IAF-98-A.4.05

THIRD-BODY PERTURBATION IN SPACECRAFT TRAJECTORY

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ABSTRACT

This paper has the goal of developing an analytical and a numerical study of the perturbation caused in a spacecraft by a third body involved in the dynamics. One of the important applications of the present research is to calculate the effect of Lunar and solar perturbations on high-altitude Earth satellites. There is a special interest to see under which conditions a near-circular orbit remains near circular. The so called "critical angle of the third-body perturbation," that is a value for the inclination such that any near-circular orbit with inclination below this value remains near-circular, is discussed in detail. The assumptions of our model are very similar to the ones made in the restricted three-body problem: a) There are only three bodies involved in the system: a main body with mass m_0 fixed in the origin of the reference system; a massless spacecraft in a generic orbit around the main body and a third body in a circular orbit around the main body in the plane x-y; b) The motion of the spacecraft is supposed to be a three-dimensional Keplerian orbit with its orbital elements disturbed by the third body.

The motion of the spacecraft is studied under two different models: i) A double-averaged analytical model with the disturbing function expanded in Legendre polynomials; ii) A full unaveraged three-body problem in three dimensions, without any truncation or approximation. The double-averaged models make the averages over the short period of the spacecraft and the long period of the distant third-body. Next, the theory developed here is used to study the behavior of a lunar satellite, where the Earth is the disturbing body. Several plots show the time-histories of the Keplerian elements of the orbits involved.

THE MATHEMATICAL MODELS

This problem has been under study before by several researchers, like Costa[1], Broucke[2], Kozai[3,4,5,6,7], Giacaglia[8], Kaula[9] and Prado and Broucke[10].

This section derives the equations required by the mathematical models used during the simulations made in this research. It is assumed that the main body with mass m_0 is fixed in the center of the reference system x-y. The perturbing body with mass m' is in a circular orbit with semi-major axis a' and mean motion n' (given by the expression $n'^2 a'^3 = G[m_0 + m']$). The massless spacecraft m is in a generic three-dimensional orbit which orbital elements are: a, e, i, ω, Ω and the mean motion is n (given by the expression $n^2 a^3 = Gm_0$). In this situation, the disturbing potential that the spacecraft has from the action of the disturbing body is given by:

$$R = \frac{\mu'}{\sqrt{r^2 + r'^2 - 2rr'\mathrm{Cos}(S)}} \tag{1}$$

Using the traditional expansion in Legendre polynomials (assuming that r' >> r) the following expression can be found:

$$R = \frac{\mu'}{r'}\sum_{n=2}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\mathrm{Cos}(S)) \tag{2}$$

The next step is to average those quantities over the short period satellite as well as with respect to the distant perturbing body. The standard definition for average used in this research is: Average of f = $\langle f \rangle = \frac{1}{2\pi}\int_0^{2\pi}(f)dM$, where M is the mean anomaly, that is proportional to time.

The equations obtained for the first four terms are[1]:

$$\langle \overline{R}_2 \rangle = \frac{\mu' a^2 n'^2}{16}\left[(2+3e^2)(3\mathrm{Cos}^2(i)-1)+15e^2\mathrm{Sin}^2(i)\mathrm{Cos}(2\omega)\right] \tag{3}$$

$$\langle \overline{R}_3 \rangle = 0 \tag{4}$$

$$\begin{aligned}\langle \overline{R}_4 \rangle = \frac{9n'^2 a^4}{65536a'^2}\Big[&144+720e^2+270e^4+(320+1600e^2+600e^4)\mathrm{Cos}(2i)+(560+2800e^2+1050e^4)\mathrm{Cos}(4i)+\ldots\\ &\ldots+(1680e^2+840e^4)\mathrm{Cos}(2\omega)+4410e^4\mathrm{Cos}(4\omega)+(2240e^2+1120e^4)\mathrm{Cos}(2i)\mathrm{Cos}(2\omega)+\ldots\\ &\ldots+(3920e^2+1960e^4)\mathrm{Cos}(4i)\mathrm{Cos}(2\omega)+5880e^4\mathrm{Cos}(2i)\mathrm{Cos}(4\omega)+1470e^4\mathrm{Cos}(4i)\mathrm{Cos}(4\omega)\Big]\end{aligned} \tag{5}$$

The equations were developed up to the order 8, but they are too long to be shown here.

After calculating $\langle \overline{R}_2 \rangle = \overline{\overline{R}}_2$, $\langle \overline{R}_3 \rangle = \overline{\overline{R}}_3$ and $\langle \overline{R}_4 \rangle = \overline{\overline{R}}_4$, the next step is to obtain the equations of motion of the spacecraft. They come from the Lagrange's planetary equations in the form that depends on the derivatives of the disturbing function R with respect to the Keplerian elements. Those equations are not developed here, due to space limitations.

THE FIRST INTEGRALS C_1 AND C_2

Two important first integrals of the double-averaged model of second order are C_1 and C_2. They are given by the expressions:

$$C_1 = (1-e^2)\mathrm{Cos}^2(i) \tag{6}$$

$$C_2 = e^2\left(\frac{2}{5} - \mathrm{Sin}^2(i)\mathrm{Sin}^2(\omega)\right) \tag{7}$$

C_1 can be recognized as the square of the z-component of the angular momentum and C_2 is a combination of C_1 and the law of conservation of energy of the system. The proof that they are constant is omitted here to save space, but they can be obtained by direct derivation of equations 6 and 7 with respect to time. From those equations it is also possible to see that $0 \le C_1 \le 1$ and $-0.6 \le C_1 \le -0.4$. An important consequence of the first integral C_1 comes directly from the equation 6. The inclination and eccentricity always vary in opposite directions, when the inclination is in the first quadrant, to keep C_1 constant. Those first integrals are also important in defining the properties of the system.

RESULTS

In this section some results are shown related to the third body perturbation problem. This section is divided in several sub-sections to show clearly several aspects of the problem.

The Near-Circular Orbits and the Critical Inclination

One of the most important properties of the third body perturbation is the existence of a critical value for the inclination between the perturbed and the perturbing body. This critical inclination is related to the stability of near-circular orbits. The problem considered here is to discover under what conditions a spacecraft that starts in a near-circular orbit around the main body remains in a near-circular orbit after some time. The answer for this question depends on the initial inclination i_0. There is a specific critical value such that if the inclination is higher than that the eccentricity increases and the near-circular orbit becomes very elliptic. Alternatively, if the inclination is lower than this critical value the orbit stays nearly circular.

The problem considering an exact circular orbit (e = 0.0) is studied separately in the next section. The problem of near-circular orbits is considered very important because usually a spacecraft that is nominally in a circular orbit has perturbations from other sources that makes its eccentricity to move way from the nominal value 0.0.

In the double-averaged second-order model this critical situation occurs when $\mathrm{Cos}^2(i) = 0.60$ (i = 39.2315 degree). The behavior of the inclination and the eccentricity with time is studied for near-circular orbits covering a large range of initial inclination ($0° < i_0 \leq 80°$). Figs. 1 and 2 show the results. For those simulations the initial orbit used always have Keplerian elements a = 0.1, $e_0 = 0.01$, $\omega = \Omega = 0$. The system of primaries is the Earth-Moon. The initial inclination i_0 vary as shown in the figures. Remember that the time is defined such that the period of the disturbing body is 2π. In that way 1000 units of time in those figures correspond to about 160 orbits of the disturbing body. The mathematical models used in those simulations have order four. The Figs. are divided in three parts: values of i_0 below the critical value ($i_0 < 35°$), in the region of the critical value ($38° < i_0 < 43°$) and above the critical value ($i_0 > 47°$). Fig. 1 shows the behavior of the inclination. The region below the critical value is not shown, because the inclination remains constant.

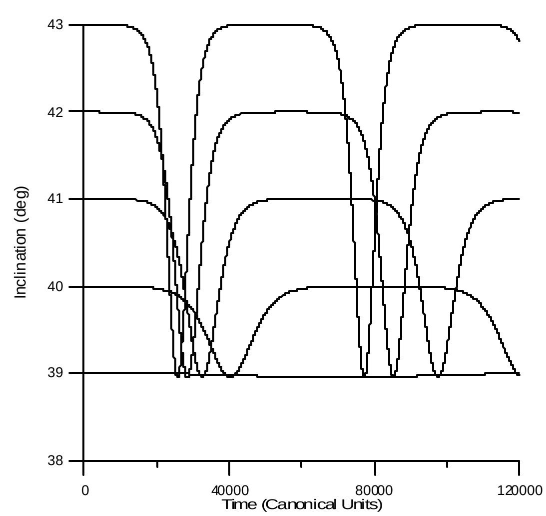


Fig. 1 - Time-History for the Inclination.

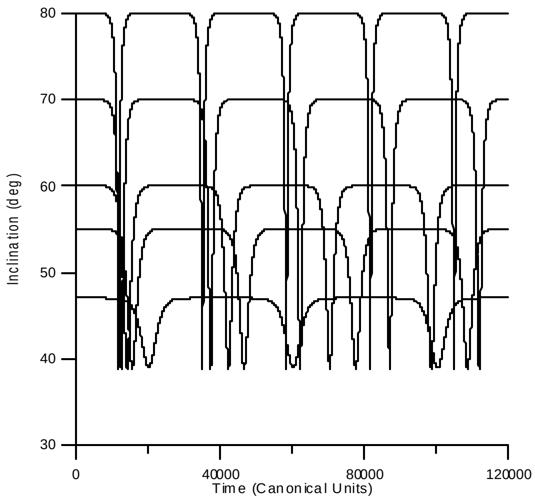


Fig. 1 (Cont.) - Time-History for the Inclination.

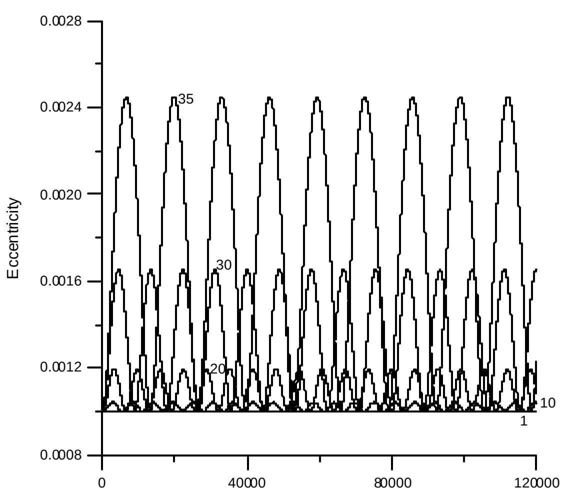
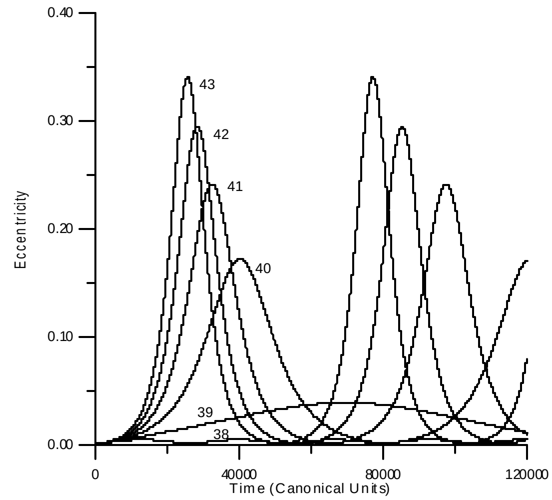


Fig. 2 - Time-History for the Eccentricity.



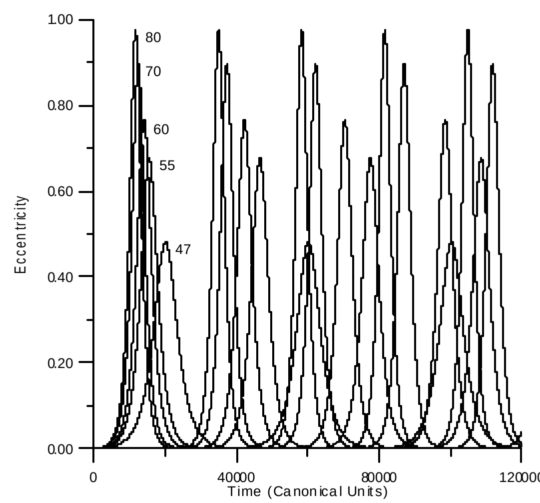
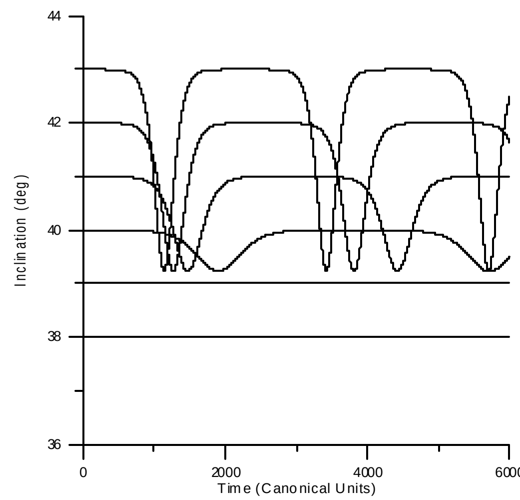


Fig. 2 (Cont.) - Time-History for the Eccentricity.

The results show that for values of the initial inclination i_0 below the critical angle ($i_0 <$ 35°) the eccentricity oscillates with a very small amplitude (less then 0.025 in most of the cases) that decreases fast when i_0 decreases. The inclination remains constant in this situation. For values of i_0 around the critical value ($39° < i_0 < 43°$) it is possible to see that the eccentricity oscillates with a larger amplitude (about 0.35) that increases with the increase of i_0. The inclination has a very characteristic behavior in this region of i_0. For values of i_0 slightly below the critical angle the inclination stays close to i_0 with an oscillation of small amplitude and large period. For values of i_0 slightly above the critical value the inclination starts at i_0, decreases until the critical value and then it returns to its original value i_0. For values of i_0 well above the critical value ($i_0 \geq 47°$) the eccentricity oscillates with increasing amplitudes that goes close to 1.0. The inclination keeps its characteristic behavior of starting at i_0, decreasing to the critical value and then returning to its original value i_0. The figure also shows that this behavior repeats itself in an endless cycle. The time required to reach the critical value decreases when i_0 increase. Those results show that the critical angle is not a sharp separation between stable and unstable near-circular orbits. This region has a gradual transition where the eccentricity oscillates with an amplitude that increases fast with i_0, reaching the value 1.0 only in the case $i_0 = 90°$.

The practical application of those results is that only near-circular orbits with inclination lower than the critical value are stable in the long range, since above this value the orbit looses its characteristics of near-circularity.

Figs. 3 and 4 reproduces the same study for the case of a lunar satellite. The initial orbital elements used are: a = 0.01, $e_0 = 0.01$, $\omega = \Omega = 0$. The initial inclination i_0 has the same values used for the Earth's satellite studied before. The results are similar to the ones obtained in the previous case. The differences are: i) the Keplerian elements of the lunar satellite oscillates faster than the Earth's satellite; ii) the maximum values reached by the eccentricities are a little smaller in all the situations; iii) The minimum reached by the inclination are a little higher for the lunar satellite, specially when the inclinations are closer to the critical value.



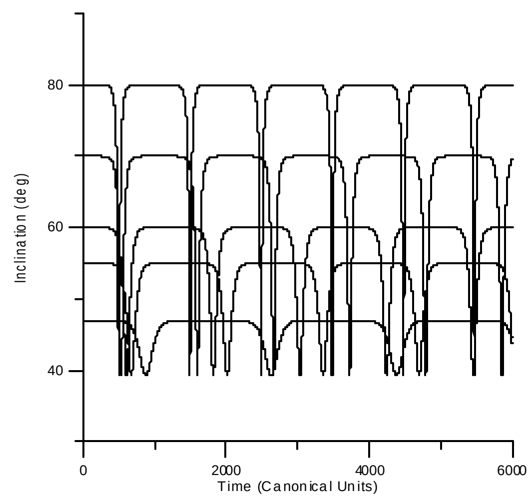


Fig. 3 - Time-History for the Inclination for a Lunar Satellite.

Finally, we studied both cases in the region close to the critical angle under the model given by the restricted three-body problem, that has no approximation or truncation of any type. The result is the existence of short period terms, that adds an oscillation following the lines given by the truncated models, initially. After some time the behavior becomes completely oscillatory with a very fast period.

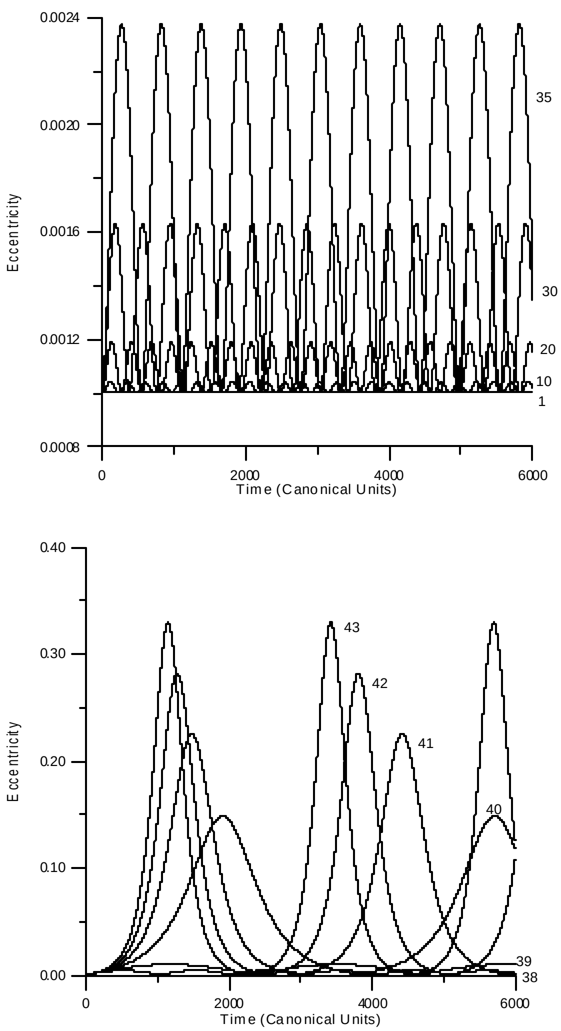


Fig. 3 - Time-History for the Eccentricity for a Lunar Satellite.

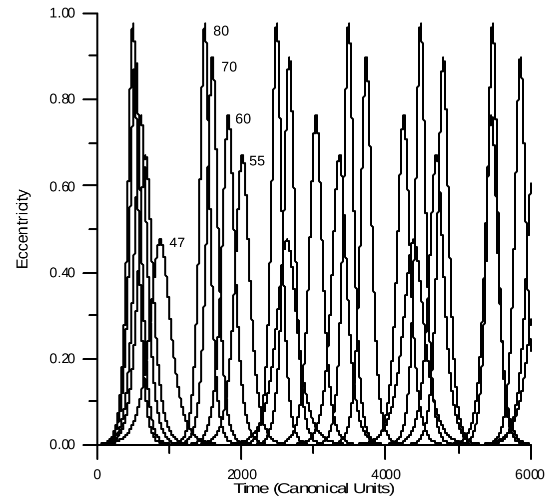


Fig. 3 (Cont.) - Time-History for the Eccentricity for a Lunar Satellite.

<u>The Circular Orbits</u>

Directly from the equations of motion for the double-averaged second order model it is possible to identify the existence of circular solutions for this problem. It means that, in the ideal case of an orbit that starts with eccentricity zero, its eccentricity remains always zero. This occurs because the right-hand side of the equation for the time derivative of the eccentricity is zero (it is proportional to the eccentricity). Another property of those orbits is that the inclination is also constant for the same reason (the time derivative of the inclination is proportional to the square of the eccentricity).

This is not true for high order models. The expressions for the equations of motion have terms that are independent of the eccentricity that generate a term with the eccentricity in the denominator for the expression for $\frac{de}{dt}$. This fact does not allow the right-hand side of those equations to vanish.

The same occurs for the inclination, since its variation also depends on the derivative $\frac{\partial \overline{\overline{R_4}}}{\partial \omega}$.

The evolutions of these two quantities (eccentricity and inclination) were studied under the full-unaveraged three-body problem. The results show that the circular solutions with constant inclination do not exist in this more realistic model. The eccentricity oscillates with a large amplitude. The inclination remains constant most of the time, but from time to time it decreases to the value of the critical inclination and then it returns to its initial value. The minimums in inclination occur in the same time with the maximums in eccentricity. The figures are not shown here to save space.

The general conclusion is that the circular solutions with constant inclination appear due to the truncation of the Legendre polynomial in terms of second-order and it is not a physical phenomenon.

The Equatorial Orbits

Another property of this system that comes directly from the inspection of the equations of motion is the existence of equatorial orbits. It means that if an orbit starts with i_0 = 0, the inclination and eccentricity remain constant and the orbits remain in the equatorial plane. In the second-order model this property is evident from the equations of motion. If i_0 = 0, then the right-hand sides of the expressions for $\frac{de}{dt}$ and $\frac{di}{dt}$ are also zero, because they are proportional to $Sin^2(i)$ and Sin(2i), respectively.

The high order models has terms in Sin(i) in the denominator of the expressions for $\frac{de}{dt}$ and $\frac{di}{dt}$. This is due to the existence of several terms independent of the inclination in the expression for $\frac{\partial \overline{\overline{R_4}}}{\partial \omega}$, which is part of those expressions.

The numerical integration of the full-unaveraged model shows the existence of equatorial solutions also in this more general model. The inclination remains zero all the time and the eccentricity has only a short period oscillation with very small amplitude.

Frozen Orbits

From the equations of motion for the second-order averaged model it is possible to detect the existence of a new family of special orbits. They are the orbits called "Frozen Orbits." This family is composed by the orbits that have $\frac{de}{dt} = \frac{di}{dt} = \frac{d\omega}{dt} = 0$, what means that the eccentricity, inclination and argument of periapse are constants.

From the equations of motion it is possible to derive the conditions for this situation. They are: $Sin(2\omega) = 0$, $Cos(2\omega) = -1$. It implies that $\omega = 90°$ or $\omega = 270°$. From the equation $\frac{d\omega}{dt} = 0$ one more condition is available. It is:

$$5Cos^2(i) - 1 + e^2 - 5\left(1 - e^2 - Cos^2(i)\right) = 0 \quad \Rightarrow$$

$$\Rightarrow \quad e^2 = 1 - \frac{5}{3} Cos^2(i) \qquad (8)$$

From the equation 8 the condition $Cos^2(i) < 3/5$ is obtained. This condition sets a minimum value for the inclination. Those conditions are valid only for the second-order-averaged model. When submitted to the fourth-order averaged model a frozen orbit is destroyed and it shows oscillations in all the three orbital elements.

CONCLUSIONS

This paper develops mathematical models to study the third-body perturbation: the double-averaged in several orders and the full unaveraged three-body problem.

The results show in detail the behavior of the orbits with respect to the initial inclination and the rule of the critical inclination in the stability of near-circular orbits. They show that this critical value is a transition region where the eccentricity has an oscillation that increases in amplitude.

It is also shown the existence of equatorial solutions and the non-existence of circular solutions in the unaveraged problem.

The "Frozen Orbits" found in the double-averaged second-order model are studied in the fourth-order model. It is shown that they have their Keplerian elements disturbed by an oscillation.

The study of a lunar satellite completes the paper. It showed that similar results are obtained, but the Keplerian elements of the lunar satellite oscillates faster and shows higher minimum for the inclination and smaller maximum for the eccentricity.

REFERENCES

1. Costa, I. V., "Estudo da Perturbação de Terceiro Corpo em Um Satélite Artificial da Terra"; Master Thesis, INPE, São José dos Campos-SP-Brazil, 1998.

2. Broucke, R. A. "The Double Averaging of the Third Body Perturbations"; Classnotes, Texas University, Austin-TX-USA, 1992.

3. Kozai, Y., "Smithsonian Astrophysical"; The Astronomical Journal, Cambridge-Massachusetts, 1959. Vol. 64, nº 22, pg. 367.

4. Kozai, Y., "Second-Order Solution of Artificial Satellite Theory Without Air Drag"; The Geophysical Journal, Cambridge-Massachusetts, 1962.

5. Kozai, Y., "Secular Perturbations of Asteroids with High Inclination and Eccentricity"; The Astronomical Journal, Cambridge-Massachusetts, 1962, Vol. 67, nº 09, pg. 591, 598.

6. Kozai, Y., "Effects of the Tidal Deformation on the Motion of Satellites", Publ. Astron. Soc., Japan, 1965, vol. 17, pp. 395 – 402.

7. Kozai, Y., "A New Method to Compute Lunisolar Perturbations in Satellite Motions", Smithsonian Astrophyical Observatory, Cambridge, MA., SAOSR – 349, 1973.

8. Giacaglia, G., "Lunar Perturbations of Artificial Satellites of the Earth", Smithsonian Astrophysical Observatory, Cambridge, MA, SAOSR – 352, 1973.

9. Kaula, W. W., "Development of the Lunar and Solar Disturbing Functions for a Close Satellite". Astonautical Journal, vol. 67, pg. 300, 1962.

10. Prado, A. F. B. A., Broucke, R. A., "An Analytical Study of the Third Body Perturbation"; Research report, Texas University, Austin-TX-USA, 1993.