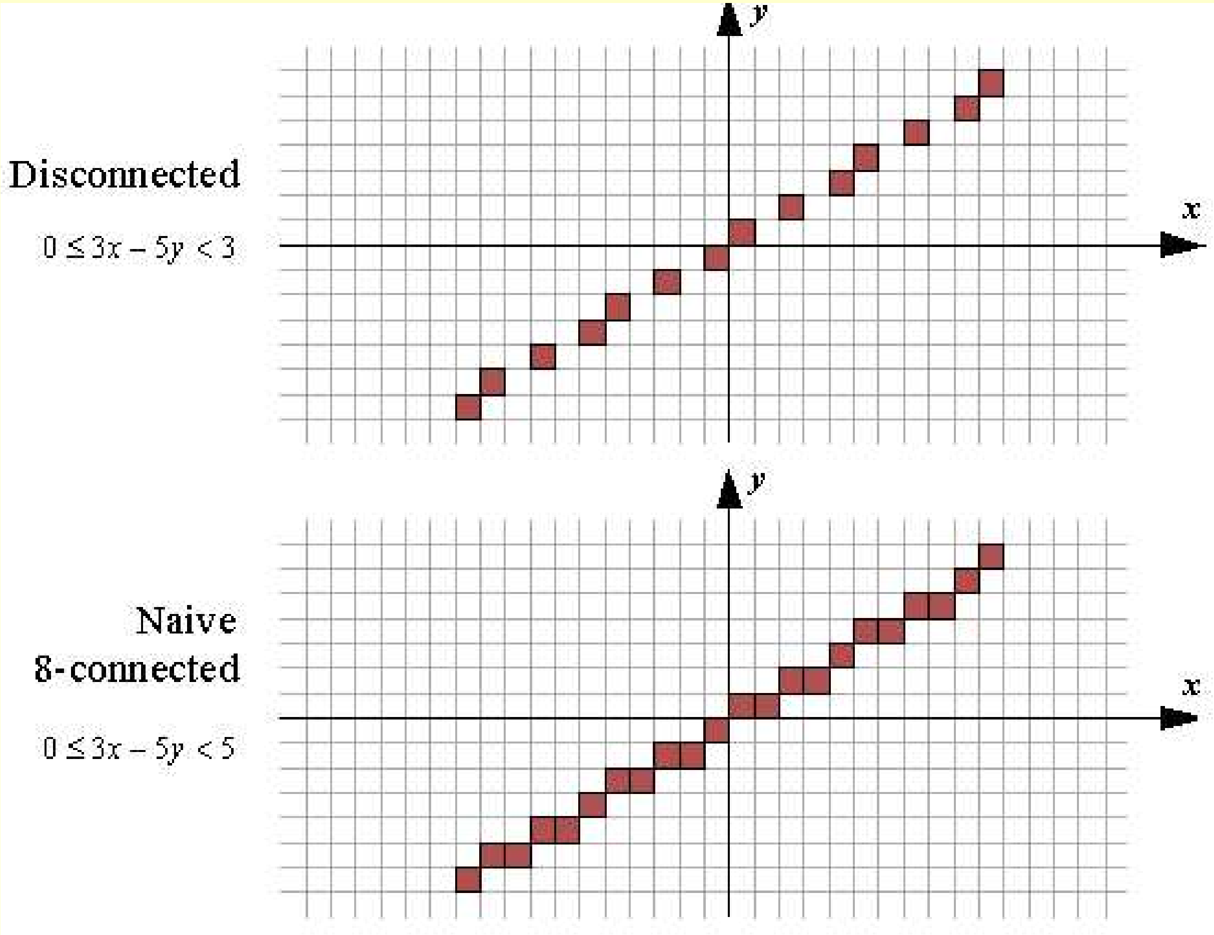


$$-4 \leq 3x - 5y < -4 + 5$$

Thickness of the Réveillès lines



$$\rho < \max(|a|,|b|)$$

Naive line

$$\rho = \max(|a|,|b|)$$

Decomposition

Decomposition of Réveillès straight lines into Bezout ones

$$D: \quad \gamma \leq ax + by < \gamma + \rho$$

$$D: \quad \bigcup_{\gamma \leq c < \gamma + \rho} \{ax + by = c\}$$

$0 \leq 3x - 5y < 5$

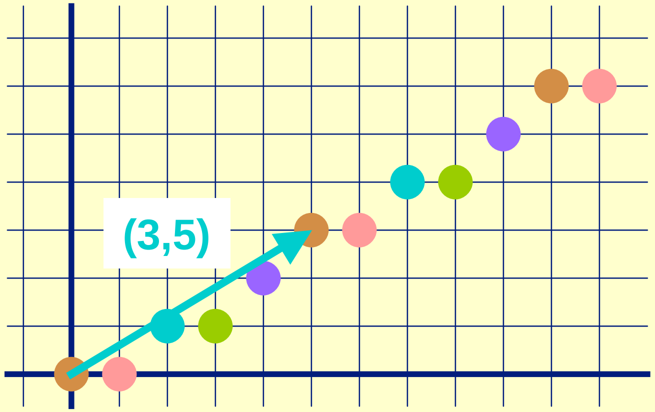
$3x - 5y = 0$

$3x - 5y = 1$

$3x - 5y = 2$

$3x - 5y = 3$

$3x - 5y = 4$



Convexity for Steiner sets

- ***Theorem 3*** : In Z^2, a Steiner set X of measure

$$\{ k_i\, \omega_\iota,\ 1 \leq i \leq p \}$$

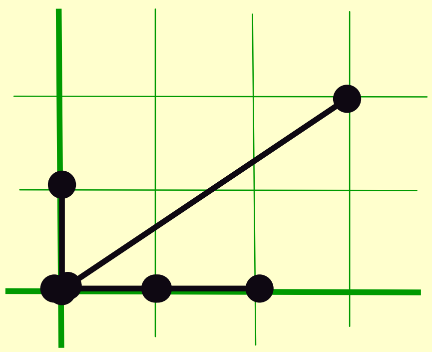
is ***convex*** iff for one direction, p say, the dilate of the Bezout line D_p by the other segments, i.e.

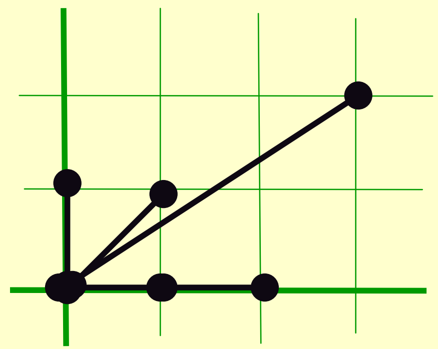
$$D_p \oplus L_1 \oplus L_2 \oplus \ldots. \oplus L_{p-1}$$

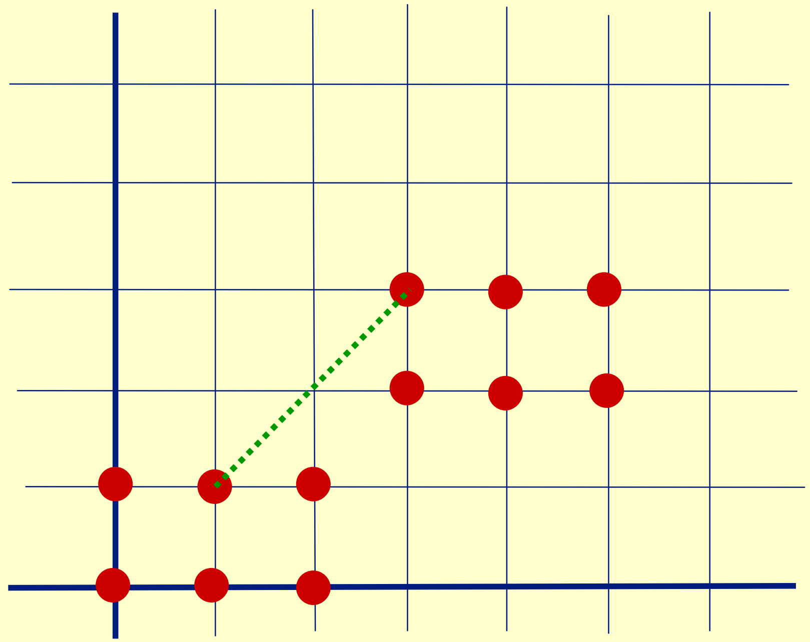
is a ***Réveillès straight line***

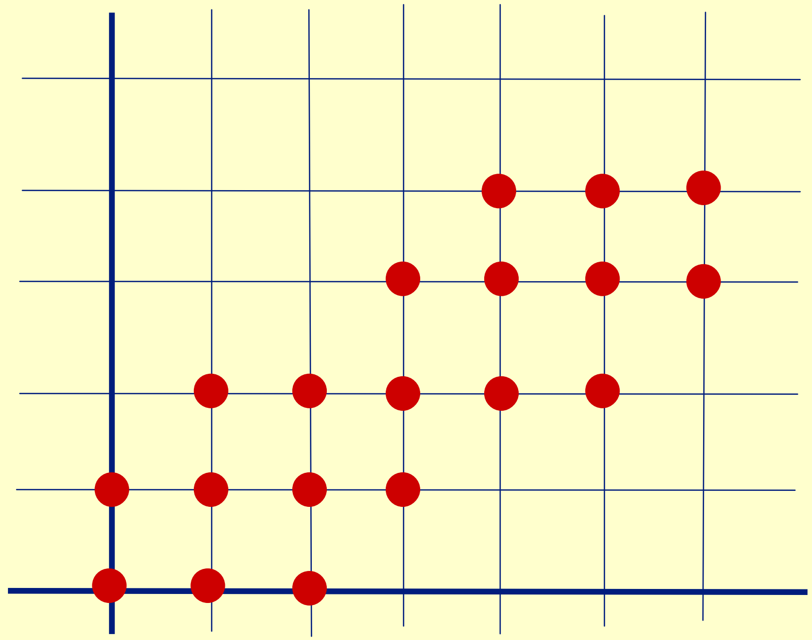
- Similar statement in Z^n .

Steiner convex sets





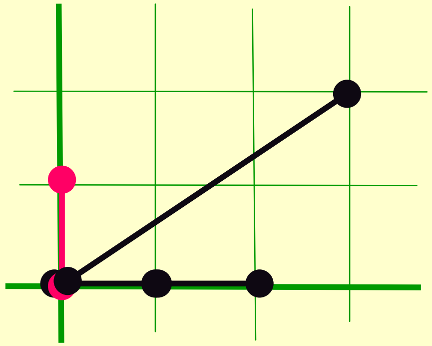




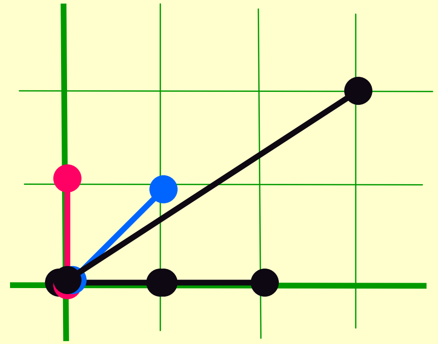
a)

b)

Steiner convex sets

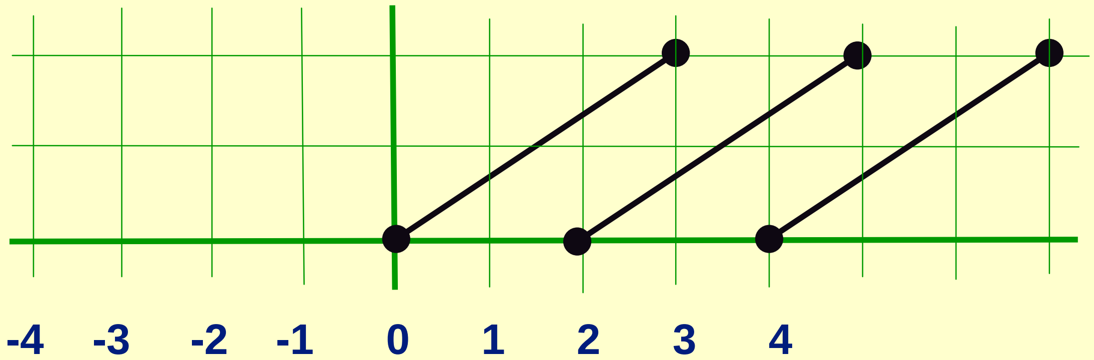


1rst directional measure

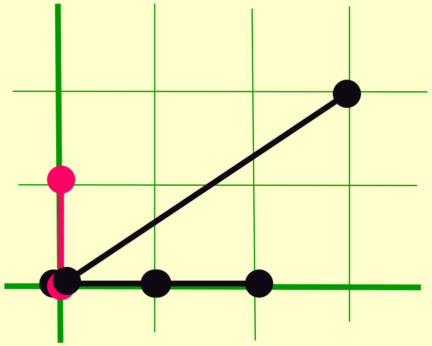


2nd directional measure

$2x - 3y = 0$, shifted by ●—● gives $2x - 3y = 2$



Steiner convex sets



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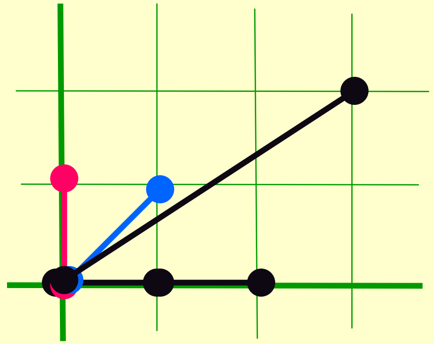

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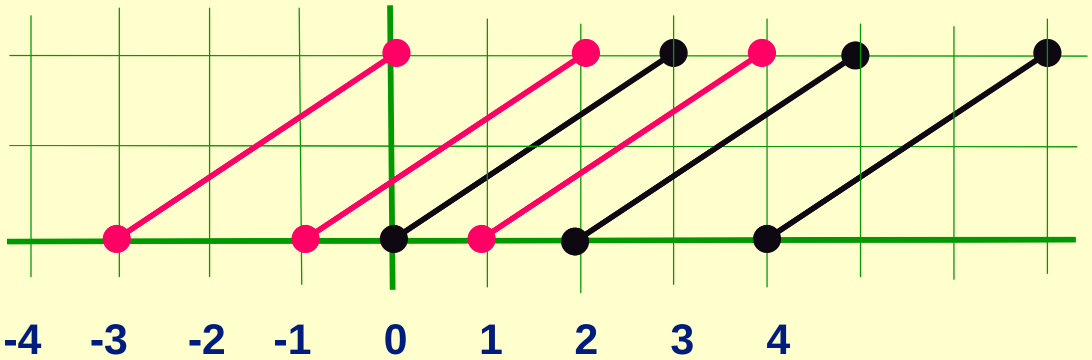
shifted by gives $2x - 3y = -3$



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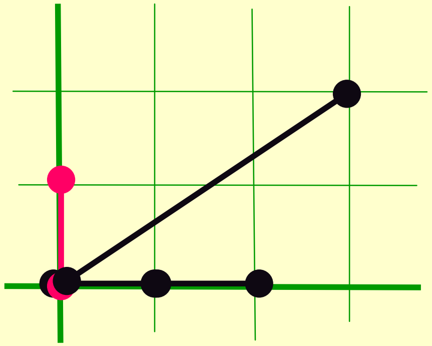

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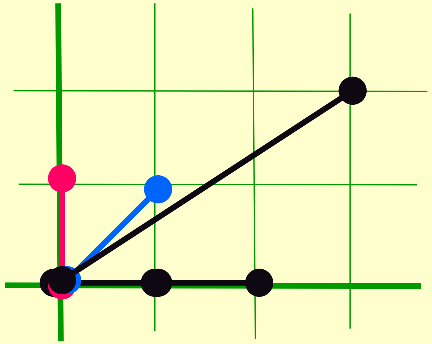

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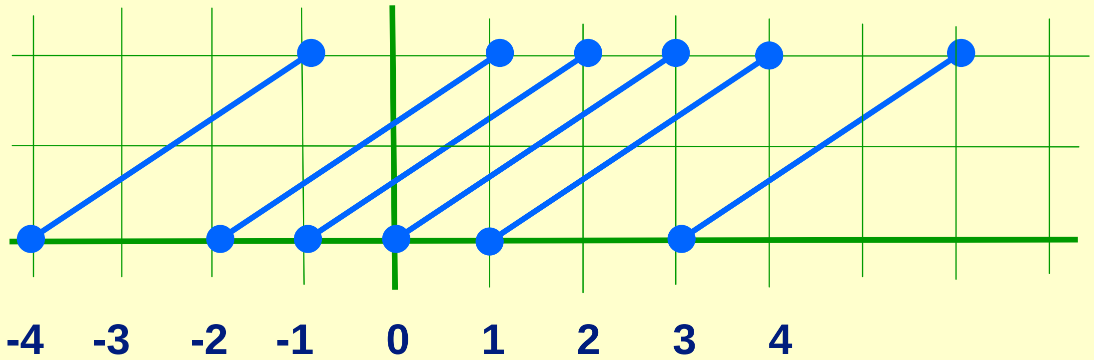
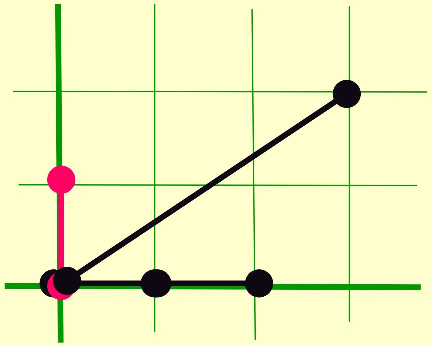
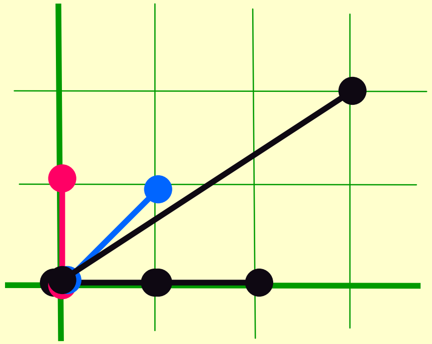
2x – 3y = 0 , shifted by  gives 2x – 3y = – 1



Steiner convex sets

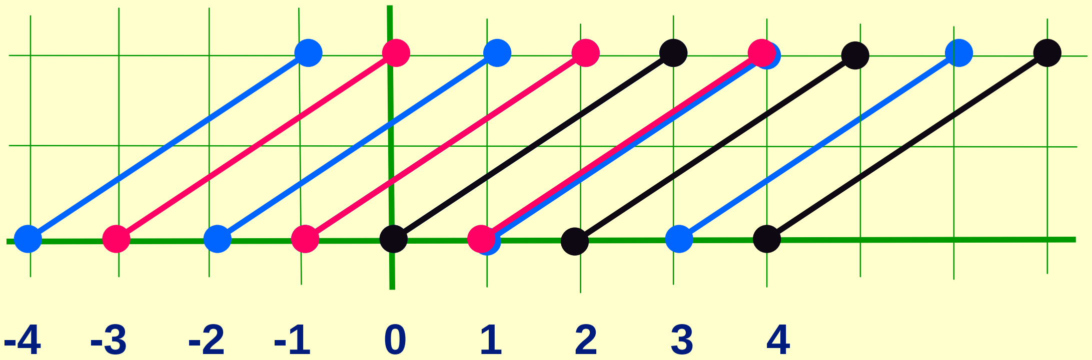


1rst directional measure

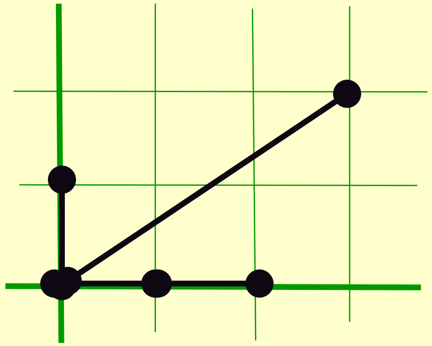


2nd directional measure

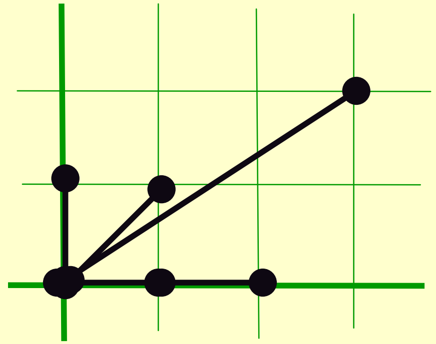
And with the previous shifts



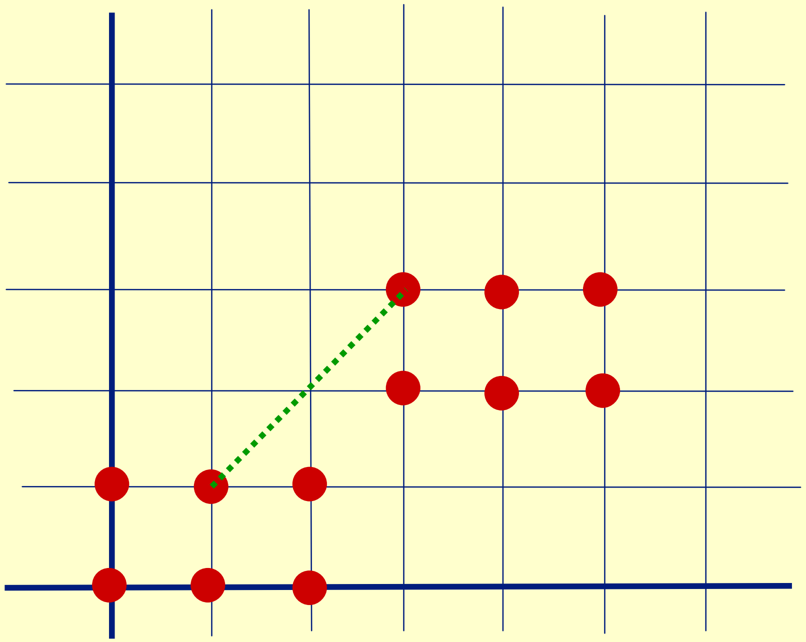
Steiner convex sets



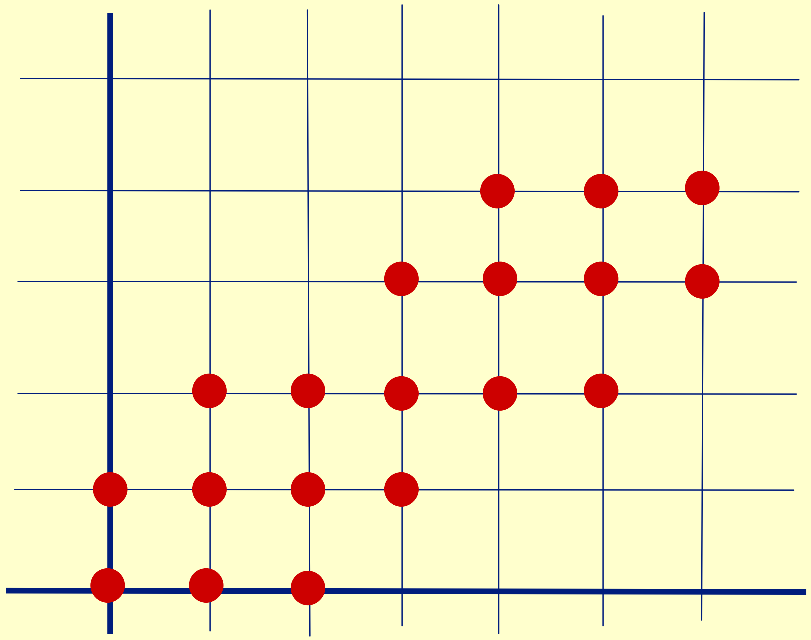
1rst directional measure



2nd directional measure

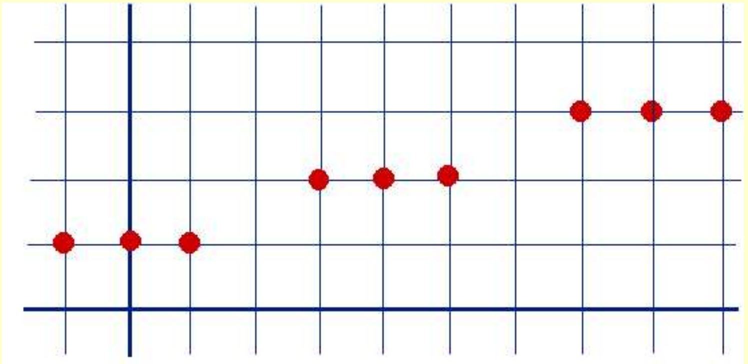


a)



b)

Steiner sets and connectivity



Theorem 4

In Z^n, the Steiner set X of measure $\{ k_i, 1 \leq i \leq p \}$ with $n \leq p$, is ***connected*** if and only if for each j such that $n < j \leq p$, the component ω_ι^j of direction ω_j w.r.t. axis ω_ι satisfies the inequality

$$k_j \;\; \omega_\iota^j \leq k_i$$

Anamorphoses

- An ***anamorphosis*** between two lattices $\mathcal{L}$ and $\mathcal{L}'$ is a mapping α such that

  α is a bijection from $\mathcal{L}$ and $\mathcal{L}'$

  α and α^{-1} are both erosions and dilations.

- ***Semi-anamorphosis*** When $\alpha : \mathcal{L} \rightarrow \mathcal{L}'$ is a dilation, every granulometry $\{\gamma_\lambda\}$ on $\mathcal{L}$ induces a granulometry $\{\zeta_\lambda\}$ on $\mathcal{L}'$ and we have

$$\alpha\, \gamma_\lambda(X) \leq \zeta_\lambda(\alpha\, X) \,,$$

  with the equality when is an anamorphosis.

- Example : α maps the plane R^n on a torus.

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$$\lambda B \oplus \mu B = (\lambda + \mu) B \quad \Leftrightarrow \quad B \text{ is } \textit{compact convex}$$

for ***defining*** homothetics, and we extended this type of decomposition to that of B into dilations of segments.

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- Arcwise connectivity turns out to be a very specific requirement, that one can add, but which plays no role in the theory.
- Though figures are 2-D, the whole approach works in Z^n.

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