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FUNDAMENTALS OF ASTROPHYSICAL PLASMAS

Vinod krishan

*Indian Institute of Astrophysics
Bangalore 560034, India

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Preface

Life was simple when the dynamic, the spectral and the resolving powers of our instruments were small. One observed whole objects – planets, stars, sunspots, galaxies, often in rainbow colors. Then the revolution occurred: we acquired the centimetric eyes, the millimetric eyes, the infrared eyes, the ultraviolet eyes, the X-ray eyes and the γ-ray eyes. With these we see mottles on the surface of stars, streams in sunspots, and spirals in nuclei of galaxies. We see regions of multiple mass densities and temperatures in a precarious balance, losing it occasionally, exhaling flares. The universe is timed, cosmic phenomena are clocked; eternity is lost and variability is bought. Microarcsecond resolutions revealed stirring and sizzling interiors underneath serene surfaces. Short durations and small scales demanded employing a discipline with similar attributes – the discipline of Plasmas and Fluids – known more for its complexity than for its felicity.

These lecture notes are based on a course on Fundamentals of Astrophysical Plasmas designed for graduate students of astrophysics and astronomy, geophysics, and plasma physics at the INPE. The background of the students varied from a faint familiarity with the subject to a working knowledge of some aspects. However, the will to learn and understand more of it was writ large on every countenance. Each session of lectures lasted for more than two hours with a very brief coffee break and continued discussions. I was guided, in my choice of material, by the desire to present a systematized and logical development of this complex field without apologizing for the rigour. The students, as my perception goes, lapped it up and responded enthusiastically to the suggestion of preparing these lecture notes.

Jorge Albuquerque de Souza Corrêa, one of the students of the "Divisão de Astrofísica", was lured into coordinating the production of the lecture notes and he has done this job admirably well. I appreciate and thank Jorge and his team mates Jean Carlos Santos, Marcelo Henrique Gonçalves do Nascimento and Marcia Oliveira for their interest and industry, without which my transparencies could never be converted into this user friendly form. I particularly thank Alessandra de Mello Stocco for her impeccable typing of the major portions of the non-mathematical material.

I am glad that I got to know Dr. José Williams dos Santos Vilas Boas, the organizer of the course. In fact, it was he who conceived the idea of producing these lecture notes and we fell into his trap! We discussed matters outside this course and I particularly enjoyed learning about his research interests in Meteor Showers and trails which might have electrically charged dust component.

I wish to put on record my sincere thanks to FAPESP (process n° 01/06031-8) for supporting my visit to the Solar Physics Group of the INPE.

Special thanks are due to my host, collaborator and friend Prof. H.S. Sawant for his continued support throughout my year long (August 2001 – August 2002) stay at INPE. And this brings me finally to express my deep gratitude to the Institute (INPE), especially to the "Divisão de Astrofísica" (DAS) for providing me whatever help I needed from time to time. I shared a warm and friendly relation with many of the DAS members. I do hope that these notes will help the uninitiated to venture deeper into the complexities and challenges of astrophysical plasmas.

Vinod Krishan.

Chapter 1
INTRODUCTION TO PLASMA PHENOMENA

1.1. The Agenda

Out of the four fundamental forces which have shaped the universe as we see it today, the gravitational force dominates the macrocosmos. Newton's law of gravitation along with Einstein's general and special theories of relativity was the major preoccupation of the astronomers of the last century. After obtaining a decent understanding of the motion of the heavenly bodies, it was natural that the investigator turned her attention to the investigation of their working. How and why do Cosmic objects shine or not shine were the questions whose answers lay in the interplay of the other three forces. The Electromagnetism with its myriad manifestations turned out to be the next dominant occupation of the astronomers as they learnt to see under the lamp-post! And plasmas appeared as one more consequence of the electromagnetism. It did not take the inquisitive astronomer long to apprehend the importance of plasmas in their diverse abodes and forms, be it in the humble environs of the planets or the dark cores of stars or the vastness of the intergalactic space. Plasmas are here to stay so we better get to know them in all their glory if we wish to sustain any pretentions of knowing our universe. This appears to be the agenda of the present century astrophysicists!

1.2. Description of a Field

On being asked how should one describe one's field of activity, the Nobel Laureate Subramanyan Chandrasekhar replied: The description should answer the following questions (i) What is the nature of the system? (ii) towards what purpose is its study and (iii) what are the techniques for its investigation? Taking inspiration from the 'man who knew the best', we shall in this chapter, begin with the nature of a plasma and continue to explore it in other chapters, pausing in Chapter 2 to develop the techniques. The purpose unfolds itself as we learn more and more about the nature of plasma. However, if you like one-liners, we can say: The nature of a plasma is **Hyperactive,** the purpose is to **Comprehend the Universe** and the techniques employed are **Many-Body.**

1.3. Birth of Plasma Physics

In 1879, William Crookes was investigating the phenomenon of electrical discharge in neutral gases. He coined the term – "Fourth State of Matter" in order to describe the ionized component of the gas generated by the electrical discharge. However, it was Irving Langmuir who while studying electronic devices based on ionized gases, in 1927, christened the ionized gas as plasma, since the ionized conducting fluid carrying high speed electrons, ions and atoms reminded him of the blood plasma carrying red and white corpuscles along with germs!

The research in plasma physics spread with the development of the radio and the subsequent discovery of the ionosphere. The reflection of radio waves from the ionospheric plasma made the short-wave communication possible around the world. The astrophysicists recognized the universal presence of plasmas and their role in the production and propagation of radiation from our nearest star, the Sun, to the most distant quasars. The laboratory studies of plasmas received a big boost with the promise of a nearly free, and pollution free sources of energy from the controlled thermonuclear fusion. The plasma physicists dreamt of creating a Sun in the laboratory. Alas they are still dreaming!

1.4. What's a plasma?

We are fairly well familiar with the three states of matter – the solid, the liquid and the gas. Some of us have some familiarity with a fourth state known as the plasma. This plasma state of matter is so prevalent throughout the cosmos that it is the **Plasma Universe**, a term coined by the Nobel Laureate Hannes Alfven, that we observe and are in awe of. We may find it disconcerting that our expertise and experience in the three common states of matter are barely adequate to comprehend only about one percent of the universe, the remaining 99% being in the plasma state. Of course, if 90% of the universe is actually made up of the so called **Dark Matter** (gravitating but nonradiating matter), as some people believe, then the remaining 10% is the radiating and reflecting and, therefore, observable matter; plasmas then form the silver lining of this dark universe. This, in no way exempts us from the study of plasmas since it is the visible radiating matter that telleth of the unseen! Let us forget about fractions and start exploring the nature of a plasma. So, what is a plasma?

A solid can be converted into a liquid and a liquid can be converted into a gas by heating them. With an increase in the temperature of a solid or a liquid, the freedom of movement of the constituent atoms or molecules increases. What happens if we continue to heat a gas? The gas is ionized. The atoms or molecules lose some of their electrons and a sizzling sea of positively charged ions and negatively charged electrons is created. The net electric charge of the system is zero. Some neutral atoms and molecules may still be present. The electric charges in motion produce electric currents. These are the makings of a plasma. A plasma is an electrically conducting gas, though the converse need not be true.

1.5. Plasmas as Cleansers

The Plasma technology has entered our very homes. The plasma based ultraviolet and X-ray sources as well as the energetic electron beams have found applications for protection against diverse environmental hazards. The plasma ultraviolet sources incapacitate the DNA of the microorganisms in water in a matter of a few seconds at a small fraction of the cost of the traditional water purification methods such as boiling. A plasma with iron filings injected into it produces excited states of atoms, the emissions from which carry their identification. Such an atomic metal emission detector is placed in a smokestack where it detects dangerous fuel impurities. The plasma technology is turning out to be all encompassing from depositing a few angstrom thick layer of gold on brass to creating space-specification materials to plasma crystals with yet undiscovered characteristics!

1.6. Plasmas in the Universe

Plasmas are found everywhere in the universe. Infact our earth and all the other planets of the solar system are in a continuous flux of plasma – the solar wind. Plasmas occur with densities as small as or smaller than those in the highest vacuum that can be produced in a terrestrial laboratory and as large as and larger than those existing in the atomic nuclei. An idea of the range of densities, temperatures and the not so well known magnetic fields is given below:

(1) Earth's Ionosphere

The Solar ultraviolet radiation ionizes the upper atmosphere in the altitude range of ~70~1500 km. Here electron density varies from 10^3 cm^{-3} to 10^6 cm^{-3}, the temperature from 10^2 K to 10^4 K and the magnetic field is of the order of a fraction of a Gauss. These are only fiducial numbers. The ionosphere undergoes large variations in all its properties on various time scales. Due to its ability to guide radio waves and its coupling to the overlying magnetosphere, the ionospheric studies are an active field of research. The electrical conductivity of the ionosphere is another not so well known quantity. In addition to being inhomogeneous, it can also become anomalously small due to the intervention of plasma instabilities to which the ionosphere is very much vulnerable.

(2) The Solar Wind

By observing the deflection of the ionized part of the cometary tails, Biermann in the fifties conjectured that it must be due to a flux of protons from the Sun. Parker, later developed a model of the solar wind and inferred that the corona is at 10^6 K and expanding. The solar wind has an electron density varying from 0.5 cm^{-3} to 10^2 cm^{-3}, a temperature from 10^5 K to 10^6 K, a velocity anywhere from 200 km s^{-1} to 900 km s^{-1} and a magnetic field of 0.2 to 80 nanotesla. Again the solar flares cause drastic changes in all these parameters.

(3) The Solar Corona

The million degree solar corona exhibits a host of structures in the form of loops, arches and filaments. These structures have an electron density around 10^9-10^{10} cm^{-3} and are magnetically dominated. In other words, the magnetic pressure is much larger than the kinetic pressure. The heating of the solar corona is a big puzzle and has remained an active field of the solar physics research.

(4) Stellar Interiors

The stellar cores harboring thermonuclear reactions have higher than metallic densities and temperatures greater than tens of million degrees Kelvin. Some of plasma shielding effects play an essential role in the physics of nuclear reactions and energy generation.

(5) Pulsars

Pulsars are highly compact objects with neutrons as the major constituent and a small fraction in the form of electrons and protons. The densities are higher than nuclear densities in the core. However the atmosphere of the pulsars, especially the radio emitting region, has typically solar coronal densities but with magnetic field of 10^5 - 10^6 Gauss. The relativistic electron-positron plasma pulled out from the surface of a pulsar by inductive electric field has densities of the order of 10^{13} - 10^{15} cm^{-3}.

(6) Other Astrophysical Objects

A variety of compact objects such as nuclei of active galaxies, extragalactic jets accretion disks around black holes, X-ray binaries, intracluster region of galaxies and many more contain hot and dense and hot and tenuous plasmas which are detected though their emission in different bands of the electromagnetic spectrum.

1.7. Making Plasmas

Astrophysical plasmas consist mainly of protons, electrons, He ions and traces of heavy ions and atoms. In equilibrium, these particles obey the **Boltzmann Distribution**, so that the number density n_e of particles in a given energy state E_l, at a temperature T is determined from:

$$n_l = g_l \exp\left(-\frac{E_l}{K_B T}\right) \tag{1.1}$$

where g_l is the degeneracy factor describing the number of states with the energy E_l and K_B is the Boltzmann constant. The ratio p of the number density of particles in states l and m is, therefore, given by:

$$p = \frac{n_l}{n_m} = \frac{g_l}{g_m} \exp\left[-\left(\frac{E_l - E_m}{K_B T}\right)\right] \tag{1.2}$$

If we identify the state l with the electron-ion pair and the state m with a neutral atom, we can find the fractional ionization from equation (1.2) provided the energy difference

(E_l-E_m) is equal to the ionization energy I of the atom. Thus the number density of ions n_i is found to be:

$$n_i = n_m \left(\frac{g_i}{g_m} \right) \exp\left[-\frac{I}{K_B T} \right] \tag{1.3}$$

The values of g_i and g_m are found from quantum – mechanical calculations and here, we give an approximate formula for the ratio

$$\frac{g_i}{g_m} \cong \left(\frac{2\pi m_e K_B T}{h^2} \right)^{3/2} \frac{1}{n_i} \cong 2.4x10^{21} T^{3/2} n_i^{-1} \tag{1.4}$$

where h is Planck's constant, m_e the electron mass, and the temperature T is in degrees Kelvin. Substituting Equation (1.4) in Equation (1.3), we obtain **Saha's Ionization Equation**:

$$\frac{n_i}{n_m} = 2.4x10^{21} T^{3/2} n_i^{-1} \exp\left[-\frac{I}{K_B T} \right] \tag{1.5}$$

We find that a significantly large degree of ionization is achieved for hydrogen even at temperatures much below that corresponding to the ionization energy of 13.6 eV. This is **Thermal Ionization**. Stellar plasmas are mostly produced through thermal ionization.

Another way of producing ionization is by applying electric fields for example through an Electric Discharge, in terrestrial laboratories or during lightening in planetary atmospheres. Matter can also be ionized by the action of electromagnetic radiation. The ultraviolet radiation from hot stars ionizes most of the interstellar medium. The solar ultraviolet radiation creates our ionosphere.

In high density regions, collisions among electrons and atoms can also cause ionization. At even higher densities, the phenomenon of **Pressure Ionization** occurs. During this process, the matter is so tightly packed that electrons are squeezed out of their energy levels if their energy, the **Fermi Energy**, exceeds the ionization energy.

Interiors of large gaseous planets such as Jupiter, and the ultradense stars called White Dwarfs are some of the probable sites of pressure-ionized plasmas.

1.8. Qualifications of a Plasma

Ionization is a necessary but not a sufficient condition for a plasma. In order to qualify as a plasma, an ionized gas must admit **Quasineutrality** and exhibit **Collective** behavior.

Two isolated charges separated by a distance experience the Coulomb force. In an ionized gas, there are many other charges between these two charges, influencing and being influenced by them. Thus, the motion of a given charge is determined collectively by the entire system of charges. There are no free charges in a plasma. As these charges move around, local concentrations of positive and negative charges can develop, producing electric fields. This charge separation, though, can occur only on microscopic scales, for otherwise forbiddingly large electric fields would be generated. The existence of a very small amount of charge separation over a very short spatial scale for a very short time interval is what is meant by the **Quasineutrality** of a plasma. It is for this ability of a plasma to sustain a tiny difference in positive and negative charge densities that its study has acquired the status of an independent discipline. The charge separation arises due to the thermal motion of electrons and ions. Thus, electric fields of strengths such that the associated electrostatic energy per particle does not exceed the thermal energy per particle are produced spontaneously. This condition restricts the amount, the extent and the duration of charge separation in a plasma. The electric field E due to a net charge density Q existing over a region of linear extent l is $E=4\pi Ql$ and the electrostatic energy of an electron or proton is $4\pi Qel^2/2$. The average thermal energy per particle with one degree of freedom is $K_BT/2$. Therefore,

$$4\pi Qe\frac{l^2}{2} \leq \frac{K_BT}{2} \tag{1.6}$$

and l is the spatial scale of charge separation. For complete charge separation in a plasma of electron density n, $Q=en$, so that

$$l \leq \left(\frac{K_B T}{4\pi n e^2} \right)^{1/2} \equiv \lambda_{De} \tag{1.7}$$

where λ_{De} is called the **Debye Length** of electrons. It is also known as the screening distance, as it screens the charge separation from the rest of the system. The time duration, τ_e, for which this charge separation can exist is the time taken by an electron to travel the distance l with the mean thermal speed V_e, so that

$$\tau_e = \frac{l}{V_e} = \left(\frac{m_e}{4\pi n e^2} \right) \equiv \omega_{pe}{}^{-1} \tag{1.8}$$

and ω_{pe} is called the **Electron-Plasma Frequency**. Thus, the higher the density n, the shorter is the time scale τ_e and the spatial scale λ_{De}. We realize that we can have a spatial scale τ_i and a time scale λ_{Di} for ions and that $\lambda_{Di} \leq \lambda_{De}$ and $\tau_i >> \tau_e$. Therefore τ_e is the shortest time scale for which charge separate can exist and strict neutrality can be violated. It is in this sense that a plasma is **Quasi-Neutral**; on a spatial scale λ_{De} and a time scale τ_e, departures from charge neutrality exist.

In this sea of electrons and ions, there must be Coulomb collisions among these particles. The collisions produce mixing of charges, and therefore reduce the spatial scale of charge separation. In order that there still exists a finite spatial scale of charge separation, the mean free path λ_e of the electrons must be much larger than the scale of charge separation λ_{De}. This condition gives:

$$\lambda_e = \frac{V_e}{\nu_{ei}} >> \left(\frac{K_B T}{4\pi n e^2} \right)^{1/2} \tag{1.9}$$

or

$$\omega_{pe} >> \nu_{ei}$$

i.e. the electron plasma frequency must be much larger than the electron-ion collision frequency ν_{ei}. Another important consequence of a low collision frequency ν_{ei} is that the electrons and ions take a long time to thermalize with each other and reach a common temperature. For durations smaller than the collision time $(\nu_{ei})^{-1}$, electrons and ions can remain at different temperatures T_e and T_i respectively. An ionized gas with vanishingly small λ_{De} does not qualify to be a plasma as such as a system is strictly neutral and not quasi-neutral. Further, in order to treat the part of the system enclosed in a region of linear dimension λ_{De} as a statistical system with a temperature T and number density n, it must contain a large number of particles. The number of particles N_d in a sphere of radius λ_{De} – called the **Debye Sphere**, is

$$N_d = \frac{4\pi}{3} n \lambda_{De}{}^3 >> 1 \tag{1.10}$$

We have now found all the conditions that an ionized gas must satisfy before it can be called a plasma. These are given by equations (1.9) and (1.10) along with the obvious requirement that the size of the system must be larger than all the characteristic spatial scales such as λ_e and λ_{De}. We shall learn more about the consequences of quasineutrality in other chapters.

1.9. Electrostatic Potential in a Plasma-Debye Screening

Suppose we insert a positive test charge q in a plasma. We expect that it will immediately attract a cloud of negative charges around it. The effect of the test charge is not felt outside this cloud. The rest of the plasma is screened from the test charge by the cloud. What is the size of this cloud and what is the potential due to q in a plasma? According to Poisson's equation, the electric potential φ can be determined from

$$\nabla^2 \varphi = 4\pi e(n_e - n_i) \tag{1.11}$$

where n_e and n_i are respectively the electron and singly charged ion densities in the plasma. The two densities become unequal due to the presence of the potential φ caused by the test charge q. The particles, in addition to their kinetic energy, now possess potential energy $|e\varphi|$. If we assume that at a temperature T_e the electrons follow the Boltzmann distribution given by

$$n_e = N \exp\left[-\frac{W_e}{K_B T}\right] \tag{1.12}$$

where the total energy $W_e=(m_e V^2/2) - e\varphi$, we find

$$n_e = n_0 \exp\left[\frac{e\varphi}{K_B T}\right] \tag{1.13}$$

where n_0 is the electron density in the absence of the potential. The ions at their temperature T_i also follow a similar distribution:

$$n_i = n_0 \exp\left[-\frac{e\varphi}{K_B T}\right] \tag{1.14}$$

These Boltzmann distributions are discussed again in Chapter 4. For $|e\varphi / K_B T_e| << 1$ and $|e\varphi / K_B T_i| << 1$ we can expand the exponentials in Equations (1.13) and (1.14), substitute in Equation (1.11), and find that the spherically symmetric potential φ at a distance r from the charge q obeys the linear equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\varphi}{\partial r}\right) = \frac{4\pi n_0 e^2}{K_B}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)\varphi \equiv \frac{\varphi}{\lambda_D{}^2} \tag{1.15}$$

The solution to Equation (1.15) can be easily seen to be:

$$\varphi = \frac{q}{r}\exp\left(-\frac{r}{\lambda_D}\right) \tag{1.16}$$

where

$$\frac{1}{\lambda_D{}^2} = \frac{1}{\lambda_{De}{}^2} + \frac{1}{\lambda_{Di}{}^2} \tag{1.17}$$

and λ_{De} and λ_{Di} are the electron and ion Debye lengths respectively. So, the electrostatic potential φ (Equation 1.16) is no longer like the Coulomb potential which varies as r^{-1}, but has additional exponential decay with distance r and diminishes to $1/e$ of its Coulomb value at $r = \lambda_D$, the **Effective Debye Length**. Thus, for $r >> \lambda_D$, the potential becomes vanishingly small. The potential is felt by the plasma particles within a distance $r \cong \lambda_D$ from the position of the charge q. We, therefore, find that the size of the screening or the Debye Cloud is ~ λ_D. It increases with an increase of temperature, since electrons with high kinetic energy can withstand the attraction of the positive charge q up to larger distances. On the other hand, λ_D decreases with an increase of density n_0, since a larger number of charges or electrons can now be accommodated in a shorter region to annul the effect of q. As T_e and $T_i \to 0$ or $n_0 \to$, $\lambda_D \to 0$ and all the interesting plasma phenomena disappear. A finite λ_D is the source of the entire game of electromagnetic phenomena in a plasma. We have already seen the collective behavior – the potential due to a single change q in a plasma is a function of the electron or ion density and their temperatures and does not depend on the individual properties of the plasma particles.

1.10. Coulomb Collisions Among Plasma Particles

There are several characteristic time scales in a plasma. We have already encountered three of them, τ_e and τ_i corresponding to the electron-plasma frequency ω_e and the ion-plasma frequency ω_i, and the third corresponding to the electron-ion collision frequency ν_{ei}. We shall encounter more time scales when we study a plasma in a magnetic field. Since the collision frequency is one of the defining characteristics of a plasma, it must be known before we proceed to explore the nature of a plasma any further.

In a plasma, charged particles continuously feel the Coulomb force due to other particles. Therefore, the actual trajectory of a particle is not a sum of discrete random paths. However, it is found that some aspects of thermal motion and transport processes can be well accounted for using the description of Coulomb collisions and defining an effective Coulomb cross-section. We can treat collisions resulting in large scattering angles and those resulting in small scattering angles separately. Here, we present an approximate analysis of Coulomb collisions, a more formal treatment of collisions can be seen in Chapter 2.

A particle of charge z_1e and mass m_1, undergoes a deflection in its trajectory due to the effect of the Coulomb force exerted by another particle of charge z_2e and mass m_2.

The magnitude of the Coulomb force F when the two particles are a distance r apart is given by

$$F = \frac{z_1 z_2 e^2}{r^2} \tag{1.18}$$

This two-body collision problem can be reduced to the problem of a single particle of reduced mass μ and relative velocity $\vec{V}$ moving in the force field $\vec{F}$. The particle feels this force for the duration t it spends in the neighborhood of the other particle. This is the time needed to cross the closest distance of approach b. Thus

$$t = \frac{b}{V} \tag{1.19}$$

where $\vec{V}$ is the relative velocity of the particles. The change ΔP in the momentum of the particle is, therefore

$$\Delta P = \delta(\mu V) = \left(\frac{z_1 z_2 e^2}{b^2}\right)\left(\frac{b}{V}\right), \tag{1.20}$$

with

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Now, we know that for a scattering angle of 180° the particle reverses its direction of motion and the change in the momentum is twice the original momentum. Therefore, we expect that for large deflections the change in the momentum of a particle is of the order of its momentum, i.e.,

$$\left(\frac{b}{V}\right)\frac{z_1 z_2 e^2}{b^2} \approx \mu V \tag{1.21}$$

Thus the distance b, known as the **Impact Parameter**, between the particles for a large deflection is found to be:

$$b = \frac{z_1 z_2 e^2}{\mu V^2} \tag{1.22}$$

The effective cross section for a binary interaction is the area of a circle of radius b, i.e. πb^2. The effect of all small angle scatterings is collectively included in a parameter known as the Coulomb Logarithm ln Λ. The Coulomb logarithm is the ratio of the maximum and minimum impact parameters. The maximum impact parameter is of the order of Debye screening distance λ_D because for distances larger than this, the Coulomb potential becomes vanishingly small and the particle does not undergo any deflection in its path. The minimum impact parameter b_{min} corresponds to the maximum deflection and can, therefore, be taken as the value of b given in Equation (1.22). It is inversely proportional to V^2. In the classical regime, b_{min} must be larger than the de Broglie wavelength of the particle. In the quantum-mechanical regime, b_{min} can be taken to be the de Broglie wavelength given by

$$b_{\min} = \frac{\eta}{\mu V} \tag{1.23}$$

Thus, the Coulomb logarithm in the classical regime is:

$$\ln \Lambda = \frac{\lambda_D \mu V^2}{z_1 z_2 e^2} \tag{1.24}$$

and the total Coulomb cross-section σ_c is given by:

$$\sigma_c = \left(\pi b^2\right) \ln \Lambda \tag{1.25}$$

A particle moving with velocity V, in a medium of density n undergoes $nV\sigma_c$ collisions per unit time. The collisions frequency ν_c, is therefore, given by

$$\nu_c = nV\sigma_c \tag{1.26}$$

and the mean free path by

$$l_c = \frac{1}{n\sigma_c} \tag{1.27}$$

If the particle undergoes collisions with several different types of particles, the total mean free path is the sum of all the inverse free paths and the total collision frequency is the sum of all the collision frequencies.

Three types of collisions can occur in a fully ionized plasma – among like particles such as electron-electron and ion-ion and among unlike particles such as electron-ion. Thus, the **Electron-Electron Collision Frequency** is given by

$$\nu_{ee} = \frac{4\pi n_e e^2}{m_e^{1/2}(3K_B T)^{3/2}} \ln \Lambda_{ee} \tag{1.28}$$

in a plasma of temperature *T*, electron density n_e and ion density n_i. Here *V* has been replaced by its root mean square velocity. The **Ion-Ion Collision Frequency** is given by

$$\nu_{ii} = \frac{4\pi n_i z_1 z_2 e^4}{m_i^{1/2}(3K_B T)^{3/2}} \ln \Lambda_{ii} \tag{1.29}$$

Finally, the **Electron-Ion Collision** frequency ν_{ei} is found to be

$$\nu_{ei} = \frac{\pi n_i Z^2 e^4}{m_e^{1/2}(3K_B T)^{3/2}} \ln \Lambda_{ei} \tag{1.30}$$

for the charge *ze* on the ion of mass $m_i >> m_e$. Now, collisions bring about thermal equilibrium among the various particles. During each collision, there is an energy transfer from the more energetic particle to the less energetic particle. The laws of momentum and energy conservation imply that during a large angle collision between particles of identical masses, the particles have nearly equal energies after the collision. Therefore $\tau_{ee} = \nu_{ee}^{-1}$ and $\tau_{ii} = \nu_{ii}^{-1}$ represent the time durations during which electrons and ions reach a common temperature *T*. The case of electron-ion thermalization is different. Again from kinematics, it can be easily checked that during a large angle collision an electron transfers only a fraction m_e / m_i of its energy to an ion. Therefore, equilibration among electrons and ions would take that much longer, i.e. the thermalization, also called the relaxation time τ_{ei} for electrons and ions is given by

$$\tau_{ei} \cong \frac{m_i}{m_e \nu_{ei}} \tag{1.31}$$

We find that for an electron-proton plasma $\tau_{ei} > \tau_{ii} > \tau_{ee}$ which implies that electrons-ions interaction can remain at different temperatures for much longer than can ions-ions and electrons-electrons interactions. This is why a plasma is often characterized by two temperatures, one for ions and the other for electrons. As we shall see, a magnetized plasma can have even more than two temperatures. How do the three Coulomb logarithms compare? There could be one more type of collision process – Coulomb collisions of ions with electrons with a collision frequency ν_{ie}. It may not be very clear at this stage that ν_{ie} is different from ν_{ei}. We shall learn in Chapter 2 that, in the magnetohydrodynamic description of a plasma, where it is assumed that the rate of momentum transfer between the electron fluid and the ion fluid is proportional to their relative velocity, the two collision frequencies are related as (Equation 2.111):

$$\rho_e \nu_{ei} = \rho_i \nu_{ie} \tag{1.32}$$

where ρ_e and ρ_i are the mass densities of the electron and the ion fluids respectively. Thus, in a plasma with equal particle densities, we find $\nu_{ie} << \nu_{ei}$ and therefore, electron-ion collisions are the main process of thermalization among electrons and ions. These collisions frequencies determine the rates of transport processes, such as diffusion and dissipation processes, such as Ohmic heating.

1.11. Diffusion in a Plasma

Associated with a collisional process is a mean free path $l_m = V\tau$ that a particle with velocity V traverses between two collisions in a collision period τ. The particle diffuses in the system from one position to another through this random walk. The diffusion coefficient D is defined as:

$$D = \frac{l_m^{\ 2}}{\tau} \tag{1.33}$$

If a plasma is magnetized, the particles are not so free to walk randomly. We shall learn, in detail, about the motion of charged particles in a magnetic field in Chapter 3, but for the present, it would suffice to remember that charged particles execute a circular motion in a direction perpendicular to the magnetic field and move freely along the magnetic field. The diffusion rates, consequently, differ in these two directions. Perpendicular to the magnetic field, a particle diffuses from one orbit to another only if it undergoes collisions with other particles. The mean free path, here, is then of the order of the radius of the orbit, which is the cyclotron radius $R_B = V/\Omega_B$ where Ω_B is the cyclotron frequency (Equation 3.6). The diffusion coefficient $D_\perp$ in the perpendicular direction is given by:

$$D_\perp = \frac{R_B^{\ 2}}{\tau} = \frac{V^2}{\Omega_B^{\ 2}\tau} \tag{1.34}$$

Along the magnetic field, the parallel diffusion coefficient $D_{//}$ is the same as in an unmagnetized plasma (Equation 1.33). The ratio of the two diffusion coefficients is found to be:

$$\frac{D_\perp}{D_{//}} = \frac{1}{\Omega_B^{\ 2}\tau^2} \tag{1.35}$$

In the absence of the magnetic field $D_\perp = D_{//}$. Therefore, we can write a general expression as:

$$\frac{D_\perp}{D_{//}} = \frac{1}{1+\Omega_B^{\ 2}\tau^2} \tag{1.36}$$

so that $D_\perp = D_{//}$ for $\Omega_B = 0$ and Equation (1.35) is recovered for $\Omega_B\tau^2 >> 1$. This provides us with a definition of a magnetoplasma, i.e., one where the period of circular motion is much less than the collision period. The motion of the particles is dominantly circular and not a drunkard's doodle. The perpendicular diffusion coefficient is inversely proportional to the square of the magnetic field *B*. This dependence on *B* is also followed by other transport coefficients such as thermal conductivity and electrical conductivity.

It is easy to see that in an electron-proton plasma, the parallel diffusion coefficient $D_{//e}$ for electrons is much larger than $D_{//i}$, the parallel diffusion coefficient for ions. Does this mean that electrons will diffuse away from a region and ions will be left behind there? No. The quasineutrality condition forbids that. The slow diffusing ions will pull back the fast diffusing electrons and yield conditions for a joint diffusion of both electrons and ions with a diffusion coefficient $D_{A//}$ such that $D_{//e} > D_{A//} > D_{//i}$. This phenomenon is known as **Ambipolar Diffusion**. In a plasma, therefore, the net diffusion rate is not decided by the fast diffusing particles, but rather by the slow ones.

In a magnetized plasma, the perpendicular diffusion coefficient for electrons, $D_{\perp e}$ is much smaller than $D_{\perp i}$ (Equation 1.34) for identical electron and ion temperatures *T*. Therefore, the joint diffusion coefficient $D_{A\perp}$ is determined dominantly by the slow diffusing species, the electrons. The physical reason for the slower perpendicular diffusion of electrons is their smaller mean free path, which is nothing but their cyclotron radius. We shall derive an expression for the ambipolar diffusion coefficient in Chapter 5, since it requires the two-fluid description of a plasma.

In a fully ionized plasma, the diffusion is predominantly governed by Coulomb collisions among unlike particles. The collisions among like particles do not affect a change of the center of mass and hence cause little or no diffusion.

1.12. Electrical Resistivity of a Plasma

For moderate values of the electric field $\vec{E}$, a plasma obeys Ohm's law:

$$\vec{E} = \eta \vec{J} \tag{1.37}$$

where $\vec{J}$ is the current density and η is the resistivity. Electrons, the major carries of the current, move under the combined actions of the applied field $\vec{E}$ and the frictional force due to collisions with ions. Under steady state conditions ($\partial/\partial t = 0$). We get:

$$-en_i\vec{E}-\vec{\Gamma}^{ei}=0 \tag{1.38}$$

where Γ^{ei} is the frictional force density. For a simple collisional model where the frictional force is proportional to the relative velocity between electrons and ions, we can write:

$$\vec{\Gamma}^{ei}=m_e n_e \nu_{ei}\left(\vec{V}_e-\vec{V}_i\right) \tag{1.39}$$

The current density $\vec{J}$ is by definition:

$$\vec{J}=-en_e\left[\vec{V}_e-\vec{V}_i\right], \tag{1.40}$$

and we find the resistivity

$$\eta=\frac{m_e\nu_{ei}}{n_e e^2} \tag{1.41}$$

Now, substituting for ν_{ei} from Equation (1.30), we obtain the surprising result that the resistivity is almost independent of the electron density n_e. Therefore, the current density $\vec{J}$ driven by the electric field $\vec{E}$ is independent of the concentration n_e of the charge carriers. If a plasma contains neutral atoms or molecules, the situation, however, changes. Recall that the electron-ion collisions are infrequent at high temperatures. The reason for this is that at high thermal velocities an electron spends a rather short time in the vicinity of an ion and therefore loses only a small quantity of momentum. Thus, at sufficiently high temperatures, electrons do not feel the frictional drag due to ions and the motion of the two species decouples. The plasma resistivity becomes vanishingly small. Under such conditions, electrons can gain energy from the applied electric field, unhindered. There is a critical value of the electric field, known as the **Dreicer Field,** E_D, above which electrons feel only acceleration and no frictional force. The value of E_D can be determined from equation (1.38) and is found to be:

$$E_D \cong \frac{m_e}{e} v_{ei} \left(\frac{K_B T}{m_e} \right)^{1/2} \tag{1.42}$$

where V_e has been replaced by the root mean velocity and $V_i << V_e$. This is one way of generating high energy electron beams. Of course, the acceleration cannot go on indefinitely. The charge separation, resulting from the decoupling of electron and ion motion, builds up to a value that makes the system unstable, since a plasma always tends to remain quasi-neutral. These instabilities give rise to waves with amplitudes growing with time. The electrons now feel a kind of frictional drag due to these waves. It is found that the resistivity under such circumstances becomes anomalously large, sometimes larger by several orders of magnitude. The effective Ohmic heating, correspondingly, take place at a much higher rate than that due to Coulomb resistivity. The anomalously large resistivity turns out to be extremely useful in explaining phenomenon such as solar flares, where large amounts of energy are released in a very short time interval.

1.13. Plasma as a Dielectric Material

In a plasma, the charged particles not only move in response to the externally applied electric and magnetic fields, but during their motion, they also continuously produce electric and magnetic fields. Thus, a very complex interplay of fields and motion takes place and we face a difficult task of determining these fields in a self-consistent manner.

In a material of macroscopic dimensions, the average charge density is made up of two parts: (i) that due to the average charge of the atomic or molecular ions or the average free charge residing on the macroscopic body; (ii) that due to induced charges produced by polarization. In the absence of external fields, an atom or a molecule may or may not have a permanent electric dipole moment. Even if they have, due to their random thermal motion, the electric dipole moments are directed in a random manner, so that the average dipole moment of the entire system is zero. In the presence of an external field, there is a net dipole moment, which tends to align itself with the external field. This net dipole moment produces a charge density since the external field causes displacement and redistribution of the charges. If the polarization is uniform, there is no net change in the charge density. This is why the polarization charge density is expressed as the divergence of the dipole moment per unit volume $\vec{P}_E$. Poisson's equation including free ρ_f and induced ρ_I charge densities becomes:

$$\vec{\nabla}.\vec{E} = 4\pi\rho_f + 4\pi\rho_I ,$$

but

$$\vec{\nabla}.\vec{P}_E = -\rho_I \tag{1.47}$$

so that

$$\vec{\nabla} \cdot \left(\vec{E} + 4\pi\vec{P}_E\right) \equiv \vec{\nabla} \cdot \vec{D} = 4\pi\rho_f$$

where $\vec{D}$ is the displacement vector. Since, $\vec{P}_E$ is produced only by the application of $\vec{E}$, we can write $\vec{P}$ as a power-series in $\vec{E}$ as:

$$P_{Ei} = \sum_j \alpha_{ij} E_j + \sum_{j,k} \beta_{ijk} E_j E_k + \ldots \tag{1.48}$$

Experiments tell us that the linear term in $\vec{E}$ is quite adequate at moderate temperatures and electric fields. For isotropic conditions, Equation (1.48) can be recast as:

$$\vec{P}_E = \chi_e \vec{E} , \tag{1.49}$$

where χ_e is known as the e**lectric susceptibility** of the medium. The displacement vector $\vec{D}$ is related to the electric field $\vec{E}$ through the dielectric constant $\in$ of the medium as:

$$\vec{D} = \in \vec{E} \tag{1.50}$$

or

$$\vec{E} + 4\pi\chi_e\vec{E} = \in \vec{E},$$

so that

$$\in = 1 + 4\pi\chi_e \tag{1.51}$$

For a uniform medium $\in$ is independent of space, Poisson's equation takes the form:

$$\vec{\nabla}.\vec{E} = \frac{4\pi}{\in}\rho_f \tag{1.52}$$

Thus, for a given free charge density ρ_f, the electric field inside a plasma is reduced by the factor $\in$. The reduction results because the direction of the electric field produced by the induced charge density is opposite to that of the applied field. A plasma with a large dielectric constant screens AC electric fields the way a plasma with small Debye length screens DC electric fields.

The resistive properties of a plasma are included in the dielectric function $\in$. All aspects of electromagnetic wave propagation through a plasma are studied using the dielectric function $\in$ of a plasma. We can thereby learn about the frequency pass bands, cutoffs, absorption, refraction and reflection properties of a plasma.

When high intensity radiation propagates through a plasma, it modifies the plasma characteristics completely. The reflection region may become absorbing and a transparent plasma may become a scattering medium. All this is accomplished through the dielectric constant $\in$, which in addition to depending upon plasma parameters, becomes a function of the intensity and the frequency of the incident radiation. Calculation of the dielectric constant, which is anything but a constant, is a major occupation of plasma physicists.

1.14. Plasma as a Source of Coherent Radiation

A source of radiation is coherent if its size is smaller than the wavelength of the radiation it emits, for then, the differences in the retarded times of the different parts of the source can be neglected. Plasmas, due to their cooperative nature are found to be very efficient sources of strong and coherent radiation over a huge range of the electromagnetic spectrum. The source of energy lies in the non-thermal distributions of particles such as an electron beam traversing a plasma, or in an anisotropic velocity distribution, such as a loss-cone distribution arising in an inhomogeneous magnetic field. These distributions, in an attempt to relax to an equilibrium, produce electric and magnetic fields which may be in the form of electrostatic or electromagnetic waves. Since these waves are produced by the induced charge densities and electric currents, the bulk of the plasma particles participate in this process. We have already learned that in a plasma, charge densities can exist only over distances of the order or less than the Debye length. The condition for the production of coherent radiation is that its wavelength must be greater than the Debye length. Under these conditions, all the particles contained in the Debye sphere are in phase with each other and participate collectively in the emission process. Thus, the typical size of a coherent plasma source is of the order of the Debye length. Of course it will never be possible to resolve a source of this size in cosmic circumstances. That there is a coherent process in action, is inferred, for example, from observations of time variability of intensity and polarization of radiation among other possible consequences.

1.15. Strongly Coupled Plasmas

Strongly coupled plasmas are a more common occurrence in celestial compact objects than in terrestrial systems. An example close to home is provided by the planet Jupiter. The interior of the planet is made up of hydrogen with a few percent helium with average mass density ρ = 1-10 gcm^{-3} at a temperature $T \sim 10^4$ K. It is found to be a strongly coupled plasma with r_s = 0.6-1 and Γ = 20-50, where r_s, the ratio of the Wigner-Seitz, is the radius of an electron and the Bohr radius is given by $r_s = (3/4\pi n_e)^{1/3}(m_e c^2/\eta^2)$. The observation that Jupiter and Saturn emit 2-3 times more infrared radiation than they absorb from the Sun, has been interpreted to be the result of the energy release due to a phase transition of the strongly coupled plasma in the interiors of these planets.

The interiors of Sun-like stars have $\Gamma \sim 0.05$ and therefore do not qualify as strongly coupled plasmas, but strong coupling effects need to be included in the study of the state of heavy elements and their mixing.

A neutron star contains about the mass of the Sun in a sphere of radius of about 10 km. It is, perhaps, the most condensed state of matter. Theoretical models show that a crust of 1-2 km radius and a mass density of 10^4 - 10^7 gcm^{-3} consisting mostly of iron exists just below the surface of a neutron star. With an expected coupling constant $\Gamma \sim 10\text{-}10^3$, the tendencies of this plasma of iron nuclei towards Wigner-Crystallization and a glassy transition are being investigated since the knowledge of the state of crystal matter is crucial for understanding the cooling rate of a neutron star.

The interior of a white dwarf consists of dense material with mass density and temperature similar to what exists in the crust of a neutron star. White dwarfs with interiors made up of carbon-oxygen mixture are believed to be the progenitors of some type 1 supernovae. The possibilities of phase separation and formation of alloys in this carbon-oxygen mixture have important bearing on the cooling rates, mechanism of supernovae explosion, transport processes and neutrino emission processes. A lot of work, theoretical and experimental, is being carried out in the field of strongly coupled plasmas and it will not be an exaggeration to say that the motivation comes from the 'Heavens' and from our eternal search for new materials.

1.16. Dusty Plasmas

Though, so far, we have been mainly referring to a two component plasma with electrons and ions, in astrophysical situation, a third component, called the dust, is often present. This three component system is quite different from the one with three species of charged particles. For, one thing, the dust particles acquire electric charge, like a capacitor, when they are inserted in a plasma. Their charge is not a constant. It is a function of plasma parameters, varies with time, and it takes finite time for a dust particle to acquire charge. Dust particles are of macroscopic dimensions, of the order of a micron and smaller. The composition of the astrophysical dust varies from one environ to another. It could be carbonaceous, silicates, ferrites or any alloy of them. Heavy molecules and frozen ices are also a part of the family of dust grains. The composition of dust is determined from its response to the radiation that falls on it. The absorbed, the scattered and the re-emitted radiation carries diagnostics of the dust grains. So, what happens when a plasma of electrons and protons is impregnated with dust grains?

The grains suffer collisions with electrons and protons and acquire electric charge in the process. The electrostatic potential due to the charged grains affects the electron and proton density distributions, which in turn modify the electron and proton fluxes impinging upon the grains. Thus, the charging of the grain and plasma particle distribution must be determined self-consistently. The system is complex, therefore, we need to make some simplifying assumption. The first assumption is that the number density n_d of dust grains is much less than the electron or proton densities. Two cases are identified in this connection: a plasma is called a Dusty Plasma if the number of grains N_d in a Debye Sphere is much larger than unity. The opposite case with $N_d \leq 1$ is referred to as **Dust in a Plasma**. In addition to the charging of a grain by electron and ion currents directed on to it, its charged state may also change due to photoemission if subjected to radiation. The secondary emission of electrons when a grain is bombarded by electrons as well as the field emitted electrons, further, deprive a grain of its negative charge. The rate of change of the charge Q of a grain due to all possible causes can be expressed as:

$$\frac{dQ}{dt} = \sum_s I_s \ , \tag{1.70}$$

where I_s is the current due to a process s. Let us first consider the charging of a grain only due to electron and proton fluxes. Thus, if $\vec{J}_e$ and $\vec{J}_i$ are the current densities due to electrons and protons respectively, then the total currents I_e and I_i are given by:

$$I_e = -4\pi a^2 n_e e V_e \alpha_e \tag{1.71}$$

and

$$I_i = 4\pi a^2 n_i e V_i \alpha_i$$

where a is the radius of a spherical grain, V_e and V_i are the velocities of the electrons and protons relative to that of the grain velocity V_d, and α_e and α_i are known as the sticking coefficients. In a Maxwellian plasma, V_e and V_i can be replaced by the corresponding thermal velocities. If V_d is larger than the thermal speed of protons, than V_i is replaced by V_d. For constant n_e, n_i, and temperatures T_e, T_i, we see from Equations (1.70) and (1.71), that the charge Q increases linearly with time. Of course, this cannot continue for long. As the charge Q builds up, the negative charge on the grain begins to repel electrons and attract protons, so that in the neighborhood of the grain, n_e decreases and n_i increases. A steady state is reached when $I_e = I_i$ and $Q = Q_0$. Realizing that at a common temperature T, the electron thermal velocity $V_{Te} >> V_{Ti}$, the ion thermal velocity, we find that the time taken by the grain to accumulate a charge Q_0 is given by:

$$t_0 = \frac{Q_0}{4\pi a^2 e n_0 \left(K_B T / m_e\right)^{1/2}} \tag{1.72}$$

where we have taken $n_e = n_i = n_0$ and $\alpha_e \sim 1$. We know that electrostatic potential $\varphi \cong -K_B T/e$ can exist in a plasma. Therefore, the grain charge Q_0 produces the potential φ_0 due to which the electron current to the grain ceases. If the grain is treated as a capacitor of capacitance C_0 then the charge Q_0 is related to C_0 and φ_0 as

$$Q_0 = C_0 \varphi_0 \tag{1.73}$$

and $C_0 = a$ for a spherical capacitor. Thus, we find:

$$t_0 = \frac{\lambda_D}{a\omega_{pe}} \tag{1.74}$$

and

$$Q_0 = -\frac{K_B T a}{e}$$

Thus the charge relaxation time in a plasma is ~ ω_{pe}^{-1}, and the smaller the size of the grain, the smaller the cross-section for an electron encounter, and the longer charging time t_0, the larger temperature, the larger electron flux to the grain and, finally the larger capacitance a, the larger charge Q_0. For a more accurate derivation of these results, we need to know the particle distribution in the presence of the grain potential and determine I_e and I_i. It is found that for equal electron and ion temperatures, the grain potential is given by: $\varphi \cong -2.5K_B T/e$ which is not too different from the value obtained from the approximate treatment given above.

In astrophysical situations such as comets whizzing through the solar wind or interstellar grains braving the ultraviolet radiation of stars, photoemission of electrons from the grains must be included in the list of charging processes. The electron emission takes place via the well known photoelectric effect described by the Einstein relation:

$$\eta\omega = W + E \ ,$$

where ω is the frequency of the radiation falling on a material of work function W, and E is the kinetic energy of the emitted electron. For astrophysical grains, the work function is of the order of a few electron-volts, consequently the radiation frequency ω corresponds to the ultraviolet part of the electromagnetic spectrum. Thus a knowledge of ω and W provide us with an estimate of E and therefore of the current I_p. The electron emission endows the grain with a positive charge which may try to pull back the emitted electrons. Again, in principle, we could determine the steady state by requiring the total current

$$I_e + I_i + I_p = 0 \tag{1.75}$$

The steady state potential due to the charged grain could be positive or negative. In reality, however, a lot of work and a lot more guess work goes into the determination of the work functions for which a good idea of the composition is a prerequisite. The sizes and shapes and their distributions are the other important parameters, which are intimately connected with the formation mechanisms of dust grains.

Far from a charged grain, the electrostatic potential in a plasma tends to vanish due to screening effects. What if there are many grains within the Debbye sphere? Each grain has a potential ~ (K_BT/e) with an e-falling distance of the order of the Debbye length. But now, due to the presence of other charged grains, the potential far from the grain is not vanishingly small, but has a finite value, say, φ_f. So, the net potential $(\varphi - \varphi_f)$ will now be supported by a lesser value of Q given by:

$$Q \cong a(\varphi - \varphi_f), \tag{1.76}$$

where the capacitance of the spherical grain still remains close to a for $a << \lambda_D$. As we realize by now that charging of grains constitutes a host of complex processes and therefore, can be addressed with any thoroughness only in a specific circumstance.

1.17. Study of Plasmas: Towards what Purpose?

Since most of the visible universe is in the plasma state, knowing about plasmas would help us to understand some of the workings of the universe. Of course, it is the controlled thermonuclear fusion – that pollution free, nearly free, eternal source of energy – that is the ultimate goal of most of the laboratory plasma physicists. In the meantime, plasmas have been serving mankind through a host of technological applications – from communication to dyeing, to deposition of ions on metals, and fabrication of new materials. Plasma physics has also drawn the attention of people seeking the ultimate accelerators, the keyholes to the structure of matter and the origin of the universe.

Filamentary structures of all sizes and shapes are observed on all scales in the universe – be it on planetary and stellar atmospheres, supernovae ejecta, planetary nebulae, galactic environs or extragalactic realms. The macroscopic stability of these structures is studied using single and two-fluid descriptions of a plasma. These descriptions relate the size, the pressure, the fields and the flows in a plasma structure. In addition, we would like to know how do the characteristics of radiation that propagates or originates in these structures depend upon their defining parameters, such as, density, temperature magnetic and velocity fields. For example, quasi-periodic time variations in the radiation flux may indicate that the emitting region is in a state of oscillation.

In the same manner, we can learn about the medium through which the radiation propagates. For example, the observed delay in the arrival times of pulses of different frequencies from a pulsar, is attributed to the dispersion properties of the interstellar plasma. This time delay can be related to the electron density, the magnetic field and the size of the intervening interstellar medium. The observations and modeling of the non-thermal radio emission from the Sun provides us with estimates of density, temperature, magnetic field and geometric configuration of the solar corona. Through the absorption and scattering of electromagnetic radiation in the emission line regions of a quasar we hope to learn about the invisible central object, suspected to be a back hole.

It is in the realm of coherent sources of electromagnetic radiation that plasmas exhibit their versatility the most. Plasmas are good at fast and large releases of energy. This is possible as they can store free energy in several forms, as gradients in configuration and/or in velocity space. Thus, large departures from equilibrium are first allowed to grow; this is the state of instability. After attaining a critical stage the plasma undergoes relaxation, either in an explosive manner, or in a more gentle way. Solar flares are one such phenomenon where a complex configuration of magnetic and velocity fields becomes unstable and relaxation takes place with the release of electromagnetic and mechanical energy. Most of the strong extragalactic radio sources are associated with non-thermal (non-Maxwellian) distributions of energetic particles which thermalize through single particle and collective plasma processes, the latter being always more efficient and faster, if and when they happen. Often, the radiation observed from astrophysical sources has several components in it. There may be a steady emission over which is superimposed a rapidly varying quasi-periodic component; or the contribution of thermal to non-thermal processes may vary in different parts of a single source; or the emission may appear as absorption at some parts of the spectral region. All these situations can be a result of wave-particle and wave-wave interaction processes which can enhance or eliminate certain spectral regions. The generation and propagation characteristics of cosmic radiation bring us the diagnostics of the physical conditions in distant objects.

1.18.Techniques of Studying Plasmas

As for any many body system, the statistical methods are the most suitable for studying plasmas. The use of statistical methods have provided us with three levels of description of plasma particles and their attendant and externally imposed electromagnetic and other fields. For moderately dense plasmas, the particle-particle correlations can be ignored and the N-body system can be described though an N – particle distribution function, which is a function of positions and velocities of N-particles at a given instant of time and describes the number density of particles with given velocities at given space-time points. The only condition that this function has to satisfy is contained in the **Liouville Equation** which expresses its constancy in the phase space of positions and velocities. By integrating over all positions and velocities except one we get a single particle distribution function, and the condition of its constancy in the phase space is nothing but the **Boltzmann Equation**. Combined with Maxwell's equation, the collisionless Boltzmann or the **Vlasov Equation** describes the entire range of plasma phenomena including stability, heating and radiative processes – essentially the microscopic aspects of the plasma. This constitutes the **Kinetic Description** of plasmas.

Further simplification is achieved by taking the velocity moments of the Vlasov equation. Averaged macroscopic quantities, such as number density, velocity and pressure are obtained for each species of plasma particles. Each species is now treated as a fluid. A plasma consisting of electrons and protons has two interpenetrating fluids – the electron fluid and the proton fluid. Each fluid moves with a single velocity, has one single temperature and behaves like a conducting fluid in the presence of electromagnetic fields. This is known as the Two-Fluid Description and is very handy for describing phenomena in which electrons and ions play differential roles.

A third level of description is obtained by combining the equations of electron fluid and ion fluid. Here, the electrons, and ions lose their identities. Instead, a single fluid with a specific mass density, velocity, a current density and pressure is the outcome. This description of a plasma covers a wide variety of phenomena and has earned itself an independent title – Magnetohydrodynamics (MHD). At the root of MHD lies the mutual interaction of the fluid flow and the magnetic field. The magnetic field and its associated current produce Lorentz force which accelerates the fluid across the magnetic field, which in turn creates an electromotive force resulting in currents that modify the field. Macroscopic configurational stability, generation of magnetic fields – in fact all phenomena not dependent upon charge separation are studied using MHD. Each of the three descriptions has its region of applicability and can be deployed for linear and nonlinear problems.

1.19. Waves in Plasmas

After ascertaining the equilibrium of a plasma, its response to a small disturbance must be investigated. A plasma is said to support linear waves if the space-time variations of its defining parameters, such as density, velocity, magnetic field, etc take sinusoidal forms. Linear – because under small disturbances, the equations describing these oscillations are linear. Their nontrivial solutions provide us with a relation, known as the **Dispersion Relation**, between the frequency and the propagation wave-vector of a wave. The dispersion relation may have more than one root. Each root represents a wave with definitive phase and group velocities. By substituting the dispersion relation back into the plasma equations, we can determine the relative magnitudes of the various parameters and fields, albeit, not the absolute strengths, as well as the polarization of the waves. This much knowledge of waves is enough to classify them as **MHD Waves,**

Drift Waves and **Electromagnetic** and **Electrostatic Waves**. These waves have finite lifetimes, they suffer damping due to Coulomb collisions and other non-ideal effects. But a plasma can be remarkably collisionless. Under the circumstances, wave-particle and wave-wave interactions can drain out the energy from a given wave. These processes, together, are clubbed as collisionless damping mechanisms, of which the **Landau Damping** in unmagnetized and **Cyclotron Absorption** in a magnetized plasma are the most effective.

1.20. Instabilities In Plasmas

In a plasma, the waves can go unstable, i.e., their amplitudes grow with time, usually in an exponential way. There are three broad classes of plasma instabilities. The first class consists of instabilities which are studied using single or two-fluid descriptions. The driving force and energy for the excitation of this class of instabilities is contained in the non-equilibrium or non-thermal arrangement of the plasma fluid and the magnetic field in configuration space. For example, the Rayleigh-Taylor instability, which comes into play when a heavy fluid lies over a light fluid. The velocity shear between fluids of different mass densities could also excite instabilities as in the solar wind – comet tail interaction. A current carrying conducting fluid may undergo bending or twisting due to the excitation of these instabilities.

The second major class of instabilities involves a collisionless transfer of energy and affects the plasma at a microscopic level. These instabilities are studied using the kinetic description. The driving force and energy are contained in the non-thermal or non-Maxwellian velocity distribution functions of the electrons and the ions. For example, energetic electron and ion streams traveling through an ambient plasma medium could excite electrostatic waves, which then get converted into electromagnetic waves through several possible nonlinear processes. The other sources of energy are the density gradient, temperature anisotropy, and the current flow. Given the source of energy, the excitation of an instability requires an intermediary – one or the other of the many possible waves that a plasma supports, in order to tap the free energy. For example, it may be necessary to have an otherwise stable wave of phase velocity lying in the non-thermal part of the velocity distribution function. There is generally a threshold condition which must be satisfied before the waves go unstable. Then, once they start growing, their saturation levels and mechanisms need to be determined. These instabilities affect the transport phenomena in a very substantial manner.

The third major class of instabilities constitutes the Parametric Instabilities. Here, the driving energy is contained in a finite amplitude wave, electrostatic or electromagnetic, that impinges on a plasma. It then couples with other waves in the plasma and drives them unstable. These unstable waves may eventually undergo dissipation and heat the bulk plasma or accelerate some plasma particles. This class of instabilities also plays an important role in the generation of high frequency radiation form low frequency radiation, e.g., through a process called Stimulated Raman Scattering, where low frequency radiation, by scattering on the electron plasma wave of a high energy electron beam is converted into a high frequency radiation. This process is akin to inverse Compton scattering but with the important difference that a single electron in the Compton scattering is substituted by an electron plasma wave in the Raman scattering. Both these processes have been applied to explain the properties of Quasar non-thermal radiation. Parametric instabilities have been widely studied in the earth's ionospheric plasmas, the solar corona and extragalactic plasmas.

1.21. Plasmas in Curved Space – Time

Astrophysical plasmas are accelerated to relativistic speeds in the vicinity of compact objects such as pulsars and black holes. This necessitates the inclusion of effects due to **Special Theory of Relativity.** In addition, the strong gravitational fields of the compact objects may produce curvature in the space immediately around them. Under such circumstances, it becomes essential to study fluids and plasmas in the curved space – time using the **General Theory of Relativity** (GTR), formulated by Albert Einstein in 1916. The special theory of relativity operates in inertial frames of reference which are related to one another by the Lorentz Transformations of space-time coordinates, velocities and electromagnetic fields. The general theory of relativity is the generalization to include non-inertial frames of reference which are not related to one another by any fixed transformation laws. **The Principle of Equivalence** tells us that the properties of the motion in a non-inertial frame are the same as those in an inertial frame in the presence of a gravitational field. This implies that a system in an accelerated frame of reference is equivalent to its being in a gravitational field. We know that additional forces such as – the centrifugal and the coriolis forces arise when we go into a rotating frame of reference, for example, in the frame in which the earth is rotating. We call these forces as fictitious forces, since they disappear as soon as we go back to a non-rotating frame or inertial frame of reference. So, although, the motion in an accelerated frame of reference can be simulated by an equivalent gravitational field, this gravitational field has very different properties from a real gravitational field. The equivalent gravitational field may increase indefinitely at large distances. Whereas, the real gravitational field, as we know, must vanish at infinity. Thus, we can eliminate the gravitational field only locally, in a limited region of space. There is no transformation to a non-inertial frame by which the field can be eliminated over all space.

Chapter 2

STATISTICAL DESCRIPTION OF A MANY-BODY SYSTEM

2.1. Track them down!

Molecules in a gas or a liquid, men in a metropolis and stars in a galaxy are in a state of incessant motion resulting from the action of various forces: internal, due to other molecules or (wo)men or stars; and external forces, due to the rest of the universe. It would be a frustrating task if we had to know the position and velocity of every molecule or star at every instant of time, in order to deal with the gas or the galaxy. The saving grace is that we need not know every move of every molecule or star and we will still be able to manipulate the system to our advantage. For most purposes, it suffices to know the average properties like thermal and electrical conductivities, and mechanical and electromagnetic stresses of a large system. In this chapter, we set up a mathematical framework to describe the behavior of a macroscopic system. After establishing a transport equation for a discrete system in the phase space, continuum limits are taken to facilitate the study of fluid. The mass, momentum and energy conservation laws are derived for each species of particles in a fluid. This multi-fluid description is further simplified to a single fluid description with two component fluid of electrons and ions taken as an example. This chapter contains the entire set of mathematical tools needed to investigate 'a single particle', 'a multi-fluid' and 'a single fluid' characteristics of an electrically conducting as well as an electrically non-conducting system. In the subsequent chapters, these descriptions will be further explored under various simplifying and tractable circumstances.

2.2. The Phase Space

Consider a system with a large number of particles *N*. Its time evolution is described by Hamilton's equations of motion for given initial conditions. The Hamiltonian $H(q_1, q_2, ...; p_1, p_2, ...)$ is a function of the canonical coordinates (q_i, p_i). The mechanical state of a system can be represented by a single point in the 2N Dimensional Space. This is the **Phase Space**.

The Hamilton equations are:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{2.1}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{2.2}$$

The time derivative of any function f(q_i, p_i, t) is given by:

$$\begin{aligned}
&\frac{df}{dt} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}\right) \\
&or \\
&\frac{df}{dt} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}\right) = [f, H] + \frac{\partial f}{\partial t}
\end{aligned} \tag{2.3}$$

Where [f, H] is the Poisson Bracket. If f is a constant of motion then [f, H] = 0.

2.3. The Gibb's Ensemble

Average properties of a system are found by studying a large number of identical systems. Their location in phase space may differ, but they have the same average properties, e.g., $E = p^2 + q^2$ is the energy of a simple harmonic oscillator. The members of the ensemble will all have the same value of E but different p and q.

Each system is represented by one point in the phase space. The ensemble corresponds to a group of points. Associate a density ρ ($q_1 ... q_N$; $p_1 ... p_N$) with this group of points, the number of members in a volume ($dq_1 ... dq_N$;$dp_1 ... dp_N$) is ρ($dq_1 ... dq_N$; $dp_1 ... dp_N$), which is the statistical weight of the system.

Then the total number of members is:

$$\int \rho dq_1 ... dq_N dp_1 ... dp_N \tag{2.4}$$

The probability of finding the system in a volume ($dq_1...dq_N \, dp_1...dp_N$) is:

$$\frac{\rho dq_1...dq_N dp_1...dp_N}{\int \rho dq_1...dq_N dp_1...dp_N} \tag{2.5}$$

We use the normalization:

$$\int \rho dq_1...dq_N dp_1...dp_N = 1 \tag{2.6}$$

Now,

$$\begin{aligned} &\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \frac{\partial \rho}{\partial \vec{q}_i} \frac{d\vec{q}_i}{dt} + \frac{\partial \rho}{\partial \vec{p}_i} \frac{d\vec{p}_i}{dt} \\ &or \\ &\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H] \end{aligned} \tag{2.7}$$

The motion of the system from one point (1) to another point (2) represents the time evolution of the canonical transformation (Figure 2.1).

We know that (1) the volume element remains invariant under canonical transformations and (2) all the points in a given volume element ΔV at (1) that will end up in a volume ΔV at (2) following Newton's Laws.

Thus both the numbers of members and the volume element remain constant with time

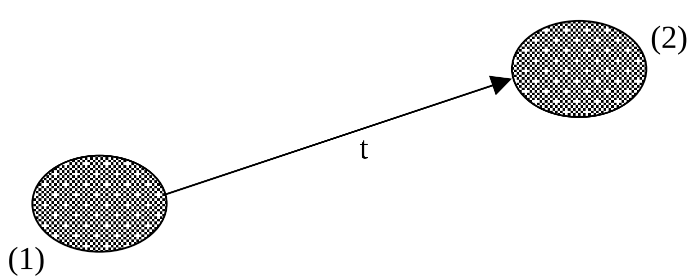

Figure 2.1.

$$\frac{d\rho}{dt} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} = -[\rho, H] \tag{2.8}$$

The equation (2.8) is known as the *Liouville Equation*. In statistical equilibrium ([ρ, H] =0)ρ is a function of only the constants of motion, e.g., the energy. The Liouville Equation is linear, i.e., $\rho_1 + \rho_2$ is also a solution – the law of superposition holds.

2.4. Distribution Function

For the given density ρ, how are the points distributed in phase space? What is the functional depedence of ρ on the constants? Is it a gaussian, power law or any other form?

When ρ is given in a special form it is called the **Distribution Function**. Thus ρ_N (q_1, ..., q_N; p_1, ..., p_N; t) is the N **Particle Distribution Function**. One can define **Reduced Distribution Function** too. A s particle distribution function ρ_s as

$$\rho_s = \int \rho_N dq_{s+1} ... dq_N dp_{s+1} ... dp_N \tag{2.9}$$

For arbitrary set of s particles the distribution function is:

$$f_s = \frac{N!}{(N-s)!} \rho_s (q_1, ... q_s ; p_1, ... p_s) \tag{2.10}$$

This is the generic distribution function used in the statistical descripition of a many body system.

2.5. One Particle Distribution Function

It is impractical to work with ρ_N. Instead we work with 1, 2 or 3 particle distribution functions. Let us begin with the Liouville Equation

$$\frac{d}{dt} f(q_1 ... q_N, \vec{V}_1 ... \vec{V}_N, t) = 0 \tag{2.11}$$

Where we have replaced momentum p by velocity $\vec{V}$. Integration over $q_2, ..., q_N$ and $p_2, ..., p_N$ gives one-particle distribution function. Integration over $q_3, ..., q_N$ and $p_3, ..., p_N$ gives two particle distribution function and so on and so forth.

The two particle distribution function is the joint probability of finding particle 1 in volume ($dq_1\ dV_1$) and particle 2 in volume ($dq_2\ dV_2$). Let us study the time evolution of one particle distribution function. We begin with

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \frac{\partial f_N}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f_N}{\partial V_i}\frac{d\vec{V}_i}{dt} = 0 \tag{2.12}$$

Integrate over all the coordinates, except q_1 and $\vec{V}_1$, to find

$$\frac{\partial f_1}{\partial t} + \vec{V}_1 \cdot \frac{\partial f_1}{\partial \vec{q}_1} + \sum_{i=2}^{N}\int V_i \frac{\partial f_N}{\partial \vec{q}_i}\prod dq_i ... dq_N dV_i ... dV_N + \int \frac{\partial f}{\partial \vec{V}_1}\frac{d\vec{V}_1}{dt} d\vec{q}_2 ... d\vec{q}_N d\vec{V}_2 ... d\vec{V}_2 = 0 \tag{2.13}$$

The partial integration $\int \frac{\partial f_N}{\partial q_i} dq_i = f_N q_i \,|_{-\infty}^{+\infty} = 0$. Since $\int f^N dq_i$ must be a bounded function.

The acceleration $\frac{d\vec{V}}{dt}$ results from (i) external forces $\vec{F}_{ext}$ and (ii) internal forces $\vec{F}_{\text{int}}$ due to the mutual interaction with other particles. The acceleration term with $\vec{F}_{ext} = m\vec{A}_{ext}$ becomes

$$\sum_{i=1}^{N}\int A_{1i}\frac{\partial f_N}{\partial \vec{V}_{1i}}\prod_{\beta=2}^{N} d^3q_\beta d^3\vec{V}_\beta + \sum_{\alpha=2}^{N}\int A_{\alpha i}\frac{\partial f_N}{\partial \vec{V}_{\alpha i}}\prod_{\beta=2}^{N} d^3q_\beta d^3\vec{V}_\beta \tag{2.14}$$

Assume $A_{\beta i}$ is independent of $V_{\beta i}$, this is true for most of the forces we encounter including the velocity dependent Lorentz Force; then each term in the sum is:

$$\int_{-\infty}^{+\infty}\frac{\partial f_N}{\partial V_{\alpha i}} dV_{\alpha i} = 0 \tag{2.15}$$

So we get

$$\sum_i A_{1i} \frac{\partial}{\partial V_{1i}} \left(\int f_N \prod_{\beta=2}^{N} d^3 q_\beta d^3 V_\beta \right) = \vec{A} \cdot \frac{\partial f_1}{\partial \vec{V}} \tag{2.16}$$

The internal force on particle α due to all the other particles can be written as:

$$F_{\text{int}} = \sum_{\alpha=1}^{N} \sum_{\substack{\gamma=1 \\ \alpha \neq \gamma}}^{N} \vec{F}_{\alpha\gamma} \tag{2.17}$$

Then the acceleration term becomes:

$$\sum_{\alpha=1}^{N} \sum_{\substack{\gamma=1 \\ \alpha \neq \gamma}}^{N} \int \vec{F}_{\alpha\gamma} \frac{\partial f_N}{\partial \vec{V}_\alpha} \prod_{\beta=2}^{N} d^3 q_\beta d^3 V_\beta \tag{2.18}$$

The internal force is a function of the separation and the perpendicular velocities, it is not a function of the velocities parallel to the force. Let us separate out the term for α=1 to get

$$\sum_{\gamma=2}^{N} \int \vec{F}_{1\gamma} \frac{\partial f_N}{\partial \vec{V}_1} \prod_{\beta=2}^{N} d^3 q_\beta d^3 V_\beta \tag{2.19}$$

It can be shown that for any other choice of α, the integral vanishes. Thus we find:

$$\sum_{\gamma=2}^{N} \frac{\partial}{\partial \vec{V}_1} \cdot \int \vec{F}_{1\gamma} f^{(2)}(1, \gamma, t) d^3 q_\gamma d^3 V_\gamma \tag{2.20}$$

Collecting all the terms, we get:

$$\frac{\partial f^{(1)}}{\partial t} + \vec{V}_1 \cdot \frac{\partial f^{(1)}}{\partial q_1} + \frac{\vec{F}_{ext}}{m} \cdot \frac{\partial f^{(1)}}{\partial \vec{V}_1} = -\int \sum_{\gamma=2}^{N} \frac{\vec{F}_{1\gamma}}{m} \cdot \frac{\partial f^{(2)}}{\partial \vec{V}_1} d^3 q_\gamma d^3 V_\gamma \tag{2.21}$$

We see that the time evolution of $f^{(1)}$ depends on $f^{(2)}$. Similarly it can be shown that the time evolution of $f^{(2)}$ depends on $f^{(3)}$ and so on and so forth. This leads to what is known as the **BBGKY Hierarchy** after Yvon (1935, 1937), Bogolioubov (1939), Born & Green (1946, 1947) and Kirkwood (1946, 1947).

BBGKY equation describes the evolution of a system under

1. Diffusion ($\partial f/\partial q$)
2. External Force ($\partial f/\partial V$)
3. Particle Interaction ($F_{1\gamma} . \partial f/\partial V$)

The internal force contribution can be generally expressed as $\delta f / \delta t |_c$ known as the collision term and different choices of the collision term gives rise to different equations such as the **Boltzmann** and the **Fokker Planck** equations.

For a collisionless system $\delta f / \delta t |_c = 0$. This applies for a low density medium. It is also applicable to stars in a galaxy which rarely collide. But there is the dynamical friction that the stars experience and can be described though the Fokker Planck equation. The hierarchy must be truncated in order to close the system. We must know $f^{(2)}$. The function $f^{(2)}$ is the joint probability of finding particle 1 at (q_1, V_1) and particle 2 at (q_2, V_2). If they move independently of each other then

$$f^{(2)} = f^{(1)}(q_1, V_1)\, f^{(1)}(q_2, V_2) \tag{2.22}$$

Then the internal force term becomes:

$$\sum_{l=2}^{N} \int \vec{F}_{1l} \cdot \frac{\partial}{\partial \vec{V}_1} \left[f^{(1)}(q_1, V_1)\, f^{(1)}(q_l, V_l) \right] dq_l dV_l = \vec{F}_s \cdot \frac{\partial f^{(1)}}{\partial \vec{V}_1} \tag{2.23}$$

Where $\vec{F}_s \equiv \int \sum \vec{F}_{1l}\, f_1(q_l, V_l) dq_l dV_l$ is known as the **self-consistent force.**

With the self-consistent force term, the one particle distribution function obeys the **Vlasov Equation** given below:

$$\frac{\partial f^{(1)}}{\partial t} + \vec{V} \cdot \frac{\partial f^{(1)}}{\partial \vec{q}} + \frac{(\vec{F}_{ext} + \vec{F}_s)}{m} \cdot \frac{\partial f^{(1)}}{\partial \vec{V}} = 0 \tag{2.24}$$

The equation (2.24) is used to study waves and instabilities in a collisionless plasma. There is one Vlasov equation for each species of particles such as electrons, protons and other ions. At high densities the particle motions are correlated and the joint probability for two particles can be written as

$$f^{(2)} = f^{(1)} f^{(1)} + f_c \tag{2.25}$$

Where f_c describes the correlations.

2.6. Collision Models

In the **Krook Collision Model**, the collision term is written as

$$\frac{\delta f_1}{\delta t}\Big|_c = -\frac{1}{\tau}\left(f_1 - f_{10}\right) \tag{2.26}$$

Where τ is the relaxation time and f_{10} is the equilibrium distribution function. In the absence of external forces and spatial variation, we find

$$f_1(V,t) = \left[f_1(V,0) - f_{10}\right]e^{-t/\tau} + f_{10} \tag{2.27}$$

The Boltzmann Collision Model used to describe binary and elastic collisions between particles treated as hard spheres. The Fokker Planck Model is derived from the Boltzmann model for small and continuous changes in the velocities of the colliding particles accounting for only grazing collisions. Stars in a galaxy provide an example of such a system.

2.7. The Kinetic Description

A high temperature plasma is nearly collisionless and can be described using the Vlasov equation in the presence of the Lorentz force. This along with the Maxwell equations provides the kinetic description of a plasma. The relevant equations are:

$$\frac{\partial f_i}{\partial t}+\vec{V}\cdot\frac{\partial f_i}{\partial \vec{r}}+\left[-\vec{\nabla}\varphi+\frac{Q_i}{m_i}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)\right]\cdot\frac{\partial f_i}{\partial \vec{V}}=0 \tag{2.28}$$

$$\vec{\nabla}\cdot\vec{E}=\sum_i 4\pi Q_i n_i \tag{2.29}$$

$$\vec{\nabla}\cdot\vec{B}=0 \tag{2.30}$$

$$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{2.31}$$

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\sum_i\vec{J}_i+\frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{2.32}$$

Where

The charge density is $Q_i n_i = Q_i\int f_i dV$.

The current density is $\vec{J}_i = Q_i\int\vec{V} f_i d\vec{V}$.

The particle density is $n_i = \int f_i dV$.

Where *i* stands for the species of particles such as electrons and ions.

This description is used to study microscopic stability or otherwise of a plasma. The cause of the instability lies in the distribution function *f*. Any departure from the equilibrium distribution function such as a relative motion between electrons and protons or temperature anisotropy such that $T_\perp \neq T_{//}$ with reference to the magnetic field can make the plasma unstable. The free energy released through instabilities can be used for heating of plasma and/or for production of radiation, the plasma ultimately reaching an equilibrium. For stellar systems, the gravitational force given by $\vec{F}_g = -\vec{\nabla}\varphi_g$ and the Poisson equation for the mass density $\nabla^2\varphi_g = 4\pi G\rho_m(\vec{r},t)$ with $\rho_m = m\int f(r,V,t)dV$, provide the kinetic description.

2.8. The Fluid Description

Under certain conditions, a system of large number of particles *N* can be described as a continuum. This continuum or a fluid has only one velocity. There is a fluid for each species of particles. Thus we have an electron fluid, a proton fluid and so on and so forth.

The fluid moves obeying a set of laws. These are the conservation of mass, momentum and energy. These conservation laws are derived by taking the moments of the Boltzmann or Vlasov equations in the velocity space. This exercise is possible only for certain forms of the distribution function *f*.

2.9. The Fluid Equations

We begin with the Vlasov equation

$$\frac{\partial f}{\partial t}+\vec{V}\cdot\frac{\partial f}{\partial \vec{r}}+\left[-\vec{\nabla}\varphi+\frac{Q_i}{m_i}\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)\right]\cdot\frac{\partial f}{\partial \vec{V}}=\frac{\delta f}{\delta t}\Big|_c \tag{2.33}$$

The first velocity moment is obtained by integrating over the velocity.

Taking the first term of equation (2.33):

$$\int\frac{\partial f}{\partial t}d\vec{V}=\frac{\partial}{\partial t}\int f(r,V,t)d\vec{V}=\frac{\partial n(r,t)}{\partial t} \tag{2.34}$$

where n is the particle density.

Taking the second term of equation (2.33):

$$\int\vec{V}\cdot\frac{\partial f}{\partial \vec{r}}d\vec{V}=\int\frac{\partial}{\partial \vec{r}}\left[\vec{V}f(r,V,t)\right]d\vec{V}=\frac{\partial\left[n(r,t)\vec{U}\right]}{\partial \vec{r}} \tag{2.35}$$

where the average velocity or the fluid velocity $\vec{U}$ is defined as:

$$\vec{U}\equiv\overline{\vec{V}}=\frac{\int\vec{V}f(r,V,t)d\vec{V}}{\int f(r,V,t)d\vec{V}} \tag{2.36}$$

Taking the third term of equation (2.33) without magnetic field:

$$\int\left(\frac{Q}{m}\vec{E}-\vec{\nabla}\varphi_g\right)\cdot\frac{\partial f}{\partial\vec{V}}d\vec{V}=\int\frac{\partial}{\partial\vec{V}}\cdot\left(\frac{Q}{m}\vec{E}-\vec{\nabla}\varphi_g\right)f(r,V,t)d\vec{V}$$

$$=\int\left(\frac{Q}{m}\vec{E}-\vec{\nabla}\varphi_g\right)f(r,V,t)\cdot d\vec{S}_V=0 \tag{2.37}$$

at $V=\pm\infty$. Notice that dS_v is proportional to V^2, then f must be proportional to $V^{-\alpha}$ (α >2). We found a restriction on the distribution function.

Now taking the third term of equation (2.33) that includes the magnetic field, we get:

$$\frac{Q}{m}\int\frac{\vec{V}\times\vec{B}}{c}\cdot\frac{\partial f}{\partial\vec{V}}=\int\frac{\partial}{\partial\vec{V}}\cdot\left[f\,\frac{\vec{V}\times\vec{B}}{c}\right]d\vec{V}-\int f\,\frac{\partial}{\partial\vec{V}}\cdot\left[\frac{\vec{V}\times\vec{B}}{c}\right]d\vec{V}=0 \tag{2.28}$$

The surface integral $\rightarrow 0$ if f is proportional to $V^{-\alpha}$ ($\alpha > 3$). This is the second condition. So what is f ? A form that satisfies these conditions is the **Maxwellian Distribution** that is given by:

$$f(V)=n\left(\frac{m}{2\pi K_BT}\right)^{3/2}\exp\left[-\frac{V^2}{2K_BT/m}\right] \tag{2.29}$$

Since the exponential fall is faster than any power law fall. The collision term describes change in n due to collisions and correlations which may result in forming aggregates and producing recombination, ionization and diffusion effects. In the absence of such processes $\int\frac{\delta f}{\delta t}|_c\ dV=0$. The exercise of taking the first moment is over! Then we find

$$\frac{\partial n(\vec{r},t)}{\partial t}+\vec{\nabla}\cdot\left[n(r,t)\vec{U}\right]=0 \tag{2.30}$$

This is the continuity equation describing particles or mass conservation.

Let us take the second velocity moment. Multiply the Vlasov equation by mV_i and integrat over $d\vec{V}$.

First term:

$$\int mV_i \frac{\partial f}{\partial t} d^3V = \frac{\partial}{\partial t}\int mV_i f d^3V = \frac{\partial}{\partial t}\left[mn\vec{U}(r,t)\right] \tag{2.31}$$

Second term:

$$\int mV_i \sum_j V_j \frac{\partial f}{\partial x_j} d\vec{V} = \int m\sum_j \frac{\partial}{\partial x_j}\left[V_i V_j f\right] d\vec{V} = \sum_j \frac{\partial}{\partial x_j}\left[\rho_m \overline{V_i}\overline{V_j}\right] \tag{2.32}$$

Define $\vec{V} = \vec{U} + \vec{u}'$ where the fluctuating part $\vec{u}'$ is such that $< \vec{u}' > = \int \vec{u}' f d^3V = 0$. The second term becomes

$$\sum_j \frac{\partial}{\partial x_j}[\rho_m \overline{V_i}\overline{V_j}] = \frac{\partial}{\partial x_j}[\rho_m U_i U_j] + \frac{\partial}{\partial x_j}[\rho_m \overline{u'_i}\overline{u'_j}] \tag{2.33}$$

The quantity

$$\prod_{ij} \equiv \rho_m(r,t)\overline{u'_i}\overline{u'_j} \tag{2.34}$$

is the stress tensor. Its diagonal components are $\rho u'^2_x, \rho u'^2_y, \rho u'^2_z$ represent pressure, the off-diagonal componets $\rho u'_x u'_y ...$ represent the shear stresses. They arise when the direction of motion of a fluid is different from the direction of the transfer of momentum. In the presence of the magnetic field, $\Pi_\perp \neq \Pi_{//}$. The fluid is said to be anisotropic.

Coming back to

$$\sum_j \frac{\partial}{\partial x_j}\left[\rho_m U_i U_j\right] = \sum_j U_i \frac{\partial}{\partial x_j}\left(\rho_m U_j\right) + \rho_m U_j \frac{\partial U_i}{\partial x_j} = U_i\left(-\frac{\partial \rho_m}{\partial t}\right) + \rho_m\left(\vec{U}\cdot\vec{\nabla}\right)U_i \tag{2.35}$$

The third term (force) $\vec{E}+\vec{V}\times\vec{B}$ becomes $\vec{E}+\vec{U}\times\vec{B}$. The collision term $\sum_{s\neq s'}\Gamma_i^{ss'}$ describes the rate of change of momentum due to collisions between different species or fluids. Stresses between different parts of the same fluid are in Π_{ij} The exercise of taking the second moment is over!

We find the momentum conservation law as:

$$\rho_s\left[\frac{\partial U_s}{\partial t}+\left(\vec{U}_s\cdot\vec{\nabla}\right)U_s\right]=\frac{Q_s\rho_s}{m_s}\left(\vec{E}+\frac{\vec{U}_s\times\vec{B}}{c}\right)-\rho_s\vec{\nabla}\varphi_g-\vec{\nabla}\cdot\vec{\Pi}_s+\sum_{s\neq s'}\Gamma^{ss'} \quad (2.36)$$

The third moment is taken with *VV*. This gives us the energy equation. In the absence of viscosity or shear, $\vec{B}=0$, thermal conductivity and collision equal to zero, we find $\frac{d}{dt}(p\rho^{-5/3})=0$ which is the adiabatic equation of state. It describes changes in pressure or volume under varying temperature conditions.

Taking moment with $\left(\vec{V}-\vec{U}\right)^2$ gives the heat transport equation. A simplified form of which is

$$\frac{\partial T}{\partial t}=\vec{\nabla}\cdot\left[k_c\vec{\nabla}T\right] \quad (2.37)$$

describing heat diffusion where the heat flux is $H=k_c\vec{\nabla}T$ with k_c the thermal conductivity.

2.10. Correlation Functions

The joint probability distributions contain information on the correlated behavior of a system, i.e., when the position, in the phase space, of one part of the system depends on distribution of other parts. We can consider the distribution of galaxies. It is well known that galaxies come in pairs, groups, clusters and super clusters. The positions of galaxies are strongly correlated. One can define the radial pair correlation function for a homogeneous and isotropic system as

$$\xi(R) = \frac{1}{4\pi \bar{n}^2} \int f_c(q_1, V_1, q_1 + R, V_2, t) dV_1 dV_2 d^2\Omega \tag{2.38}$$

where $d^2\Omega$ is the differential solid angle in the direction of separation $\vec{R}$ of the two galaxies.

The correlated part of the joint probability f_c enhances the density of particles at certain positions. The observed 3-dimensional galaxy-galaxy correlation function is found to follow

$$\xi(R) = \left(\frac{Rh}{5.4Mpc} \right)^{-1.74} \tag{2.39}$$

where the Hubble constant $\Re$ = 100h km/s Mpc^{-1}. For clusters of galaxies, the correlation function is

$$\xi(R) \approx 360 \left(\frac{Rh}{Mpc} \right)^{-1.8} \tag{2.40}$$

for R < 150 Mpc. It shows that clusters of galaxies are more strongly clustered than the galaxies. The correlation length is the distance *R* for which $\xi(R) = 1$.

2.11. The Single Fluid Description

There is a way in which the different fluids corresponding to each species of particles can be combined to form a single fluid. This single fluid is caracterized by one mass density and one velocity.

We will discuss the circustances of its validity in a later chapter. Here we derive the single fluid equations. We begin with the equation for the electron fluid and ion fluid.

$$\rho_e\left[\frac{\partial \vec{U}_e}{\partial t}+\left(\vec{U}_e\cdot\vec{\nabla}\right)U_e\right]=-\frac{e\rho_e}{m_e}\left(\vec{E}+\frac{\vec{U}_e\times\vec{B}}{c}\right)-\rho_e\vec{\nabla}\varphi_g-\vec{\nabla}\cdot\vec{\Pi}_e+\vec{\Gamma}_{ei} \quad (2.41)$$

$$\rho_i\left[\frac{\partial \vec{U}_i}{\partial t}+\left(\vec{U}_i\cdot\vec{\nabla}\right)U_i\right]=\frac{ze\rho_i}{m_i}\left(\vec{E}+\frac{\vec{U}_i\times\vec{B}}{c}\right)-\rho_i\vec{\nabla}\varphi_g-\vec{\nabla}\cdot\vec{\Pi}_i+\vec{\Gamma}_{ie} \quad (2.42)$$

We define single fluid quantities: The mass density is $\rho_m=\rho_e+\rho_i=n[m_e+m_i]$; the velocity is $\vec{U}=\frac{m_e\vec{U}_e+m_i\vec{U}_i}{m_e+m_i}$; and the current density is $\vec{J}=ne[zU_i-U_e]$. Where we have assumed that $n_e=n_i=n$.

Only the steps to the arrive at the single fluid equation will be given. They are:

1. Add the electron and ion equations
2. The single fluid continuity equation is: $\frac{\partial\rho_m}{\partial t}+\vec{\nabla}\cdot\left[\rho_m\vec{U}\right]=0$
3. For deriving the momentum conservation law, we set: $\vec{\Gamma}_{ie}=-\vec{\Gamma}_{ei}$

Which says that the momentum is conserved during inter-particle collisions. It is valid when the rate of momentum density change between two fluids is proportional to their relative velocity, i.e.:

$$\Gamma_{ei}=-\nu_{ei}\left[\vec{U}_e-\vec{U}_i\right]\rho_e \quad (2.43)$$

$$\Gamma_{ie}=-\nu_{ie}\left[\vec{U}_i-\vec{U}_e\right]\rho_i \quad (2.44)$$

$$\nu_{ei}\rho_e=\nu_{ie}\rho_i \quad (2.45)$$

The momentum conservation equation becomes

$$\rho_m=\left[\frac{\partial\vec{U}}{\partial t}+\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right]=-\vec{\nabla}p+\vec{J}\times\vec{B}-\rho_m\vec{\nabla}\varphi_g+\gamma \quad (2.46)$$

Where γ is the viscous term. So there is no electric field in an ideal electron and singly ionized ion plasma. There is only the inductive fluid in a varying magnetic field and this is given by Faraday's law of induction.

Ohm's Law

The familiar form of the ohm's law $\vec{J} = \sigma\vec{E}$, takes a rather inflated form in a non-ideal fluid. By taking the difference of the electron and the ion equations, substituting

$$U_i = U + \frac{m_e \vec{J}}{en(m_e + m_i)} \tag{2.47}$$

$$U_e = U + \frac{m_i \vec{J}}{en(m_e + m_i)} \tag{2.48}$$

into either and eliminating $\partial U/\partial t$ using the single fluid momentum equation we find

$$\frac{\dot{\vec{J}}}{\omega_{pe}{}^2 + \omega_{pi}{}^2} = \vec{E} + \vec{U} \times \vec{B} - \frac{\vec{J}}{\sigma_0} + \frac{1}{e\rho_0}\left[m_i \vec{\nabla} p_e - m_e \vec{\nabla} p_i\right] - \frac{1}{e\rho_0}\left[(m_i - m_e)\vec{J} \times \vec{B}\right] \tag{2.49}$$

The equation (2.49) is known as the generalized Ohm's Law. The last term of this equation is the Hall effect term in MHD flows. A simplified form of Ohm's law is obtained in the absence of all but the finite conductivity effects. It is

$$\vec{J} = \sigma_0 \left[\vec{E} + \frac{\vec{U} \times \vec{B}}{c}\right] \tag{2.50}$$

For ideal MHD, the electrical conducting $\sigma_0 \to \infty$ and the inductive electric field $\vec{E} = -\dfrac{\vec{U} \times \vec{B}}{c}$ is found.

In this way we arrive at the single fluid description consisting of the following equations:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\rho\vec{U}=0 \tag{2.51}$$

$$\rho\left[\frac{\partial U}{\partial t}+\left(\vec{U}\cdot\vec{\nabla}\right)U\right]=-\vec{\nabla}\cdot\vec{\Pi}+\vec{J}\times\vec{B}-\rho\vec{\nabla}\varphi_g \tag{2.52}$$

$$\vec{J}=\sigma\left[\vec{E}+\frac{\vec{U}\times\vec{B}}{c}\right] \tag{2.53}$$

$$\vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{2.54}$$

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{2.55}$$

We still need a relation between the pressure and density. Usually one uses $\vec{\nabla}p=c_s{}^2\vec{\nabla}\rho$, where $c_s{}^2$ is the sound speed. This is derived from the addition of the adiabatic energy equation for each species.

The single fluid description is used to study the configurational stability of a plasma. The cause and the source of the instability lies in the spatial gradients of density, temperature, magnetic field and velocity.

We now have mathematical framework to study a plasma as a single fluid, as two fluids and as a kinetic system.

Chapter 3

PARTICLE AND FLUID MOTIONS IN GRAVITATIONAL AND ELECTROMAGNETIC FIELDS

3.1. Back to Single Particle Motion

In Chapter 2, we presented the kinetic and the fluid descriptions of an N-particle system. We can recover the familiar single particle equations by ignoring the many body effects like collisions and stresses and the extensive quantities like pressure and temperature. For an important class of problems, the knowledge of the motion of a single particle under the action of various forces provides us with great insight at a modest effort. Under certain circumstances, the entire fluid does what each particle does. We shall study these drifts which are common to both a single particle as well as a fluid. But there are additional drifts which originate entirely due to fluid properties such as pressure forces. The two types of drifts in the presence of space and time dependent fields will be studied in this chapter.

3.2. Purpose of Studying Single Particle Motion

Yes, we must learn to walk before we can run! Identification of a simpler unit of a complex whole is an essential first step in any endeavour. For example, instead of the frustrating prospect of taking into account the gravitational forces of all the members of the Solar System, we can delineate the major component of the motion of any planet by assuming that it moves under the gravitational force of the sun alone. The finer details can be worked out later depending upon the purpose for which they are required.

The study of motion of a particle in the presence of electromagnetic and gravitational fields is inherently a nonlinear problem since the fields have to be evaluated at the instantaneous position of the particle, which is determined by the action of these bery fields. The non-linearity, however, can be broken by using perturbation methods if the spatial and temporal variations of the fields are slower than the spatial and time scales of the phenomenon under investigation. We can then determine the velocity and the trajectory of a particle. If the velocity is a function of charge and or mass of a particle, the relative motion between different species of particles like electrons and ions can result in a current flowing through the system. This current, if exceeds a certain threshold, can make the system unstable. Thus, the determination of particle drifts forms an important part of the study of the stability of a plasma. The concepts of adiabatic invariance are also found to be quite useful in analyzing the motion in complex field configurations and thereby in studying the transfer of energy from one degree of freedom to another (as we will see in a magnetic mirror geometry).

Further, the motion of a star in a galaxy, of an asteroid or a comet in the solar system, or of an atom in the ionizing electromagnetic field is being studied with a new awareness of the sensitive role of the initial conditions. This constitutes chaotic dynamics which has sprung many surprises in a seemingly well determined classical system.

The action of a strong electromagnetic wave on a particle can be described in terms of a nonlinear potential called the **Ponderomotive Potential**. In the fluid limit, then, all the particles feel this force, which is akin to a pressure gradient force. This technique comes quite handy for understanding some of the nonlinear plasma phenomena. In the following sections, we will study the variety of motions that a particle and a fluid undergo under the action of gravitational and electromagnetic forces, acting singly or jointly.

3.3. Equation of Motion of a Single Particle

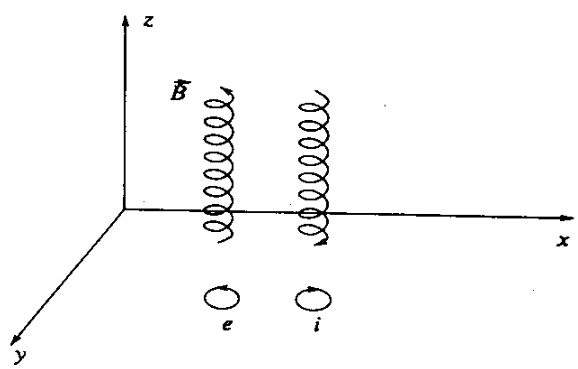

Figure 3.1. Helical Motion of Electrons and Ions in a Uniform Magnetic Field.

The equation of motion of a particle with charge Q, mass m and velocity $\vec{U}$ in the presence of an electric field $\vec{E}$, a magnetic field $\vec{B}$ and a gravitational potential φ_g is given by:

$$\frac{d\vec{U}}{dt} = \frac{Q}{m}\left[\vec{E} + \frac{\vec{U} \times \vec{B}}{c}\right] - \vec{\nabla}\varphi_g \tag{3.1}$$

3.4. Motion in Uniform Mangetic Field $\vec{B}$

Let us first study the motion of a charged particle in the presence of only the magnetic field. The equation is

$$\frac{d\vec{U}}{dt} = \frac{Q}{m}\left[\frac{\vec{U} \times \vec{B}}{c}\right] \tag{3.2}$$

Note that $\vec{B} \cdot \frac{d\vec{U}}{dt} = 0$; $\frac{d}{dt}\left(\tfrac{1}{2} mU^2\right) = 0$.

This means there is no acceleration along the magnetic field and the total energy is conserved. In a direction perpendicular to $\vec{B}$, we find

$$U_x = U_{\perp 0} \cos \Omega_B t\,, \quad U_y = -U_{\perp 0} sin \Omega_B t\,, \quad U_z = \text{constant} \tag{3.3}$$

$$\Omega_B = \frac{QB_z}{mc}\,, \quad (x - x_0)^2 + (y - y_0)^2 = \frac{U_{\perp 0}^2}{\Omega_B^2} \equiv R_B^2 \tag{3.4}$$

where $\vec{B}$ is in the z direction.

In cylindrical coordinates (r, θ, z), the phase of the particle motion is

$$\theta = \tan^{-1} \frac{y - y_0}{x - x_0} = \frac{\pi}{2} - \Omega_B t \tag{3.5}$$

Here (x_0, y_0) is the initial position, $U_{\perp 0}$ is the initial perpendicular velocity, Ω_B is the cyclotron frequency and R_B is the cyclotron radius.

In the presence of extremely strong magnetic fields such as in pulsars, the cyclotron radius R_B may become comparable to the De Broglie wavelength. Under these conditions, the motion becomes quantum mechanical.

The quantized energy levels are

$$E_l = \left(l + \frac{1}{2}\right)\hbar\Omega_B \tag{3.6}$$

where l is an integer.

The relativistic equation of motion can be written as

$$\frac{d\vec{p}}{dt} = \frac{Q}{c}\vec{U} \times \vec{B}, \quad \vec{p} = \gamma m \vec{U} \tag{3.7}$$

and $\gamma = \left(1 - \frac{U^2}{c^2}\right)^{-\frac{1}{2}}$ is the Lorentz factor.

3.5. Motion in Combined Electric and Magnetic Fields

The equation of motion is:

$$\frac{d\vec{U}}{dt} = \frac{Q}{m}\left[\vec{E} + \frac{\vec{U} \times \vec{B}}{c}\right] \tag{3.8}$$

Notice that

$$\vec{B} \cdot \frac{d\vec{U}}{dt} = \frac{Q}{m}\vec{E} \cdot \vec{B} \tag{3.9}$$

Thus there is an acceleration parallel to the magnetic field due to an electric field along $\vec{B}$. In the steady state

$$\frac{d}{dt} = 0 \tag{3.10}$$

so that

$$\vec{E} \times \vec{B} + \frac{1}{c}\left(\vec{U} \times \vec{B}\right) \times \vec{B} = 0 \tag{3.11}$$

or

$$\vec{U} \equiv \vec{U}_E = c\frac{\vec{E}\times\vec{B}}{B^2} \tag{3.12}$$

The velocity $\vec{U}_E$ is known as the $\vec{E}\times\vec{B}$ drift. It is independent of the charge, mass and the energy of the particle. Electrons and protons both move with this common velocity.

For stationary electric and the magnetic field, substituting $\vec{U} = \vec{U}_E + \vec{U}_B$ in the equation of motion, we find

$$\frac{d\vec{U}_B}{dt} = \frac{Q}{m}\left[\left(\vec{E} + \frac{\vec{U}_E\times\vec{B}}{c}\right) + \frac{\vec{U}_B\times\vec{B}}{c}\right]$$

$$\frac{d\vec{U}_B}{dt} = \frac{Q}{m}\left[\frac{\vec{U}_B\times\vec{B}}{c}\right] \tag{3.13}$$

since $\vec{E} = -\frac{\vec{U}_E}{c}\times\vec{B}$. This motion with velocity $\vec{U}_B$ is nothing but the cyclotron motion.

Thus the total motion of a charged particle in crossed electric and magnetic fields is equal to a helical motion plus the steady drift $\vec{U}_E$. It can be shown that the trajectory of the particle is given by

$$x = \left(x_0 + U_{Ex}t\right) + \frac{U_{\perp 0}}{\Omega_B} sin\Omega_B t \tag{3.14}$$

which shows that the center of gyration moves x_0 with the velocity $\vec{U}_E$. This is known as the guiding center motion. If we average over the circular motion, only the rectilinear motion of the center of gyration remains. Thus depending upon the time scale of interest, one can consider the complete motion or only the drift of the gyration center.

3.6. Motion of a Charged Particle in Magnetic and Gravitational Fields

The equation of motion in a magnetic field $\vec{B}$ and a gravitational field with acceleration due to gravity $\vec{g}$ is

$$\frac{d\vec{U}}{dt} = \frac{Q}{m}\left[\frac{\vec{U}\times\vec{B}}{c}\right] + \vec{g} \tag{3.15}$$

In analogy with the electric field, one can define a $\vec{g} \times \vec{B}$ drift $\vec{U}_g$ as:

$$\vec{U}_g = \frac{mc}{Q} \frac{\vec{g} \times \vec{B}}{B^2} \tag{3.16}$$

Note that this drift is a function of the charge and the mass of the particle. It can therefore result in an electric current density in an electron-proton plasma:

$$\vec{J}_g = -n_e e \vec{U}_{ge} + n_p e \vec{U}_{gp} \tag{3.17}$$

The current density beyond a critical value could make the plasma unstable. The circumstances of high $\vec{g}$ and low $\vec{B}$ are favorable for producing high $\vec{g} \times \vec{B}$ drift current.

3.7. Motion in Inhomogeneous Magnetic Field

Strictly speaking magnetic field is always inhomogeneous since the field lines are curved due to the divergence free nature of the magnetic field. An example is the dipole field which in spherical polar coordinates (r, θ, φ) is described as:

$$\begin{aligned} B_r &= \frac{2m_B}{r^3} \cos\theta \\ B_\theta &= -\frac{m_B}{r^3} sin\theta \\ B_\varphi &= 0 \end{aligned} \tag{3.18}$$

where m_B is the magnetic dipole moment.

Finding the trajectory of a particle in such a field is a non trivial task. We must resort to perturbation methods. We separate the field into its uniform and non uniform parts

$$\vec{B} = \vec{B}_0 + \vec{B}_1(\vec{r}, t). \tag{3.19}$$

First the particle motion in the uniform part $\vec{B}_0$ is found. Then we substitute this trajectory in the non-uniform part and find the new solution. Averaging over the circular motion gives guiding centre drifts of the particles. As an example let us take a field of the form:

$$\begin{aligned} B_x &= z\left(\frac{\partial B_x}{\partial z}\right)_0 \\ B_y &= 0 \\ B_z &= B_0 + x\left(\frac{\partial B_z}{\partial x}\right) \end{aligned} \tag{3.20}$$

We find the guiding center drift velocity

$$\vec{U}_{gc} = \frac{1}{\Omega_B}\left[\frac{U_{\perp 0}^2}{2} + U_{z0}^2\right]\frac{\vec{R}_c \times \vec{B}}{R_c^2 B}, \tag{3.21}$$

where

$$\vec{R}_C = -\left[\frac{1}{B_0}\left(\frac{\partial B_z}{\partial x}\right)_0\right]\hat{x}$$

is the radius of curvature of the magnetic field.

The component of $\vec{U}_{gc}$ proportional to $U_{\perp 0}^2$ is called the grad B drift and the component proportional to U_{z0}^2 is known as the curvature drift. The drifts provide the cross field transport of the particles.

3.8. Motion of a Charged Particle in a Magnetic Mirror or $\vec{\nabla} B // \vec{B}$ Field

Such a field has a gradient in its own direction, i. e.,

$$\frac{\partial B_z}{\partial z} \neq 0 \tag{3.22}$$

But we must have

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.23}$$

Therefore, in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \quad \text{or} \quad B_r = -\frac{1}{2}\left(\frac{\partial B_z}{\partial z}\right)_0 \tag{3.24}$$

Again let us first solve the equation of motion only in the uniform part of B. Then substitute it in the non-uniform part to find the drift velocity

$$U_z = U_{z0} - \frac{U_{r0}^2}{2}\frac{1}{B_z}\left(\frac{\partial B_z}{\partial z}\right)_0 t \tag{3.25}$$

We see that for $\frac{\partial B_z}{\partial z} > 0$, the velocity decreases with time. The particle slows down, comes to a rest and reverses its direction of motion. Now the particle encounters a negative gradient of B_z and therefore it accelerates. At any time t the total energy of the particle remains a constant. Therefore

$$U_{\perp 0}^2 + U_{zo}^2 = U_{\perp}^2(t) + U_z^2(t) \tag{3.26}$$

We notice that at the moments of reflection, the entire energy is in the perpendicular component. Thus the particles are trapped in a magnetic mirror. Other effects such as collisions may help the particles to move out of the trap. An example of such a system is found in our own magnetosphere. These are the Van Allen radiation belts which were detected through an increase in the cosmic ray intensity by a factor of 10^4 at 1.5-3 times the radius R_E of the Earth. The inner belt at 1.5 R_E is a result of the interaction of the cosmic rays with oxygen and nitrogen molecules to produce neutrons which undergo β decay to produce protons and electrons and these electrons get trapped in the earth's magnetic field. The outer belt at 3 R_E is mostly populated by the solar wind electrons.

3.9. Adiabatic Invariants of Motion of a Charged Particle in Slowly Varying Magnetic Fields

We have seen that inhomogeneities in a magnetic field can make the equations of motion difficult to solve unless we resort to perturbation methods, which can be deployed under slow variations of the field. It was proved a long time ago, that certain quantities called the Action Integrals, which are invariants of a system under homogeneous and time independent fields, remain invariant to the first order in the parameters of the slowly varying fields. This is known as the Principle of Adiabatic Invariance of Action Integrals. Many of the characteristics of a system under slowly varying fields can be discussed with the use of adiabatic invariants, without having to solve the equations of motion. The principle of adiabatic invariance has found applications in fields like plasma physics, accelerator physics and galactic astronomy.

The action integral J is defined as

$$J = \int p dq \,, \tag{3.27}$$

where p is the canonical momentum and q is the corresponding canonical conjugate coordinate. We may recall, here, the definition of the canonical momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \,, \tag{3.28}$$

where L is the Lagrangian of the system.

Let us write the Lagrangian of a particle in a magnetic field in cylindrical coordinates (r, θ, z) as:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + \frac{QBr^2\dot{\theta}}{2c} \tag{3.29}$$

where $\vec{B} = (0, 0, B)$ has been assumed. Observe that L is independent of θ and therefore θ is the cyclic coordinate. The corresponding canonical momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{QB}{2c} r^2 , \tag{3.30}$$

is a constant of the motion.

Using the results obtained above, we find

$$J = -\frac{Q\pi R_B^2 B}{c} = -\frac{2\pi mc}{Q}\frac{mU_{\perp 0}^2}{2B} = -\frac{2\pi mc}{Q}\mu = \text{constant}, \tag{3.31}$$

according to the principle of adiabatic invariance. Thus, the Magnetic Momentum $\mu = \left(mU_{\perp 0}^2 / 2B \right)$ or the Magnetic Flux $\left(BR_B^2 \right)$ are the Adiabatic Invariants of motion of a particle in a slowly varying field.

3.10. Magnetic Mirror Revisited

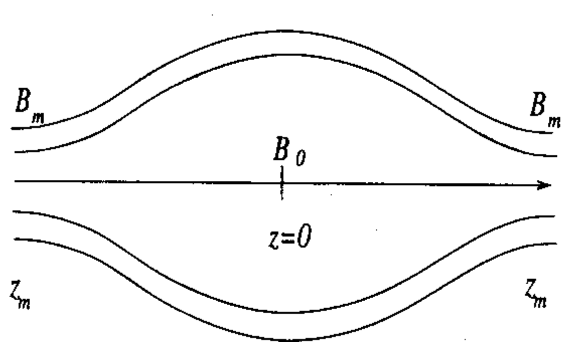

Figure 3.2. A Magnetic Mirror.

We have seen that at any instant, the total energy of a particle moving in a magnetic field remains constant. This is also true at any instantaneous position of the particle. Thus,

$$U_\perp^2(z) + U_z^2(z) = U_{\perp 0}^2 + U_{z0}^2 \equiv U^2 . \tag{3.31}$$

Since we now know that the magnetic moment μ is an adiabatic invariant, we have

$$\frac{U_\perp^2(z)}{B_z(z)} = \frac{U_{\perp 0}^2}{B_0}, \tag{3.32}$$

where $U_{\perp 0}$ and B_0 are the values at $z = 0$. From the two last equations, we get

$$U_z^2(z) = U^2 - \frac{B_z(z)}{B_0} U_{\perp 0}^2. \tag{3.33}$$

We observe that as $B_z(z)$ increases along the *z* direction, $U_z(z)$ decreases; the radius of gyration $R_B^2 \propto \left(\frac{U_\perp^2(z)}{B_z^2(z)}\right)$ decreases and the particle spirals in a continuously narrowing helical path. It also means that there is a transfer of energy from one degree of freedom to another – from parallel to perpendicular motion and vice-versa in a region of decreasing $B_z(z)$. Thus, in a magnetic mirror a particle suffers reflection at the two ends where $B_z(z)$ is a maximum and can remain trapped. The condition for trapping is $U_z(z) \leq 0$. Now $U_z(z) = 0$ at $B_z(z) = B_M(z_M)$ where $B_M(z_M)$ is the maximum value of B_z. At this point, we find

$$U^2 = \frac{B_M(z_M)}{B_0} U_{\perp 0}^2, \tag{3.34}$$

and the condition for trapping, therefore, becomes:

$$\left(\frac{U_{z0}}{U_{\perp 0}}\right)^2 \leq \frac{B_M(z_M)}{B_0} - 1 \tag{3.35}$$

or

$$sin^2\theta \geq \frac{1}{R_M} \equiv sin^2\theta_M, \tag{3.36}$$

where θ is the angle between the velocity $\vec{U}$ and the magnetic field $\vec{B}$ at $z = 0$ and $R_M = \left(B_M / B_0\right)$ is known as the Mirror Ratio. It is clear from the last equation that particles with Pitch Angles $\theta > \theta_M$ remain trapped in the magnetic mirror executing oscillations between the two regions of the highest magnetic fields. Whereas particles with pitch angles $\theta < \theta_M$ can leave the mirror. Thus, a magnetic mirror gives rise to a velocity distribution of particles which has no particles with $\theta < \theta_M$. Such a velocity distribution is known as the Loss Cone Distribution. The trapped and the untrapped particles are separated by a boundary at $sin^2\theta_M = \left(1/R_M \right)$. The trapping condition has no dependence upon mass and the charge of a particle. But the presence of collisions among particles can introduce charge and mass dependent differences.

3.11. Motion in Time Dependent Electric and Magnetic Fields

In such fields we can again use perturbation and adiabatic invariance techniques in order to determine the motion of the charged particles. Here, we describe one example where a charged particle is subjected to an intense electromagnetic radiation. The particle experiences a force known as the Ponderomotive Force. The equation of motion is:

$$m\frac{d\vec{u}}{dt} = Q\left[\vec{E}(\vec{r},t) + \frac{\vec{U}\times\vec{B}(\vec{r},t)}{c}\right] \tag{3.37}$$

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}(\vec{r})\cos\omega t \\ \vec{B}(\vec{r},t) &= -\frac{c}{\omega}\vec{\nabla}\times\vec{E}(\vec{r}) sin\omega t \end{aligned} \tag{3.38}$$

where $\vec{E}$ and $\vec{B}$ are the fields of the electromagnetic radiation. We use perturbation method as follows. We expand the position and velocity of the particle as well as the electric field as:

$$\begin{aligned} \vec{r} &= \vec{r}_0 + \vec{r}_1 + \vec{r}_2 + \ldots \\ \vec{u} &= \vec{u}_0 + \vec{u}_1 + \vec{u}_2 + \ldots \\ \vec{E}(\vec{r},t) &= E(r_0)\cos\omega t + \left(\vec{r}_1\cdot\vec{\nabla}\right)E(r)|_{r_0}\cos\omega t + \ldots \end{aligned} \tag{3.39}$$

and find the corresponding magnetic field.

To the first order, we find:

$$m\frac{du_1}{dt} = QE(r_0)\cos\omega t \tag{3.40}$$

$$u_1 = \frac{QE(r_0)}{m}\frac{\sin\omega t}{\omega}, \quad r_1 = -\frac{QE(r_0)}{m\omega^2}\cos\omega t \tag{3.41}$$

and to the second order, we get

$$\begin{aligned} m\frac{du_2}{dt} &= Q\left[\left(\vec{r}_1 \cdot \vec{\nabla}\right)E(r)|_{r_0}\ \cos\omega t + \frac{\vec{u}_1 \times B(r_0)}{c}\right] \\ &= -\frac{Q^2}{m\omega^2}\left[\left(E(r_0)\cdot\vec{\nabla}\right)E(r_2)\right]\cos^2\omega t - \frac{Q^2}{m\omega^2}\left[\vec{E}(\vec{r}_0)\times\left(\vec{\nabla}\times\vec{E}(\vec{r}_0)\right)\right]\sin^2\omega t \end{aligned} \tag{3.42}$$

Taking the average over the fast oscillatory motion at frequency ω, we get

$$\left\langle m\frac{dU_2}{dt}\right\rangle = -\frac{Q^2}{4m\omega^2}\vec{\nabla}E^2(r_0) = f_{NL} = -\vec{\nabla}\psi \tag{3.43}$$

where

$$\psi = \frac{Q^2}{4m\omega^2}E^2(r_0)$$

is known as the Ponderomotive Potential. It modifies the electron density and the temperature of the plasma.

The ponderomotive force acts like a pressure force. Thus the charged particles move from regions of high ponderomotive pressure to low ponderomotive pressure, creating a local depletion or a cavity with electron plasma oscillations. A plasma can acquire novel properties affecting the propagation of electromagnetic radiation.

3.12. Fluid Drifts

We have determined drifts of single particles under different circumstances of spatially and temporally varying electric and magnetic fields. In all cases, it was the guiding center of a particle that moved with the drift speed. When we treat an entire system of *N* particles, it acquires new characteristics like pressure, density and transport parameters like thermal and electrical conductivity, as we have seen while deriving fluid equations in Chapter two. There are, thus drifts which are specific to the fluid character of a system and do not exist for single particles.

In the presence of a pressure gradient, a neutral fluid flows from a high pressure region to a low pressure region. However, a new flow is generated when a pressure gradient force acts upon a charged fluid in the presence of a magnetic field. To see this, let us write the equation of motion of charged particles of species *s* (for example an electron or an ion fluid):

$$\rho_s\left[\frac{\partial \vec{U}_s}{\partial t}+\left(\vec{U}_s\cdot\vec{\nabla}\right)\vec{U}_s\right]=-\vec{\nabla}p_s-\rho_s\vec{\nabla}\varphi_g+\frac{\rho_s Q_s}{m_s}\left[\vec{E}+\frac{\vec{U}_s\times\vec{B}}{c}\right], \qquad (3.44)$$

where only the diagonal part p_s of the stress tensor Π_s has been retained. There are two time scales in the last equation, the inertial time scale over which $\vec{U}_s$ varies and the cyclotron time scale $(\Omega_B)^{-1}$. For variations of $\vec{U}_s$ much slower than $(\Omega_B)^{-1}$, we can put the term $\frac{\partial \vec{U}_s}{\partial t}\cong 0$. Taking the cross product of the last equation with the magnetic field $\vec{B}$ gives:

$$\rho_s\left[\left(\vec{U}_s\cdot\vec{\nabla}\right)\vec{U}_s\right]\times\vec{B}=-\vec{\nabla}p_s\times\vec{B}+\frac{\rho_s Q_s}{m_s}\left[\vec{E}\times\vec{B}-\frac{\vec{U}_{s\perp}}{c}B^2\right]-\rho_s\vec{\nabla}\varphi_g\times\vec{B} \qquad (3.45)$$

or

$$\vec{U}_{s\perp}\cong c\frac{\vec{E}\times\vec{B}}{B^2}-\frac{c}{n_s Q_s B^2}\vec{\nabla}p_s\times\vec{B}-\frac{m_s c}{Q_s B^2}\vec{\nabla}\varphi_g\times\vec{B}. \qquad (3.46)$$

You should be able to recognize the first and the third terms on the right-hand side as the $\vec{E}\times\vec{B}$ and $\vec{g}\times\vec{B}$ drifts found earlier for a single particle. The middle term is the

New Fluid Drift – The $\vec{\nabla} p \times \vec{B}$ Drift that has arisen due to the pressure gradient force in a fluid.

The $\vec{\nabla} p_s \times \vec{B}$ drift depends on the charge Q_s and the number density n_s of particles of species *s* and is known as the Diamagnetic Drift. This is akin to the $\vec{\nabla} B \times \vec{B}$ drift since, as we will show later, magnetic field has an associated pressure $\left(B^2/8\pi\right)$ with it. The diamagnetic flow is depicted in figure (3.9) for $\vec{\nabla} n$ in the *y* direction and $\vec{B}$ in the *z* direction. There are more electron orbits going up towards *x* in region A compared to the number of electron orbits coming down in region A'. Why? So, there is a net electron flux $n_e \vec{U}_{e\perp}$ in the *x* direction. For the same reason, there is a net flux $n_i \vec{U}_{i\perp}$ in the (-*x*) direction.

We must appreciate the fact that particle drifts are the drifts of their guiding centers and fluid drifts are the mass motion of the entire fluid.

We can also determine if there is any drift of a fluid in a direction parallel to the magnetic field. We write the *z* component of the equation of motion:

$$\rho_s \left[\frac{\partial U_{sz}}{\partial t} + \left(\vec{U}_s \cdot \vec{\nabla} \right) U_{sz} \right] = -\frac{\partial p_s}{\partial z} - \rho_s \frac{\partial \varphi}{\partial z} + \frac{\rho_s Q_s}{m_s} E_z \,. \qquad (3.47)$$

If we ignore the convective term, we see immediately that the fluid feels an acceleration along the direction *z* of the magnetic field. Further, neglecting the inertial terms altogether gives us a relation between the particle density, the gravitational potential φ_g and the electric potential $\varphi \left(E_z = -\partial\varphi/\partial z \right)$:

$$n_s(z) = n_{0s} \exp\left[-\frac{Q_s}{K_B T} \varphi - \frac{m_s}{K_B T} \varphi_g \right], \qquad (3.48)$$

were we have expressed $p_s = n_s K_B T$, the isothermal equation of state. This equation is the Boltzmann Relation for species *s* of particles and describes the redistribution of particles resulting due to a balance of pressure gradients, gravitational and electric fields. This distribution has important consequences for plasma phenomena as we will find out in the following chapters.

Chapter 4

MAGNETOHYDRODYNAMICS OF CONDUCTING FLUIDS

4.1. Electrically Conducting Fluids

Ionized gases or plasmas and liquid metals such as mercury or liquid sodium are electrically conducting fluids. The outer core of the earth is believed to be molten iron. Magnetospheres of planets and stars, tails of comets, extragalactic jets, accretion disks and many other astrophysical objects are studied by treating them as electrically conducting fluids. The study of magnetohydrodynamics (MHD) draws from two well known branches of physics, electrodynamics and hydrodynamics, along with a provision to include their coupling. The basic laws of electrodynamics described in the form of Maxwell's Equations supplemented by the generalized Ohm's law are sufficient for the purpose. The hydrodynamics of a fluid is expressed in the form of conservation laws of mass, momentum and energy. These laws treat the fluid as a continuum. The continuum description is valid if the mean free path of the constituent particles is much shorter than the spatial scales on which the flow is visualized. Thus, according to this criterion, any substance can be treated as a continuum at some spatial scale. The magnetohydrodynamic phenomena are a consequence of the mutual interaction of the fluid flow and the magnetic field. As is well known, a conductor crossing magnetic field lines gives rise to an induced electric field, which drives an electric current in the conducting fluid. The resulting Lorentz force accelerates the fluid across the magnetic field, which in turn creates another induced electric field and currents which modify the initial magnetic field. Thus the bulk motion of a conducting fluid and a magnetic field influence each other and must be determined self-consistently.

The interaction of fluid flows and electromagnetic and gravitational fields determines the configurational characteristics like loops, jets, tails and filaments observed on almost all scales in the universe. The configuration of the radiation emitting plasma causes variability in radiation over a host of spatial and temporal scales. The stability or otherwise of the configuration determines the lifetime of the radiating material in a particular mode. The magnetohydrodynamic instabilities help the configuration to relax with an attendant release of kinetic, electromagnetic and gravitational energy.

One of the major results of magnetohydrodynamics is the ability of conducting fluids to amplify magnetic fields, the amplification of magnetic fields being a universal necessity. This aspect of MHD reminds us that more often than not, fluids are turbulent. Turbulent fluids only permit a statistical description. We will, defer the discussion of this topic until the chapter on nonconducting fluids.

4.2. Validity of Magnetohydrodynamics

There is a well defined region of applicability of MHD. Generally MHD addresses the macroscopic, bulk or large spatial scale and large time scale processes occurring in a conducting fluid. More specifically, for example, in an electron-proton fluid, effects associated with fast variations of electric and magnetic fields are neglected. One of these effects is the **Space Charge Effect**. Therefore:

(1) **Space Charge Effects are Neglected in MHD**. In an electron-proton fluid, electrons and protons are accelerated by the applied electric and magnetic fields and decelerated by the Coulomb collisions between them. Ohm's law $\vec{J} = \sigma\vec{E}$ is a consequence of the balance of these accelerating and retarding forces. In MHD, the mean free path of the particles is very short or the collision frequency is very high. On the other hand, the frequency of the applied fields is low. Under these circumstances, there cannot result any significant charge separation since the large number of collisions can neutralize any charge separation produced by the applied fields. The applied fields of low frequency may produce a small polarization or net charge density, which is neglected in MHD. The Poisson Equation is therefore never used in MHD to determine electric fields as there is no net charge density. The electric fields are entirely produced due to time varying magnetic fields or charge distributions external to the fluid.

(2) Again, due to the slow time variation of electric and magnetic fields, the **Displacement Current** term in the modified Ampere's law is **Neglected**.

(3) The collision frequency being the highest frequency in MHD phenomena, Maxwellization of velocities of particles is ensured. Please recall that we derived the single-fluid and the two-fluid equations by taking the moments of the Boltzmannn equation. In this process, several times, we had to invoke

the velocity dependence of the distribution function which would reduce the surface integrals to zero. And we found that the Maxwellian distribution function of velocities fitted the bill very well. Collisions thermalize electrons and protons to a common temperature. Thus, a **Fluid is Characterized by a Single Temperature**.

4.3. Equations of Magnetohydrodynamics

The motion of a conducting fluid in a magnetic field is described by the usual hydrodynamic variables: mass density, velocity and pressure; Ampere's law without the displacement current; Faraday's induction law; and the generalized Ohm's law. We have derived the single fluid equations in Chapter 3. We use them here to study MHD phenomena. The MHD equations for a fluid consisting of electrons and protons are:

$$\frac{\partial \tilde{n}_m}{\partial t} + \vec{\nabla}.[\tilde{n}_m \vec{U}] = 0 \tag{4.1}$$

$$\tilde{n}_m \left[\frac{\partial \vec{U}}{\partial t} + (\vec{U}.\vec{\nabla})\vec{U} \right] = -\vec{\nabla} p + \left(\hat{\imath} + \frac{1}{3} \grave{\imath} \right) \vec{\nabla}(\vec{\nabla}.\vec{U}) + \mu \nabla^2 \vec{U} + \frac{\vec{J} \times \vec{B}}{c} - \tilde{n}_m \vec{\nabla} \varphi_g \tag{4.2}$$

$$\vec{E} + \frac{\vec{U} \times \vec{B}}{c} = \eta \vec{J} + \frac{1}{en} \left[\frac{\vec{J} \times \vec{B}}{c} - \vec{\nabla}.\vec{Đ}_e \right] \tag{4.3}$$

$$\vec{\nabla}.\vec{J} = 0 \tag{4.4}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{4.5}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \tag{4.6}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{4.7}$$

where the viscosity coefficients $\xi = \xi_e + \xi_i$,$\mu = \mu_e + \mu_i$ and pressure $p = p_e + p_i$. The generalized Ohm's law (4.3) has also been written under the assumption $(m_e/m_i) \rightarrow 0$. By adding the energy equation for each species and expressing the sum in terms of the single fluid variables $\vec{U}$ and $\vec{J}$, the energy equation for an electron-proton fluid can be determined. This is rather a long algebraic exercise. We will here give a simple form of the energy equation with the inclusion of Joule heating rate in the presence of current density $\vec{J}$ and its associated magnetic field $\vec{B}$. The rate of production of heat per unit fluid volume is given by:

$$\vec{J}.\vec{E} = \frac{1}{\sigma} J^2 = \frac{c^2}{(4\pi)^2 \sigma} (\vec{\nabla} \times \vec{B})^2$$

The rate of resulting rise in temperature,ΔT, is given by

$$\rho_m \, c_p \frac{\Delta T}{\Delta t} = \frac{c^2}{16\pi^2 \sigma} (\vec{\nabla} \times \vec{B})^2 \qquad (4.8)$$

The energy equation,therefore, in the absence of viscous effects, becomes:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{c^2}{16\pi^2 \sigma \, \mathrm{c_p} \rho_m} (\vec{\nabla} \times \vec{B})^2 \qquad (4.9)$$

Equation (4.9) does not include a contribution from the collisional term. This term tends to equalize the temperatures of the various species. Therefore under the MHD approximation, where we assume that the fluid is characterized by a single temperature, the collisional term can be neglected. We will study magnetohydrodynamics of conducting fluids by using Equations (4.1), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9).

4.4. Ideal Conducting Fluids

An ideal conducting fluid is one with infinite conductivity σ, or zero electrical resistivity η, and zero viscosity coefficients μ and ξ. In the generalized Ohm's law, Equation (4.3), the Hall term $\vec{J} \times \vec{B}$ is usually smaller than $\vec{U} \times \vec{B}$ term and if additionally the pressure forces are zero, the condition of infinite conductivity implies that

$$\vec{E} + \frac{\vec{U} \times \vec{B}}{c} = 0 \qquad (4.10)$$

Faraday's induction law Equation (4.7) becomes:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) \tag{4.11}$$

It can be easily shown that Equation (4.11) is a statement of conservation of magnetic flux, provided that the area enclosing the flux moves with the fluid with velocity $\vec{U}$. The magnetic flux φ_B can change due to (1) a change in magnetic induction $\vec{B}$ and/ or (2) a change in the area enclosing the flux as a result of the moving boundaries of the area. Thus:

$$\frac{d\,\varphi_B}{d\,t} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_S \vec{B} . \frac{\partial\, d\vec{S}}{\partial t} = \int_S \vec{\nabla} \times (\vec{U} \times \vec{B}). d\vec{S} + \int_S \vec{B}. \vec{U} \times d\vec{l}$$

since $\vec{U} \times d\vec{l}$ is the rate of change of area (Figura 4.1)

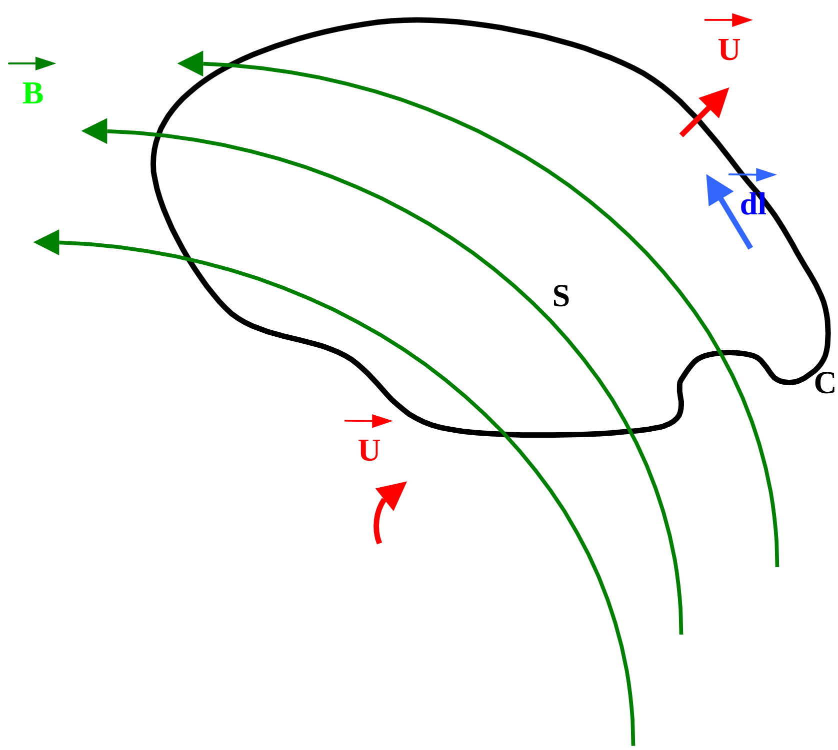

Figure 4.1. Conservation of Magnetic Flux φ_B in Moving Ideal Fluid.

Using Stokes theorem and a well known vector identity we find that:

$$\frac{d\,\varphi_B}{d\,t} = 0 \tag{4.12}$$

Thus the magnetic flux φ_B remains constant as long as the surface enclosing it moves with the fluid velocity $\vec{U}$. This characteristic of the magnetic induction $\vec{B}$ in a fluid has given rise to the term **Frozen-in Fields**, i.e., an ideal fluid and the magnetic field lines move together with a common velocity. We can easily see that this will not be true for finite conductivity, for, non-ideal fluids suffer viscous dissipation themselves, as well as cause magnetic field dissipation. From Equation (4.10), we see that this common velocity of the field and the fluid is nothing but the familiar $\vec{E} \times \vec{B}$ velocity $\vec{U}_{\vec{E}_0}$. The motion of the fluid parallel to the magnetic field is governed by non-magnetic forces. The case of infinite conductivity is not just a mathematical curiosity. We have seen earlier how induced electric fields generated by a rotating neutron star produce a plasma around the star. We can estimate that the electrical resistivity of a fully ionized plasma is given by:

$$\eta \cong 10^{-7}\, T^{-3/2} \text{ sec} \tag{4.13}$$

where T is the temperature of the plasma in Kelvin degrees. Thus for typical temperatures of the order of 10^8 - 10^{10} K, $\eta \cong 10^{-19}$ - 10^{-22} sec. Do you know that the resistivity of copper is ~ 10^{-18} sec ? Many properties of plasmas around neutron stars are studied using ideal magnetohydrodynamics. There are numerous examples of nearly ideal conducting fluids in the universe. Of course, large conductivity or small resistivity is not always an advantage. A phenomenon like solar flares , where large amounts of energy are released in a matter of few minutes, could not occur in ideal conducting fluid. Several mechanisms to reduce the conductivity and enhance the joule dissipation rate or ohmic heating have been investigated.

4.5. Pragmatic Conducting Fluids

Yes, there are fluids with large electrical conductivities and there are fluids with small conductivities but there are no fluids with infinite conductivities; the large and small are decided by the phenomena we choose to study. Let us see how a normal fluid with finite conductivity behaves. Again neglecting the viscous forces, and Hall current, we substitute for $\vec{J}$ from the generalized Ohm's law in the momentum Equation (4.2) to get

$$\rho_m \frac{\partial \vec{U}}{\partial t} = \frac{\sigma}{c}\left[\vec{E} + \frac{\vec{U}\times\vec{B}}{c}\right]\times\vec{B}$$

$$= \frac{\sigma B^2}{c}\left[\frac{\vec{E}\times\vec{B}}{B^2} - \frac{\vec{U}}{c} + \frac{(\vec{B}.\vec{U})\vec{B}}{cB^2}\right]$$

$$= \frac{\sigma B^2}{c^2}\left[\vec{U}_E - \vec{U}_\perp\right] \qquad (4.14)$$

The solution of Equation (4.21) is found to be

$$U_{//} = constant$$

and

$$\vec{U}_\perp = \vec{U}_{\perp 0}\exp\left[-\frac{\sigma B^2 t}{\rho_m c^2}\right] + \vec{U}_E \qquad (4.15)$$

That is, if there were an initial perpendicular velocity $\vec{U}_{\perp 0}$ in addition to the $\vec{E}\times\vec{B}$ velocity, it would fall exponentially with time. The *e* fall time is

$$t_e = \frac{\rho_m c^2}{\sigma B^2} \qquad (4.16)$$

and ultimately only the $\vec{E}\times\vec{B}$ motion will survive. The magnetic field retards any attempts of the fluid to cross it and offers a kind of magnetic viscosity.

What happens to the magnetic field in a non-ideal conducting fluid ? Let us substitute the generalized Ohm's law without the viscous and pressure forces and Hall current into Faraday's induction law. We find

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} \tag{4.17}$$

We have seen the effect of the first term, which established the constancy of magnetic flux in a moving ideally conducting fluid. In order to appreciate the effect of finite conductivity, let us neglect the first term. Now, Equation (4.17) assumes the form of a diffusion equation:

$$\frac{\partial \vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B} \tag{4.18}$$

This equation can be solved by the method of separation of variables. By comparing Equation (4.18) with the momentum Equation (4.2), we see that the quantity ($c^2/4\pi\sigma$) plays the same role for magnetic field as the kinematic viscosity $\nu = \mu/\rho_m$ plays for the fluid motion. Therefore we can now define the magnetic viscosity ν_m as

$$\nu_m = \frac{c^2}{4\pi\sigma} \tag{4.19}$$

From the solution of Equation (4.18), we find that at any given spatial position, the magnetic field decays exponentially with time as

$$B = B_0 \exp\left(-\frac{t}{t_d}\right) \tag{4.20}$$

where the *e*-fall time t_d is given by

$$t_d = \frac{L^2}{\nu_m} = \frac{4\pi\, \sigma\, L^2}{c^2} \tag{4.21}$$

for a typical length scale L of the spatial variation of B. Thus, for $\sigma \rightarrow \infty$, $\nu_m \rightarrow 0$ and $t_d \rightarrow \infty$, i.e., there is no decay, a result we have already seen in the conservation of magnetic flux. The relative importance of the two terms in the evolution of magnetic field (Equation 4.17) is decided by their ratio R_m given by

$$R_m = \frac{\left(UB/L \right)}{\left(\frac{\nu_m B}{L^2} \right)} = \frac{UL}{\nu_m} \tag{4.22}$$

where we have used dimensional analysis to arrive at Equation (4.22). For small values of ν_m or large values of the ratio R_m, called the **Magnetic Reynolds Number**, the magnetic field suffers very little diffusion and is simply carried away by the fluid. We can also define the **Kinetic Reynolds Number** R_k as the ratio of the convective term $(\vec{U}.\vec{\nabla})\vec{U}$ and the diffusion term $\nu\, \nabla^2 \vec{U}$ to find

$$R_k = \frac{UL}{\nu} \tag{4.23}$$

Ideal conducting fluids have infinitely large values of R_m and R_k. Astrophysical conducting fluids often satisfy the conditions $R_m >> 1$ and $R_k >> 1$, because of their large characteristic spatial scales.

4.6- Conducting Fluid in Equilibrium

We have seen that a conducting fluid experiences inertial, gravitational, pressure gradient, electromagnetic and viscous forces. A fluid can attain a state of equilibrium if the net force on it vanishes. In the equilibrium state, all physical quantities, including mass density, fluid velocity, pressure, current density and magnetic field are independent of time. From the momentum Equation (4.2), we study the various equilibria by neglecting viscous forces. The equilibrium condition is:

$$\rho_m(\vec{U}.\vec{\nabla})\vec{U} = -\vec{\nabla} p + \frac{\vec{J}\times\vec{B}}{c} - \rho_m \vec{\nabla}\varphi_g \qquad (4.24)$$

4.7. Hydrostatic Equilibrium

Perhaps the most familiar case is that of Hydrostatic Equilibrium for which $\vec{U} = 0$ and $\vec{J}\times\vec{B} = 0$ and the pressure gradient force balances the gravitational force, so that

$$\vec{\nabla} p = -\rho_m \vec{\nabla}\varphi_g \qquad (4.25)$$

This equation is of great importance for many astrophysical situations such as stars or galactic clouds. The gravitational potential φ_g for an extended mass distribution is given by

$$\varphi_g = -\frac{GM(r)}{r} \qquad (4.26)$$

where M(r) is the spherically distributed mass producing the gravitational force on a fluid element of mass density ρ_m. Equation (4.25) can be recast as:

$$\frac{d}{d\,\mathrm{r}}\left[\frac{r^2}{\rho_m}\frac{d\,\mathrm{p}}{d\,\mathrm{r}}\right] = -4\pi G\rho_m(r)r^2 \qquad (4.27)$$

by using

$$\frac{dM(r)}{d\,\mathrm{r}} = 4\pi\rho_m r^2$$

We can now take a general form of the equation of state

$$p = K_m \rho_m^{1+\frac{1}{n}} \qquad (4.28)$$

where K_m and *n* are constants and *n* is known as the **Polytropic Index**. The case $n=\infty$ represents a constant temperature fluid and this equilibrium condition is called the Isothermal Sphere. Other values of n give radial variations of temperature. After substituting for pressure from Equation (4.28), Equation (4.27) is studied by recasting it in dimensionless variables defined as:

$$\rho_m = \lambda\theta^n$$

and

$$r = \beta\xi \qquad (4.29)$$

we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] + \theta^{\mathrm{n}} = 0 \qquad (4.30)$$

and

$$\beta = \left[\left(\frac{n+1}{4\pi G}\right)K_m\lambda^{\left(\frac{1-n}{n}\right)}\right]^{1/2}$$

Equation (4.30) is known as the **Lane-Emden Equation**. Here λ is the central density so that $\theta = 1$ for $\xi = 0$ and ($d\varphi_g/dr$)=0 at r = 0 or $\theta = 1$. The constant $K_m = (1/3)U^2_{rms}$ for an isotropic velocity dispersion U^2_{rms}. Equation (4.30) can be solved for different values of *n*. For *n* = 0, we find:

$$\grave{e} = 1 - \frac{\xi^2}{6} \qquad (4.\ 31)$$

from which we learn that the boundary of zero density lies at $\xi = \sqrt{6}$, $\rho_m = \lambda$ and $p = K_m\rho_m$ = constant. This equilibrium consists of a sphere of constant density λ and radius equal to $\left(\frac{6p}{4\pi G}\right)^{1/2}$.

Traditionally, the study of the equilibrium of self-gravitating systems is not included in a section on MHD equilibrium, since these systems are believed to be mostly neutral hydrogen. However, we see that even a conducting fluid can have these mass configurations, provided the Lorentz force $\vec{J}\times\vec{B}=0$, which implies that the magnetic field must be given by:

$$(\vec{\nabla}\times\vec{B})\times\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\alpha(\vec{r})\vec{B} \tag{4.32}$$

and $\vec{B}.\vec{\nabla}\alpha(\vec{\mathrm{r}})=0$. Why ?

Such a magnetic field is known as a **Force-Free Magnetic Field** for the obvious reason that it exerts no force on a fluid. So, we reach the conclusion that a self-gravitating conducting fluid in a force-free magnetic field can attain all the configurations that a self-gravitating non-conducting fluid can!

Another type of hydrostatic equilibrium results when a fluid is in a gravitational field of another object, e.g., the earth's atmosphere in the earth's gravitational field. In such a case, one writes the gravitational potential φ_g as that due to a point mass M situated at the center of the object of radius R. The potential at any point at a distance r above the surface of the object is then

$$\varphi_g=-\frac{GM}{(R+r)} \tag{4.33}$$

and the hydrostatic balance condition gives

$$\frac{1}{\rho_m}\frac{d\,\mathrm{p}}{d\,\mathrm{r}}=-\frac{GM}{(R+r)^2} \tag{4.34}$$

4.8. Magnetohydrostatic Equilibrium

We now retain a non-zero Lorentz force, but investigate a static equilibrium, so that $\vec{U} = 0$, and Equation (4.24) then gives:

$$-\vec{\nabla} p + \frac{\vec{J} \times \vec{B}}{c} - \rho_m \vec{\nabla} \varphi_g = 0$$

or

$$\vec{\nabla}\left[p + \rho_m \varphi_g + \frac{B^2}{8\pi} \right] - \frac{1}{4\pi} (\vec{B}.\vec{\nabla})\vec{B} = 0 \qquad (4.36)$$

We see that the Lorentz force contributes to pressure balance in two ways: (1) through ($B^2/\ 8\pi$) which acts like pressure and is known as magnetic hydrostatic pressure and (2) through $(\vec{B}.\vec{\nabla})\vec{B}$ which acts like tension along the magnetic field lines. If, for a certain magnetic field configuration, the magnetic tension term $(\vec{B}.\vec{\nabla})\vec{B}$ vanishes (when B does not vary in its own direction) and if the gravitational effects can be ignored, we see that Equation (4.35) tells us that the sum of mechanical pressure p and the magnetic pressure $B^2/\ 8\pi$ must be a constant, i.e., space independent, or,

$$p + \frac{B^2}{8\pi} = \text{constant} \qquad (4.36)$$

Equation (4.36) shows us a way of confining a conducting fluid by a magnetic field, which acts like a container for the fluid. A low pressure region should have a high magnetic pressure and vice-versa. The predominance of mechanical pressure over the magnetic pressure can be expressed by a ratio, called the plasma β_p, defined as:

$$\beta_p = 8\pi \frac{p}{B^2} \qquad (4.37)$$

A fluid is said to be confined by a magnetic field if $\beta_p < 1$. A variety of loop like structures seen in the solar corona have $\beta_p < 1$, whereas in the solar photosphere $\beta_p > 1$.

In cylindrical geometry, for example, an azimuthal current density J_θ crossed with an axial magnetic field B_z can support a radial pressure gradient *p'* . Such a configuration is known as the **θ Pinch** (Figure 4.2); the

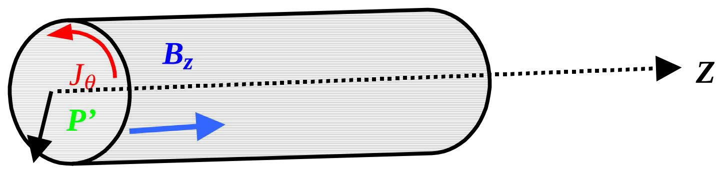

Figure 4.2. The θ Pinch

Applied magnetic field limits or pinches the radius of the fluid column. We can estimate the current required to confine a fluid from the following considerations:

$$\vec{\nabla} p \times \vec{B} = \frac{1}{c}(\vec{J} \times \vec{B}) \times \vec{B}$$

or

$$\vec{J}_\perp = c\frac{\vec{\nabla} p \times \vec{B}}{B^2} \tag{4.38}$$

where $\vec{J}_\perp$ is called the diamagnetic current, the same diamagnetic current that we discussed in Chapter 3, being associated with the diamagnetic drifts of charged particles. In a direction parallel to the magnetic field, we find

$$\vec{B}.\vec{\nabla} p = 0 \tag{4.39}$$

i.e., the pressure is constant along the magnetic field.

Another equilibrium called the **Z Pinch** (Figure 4.3) obtains when

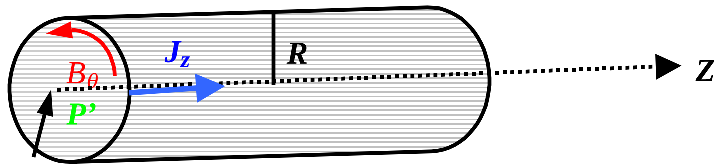

Figure 4.3. The Z Pinch

an axial current density J_z, produces an azimuthal magnetic field B_θ to support a radial pressure gradient p'. This can be seen from the equilibrium condition, Equation (4.35), with $\varphi_g = 0$:

$$\frac{\partial p}{\partial r} = -\frac{B_\theta^2}{4\pi r} - \frac{1}{8\pi}\frac{\partial B_\theta^2}{\partial r} \tag{4.40}$$

and

$$B_\theta(r) = \frac{2\pi}{c} J_z r \tag{4.41}$$

Using the boundary conditions p = 0 at r = R, the external boundary of the fluid and p = p_o at r = 0, we find

$$p(r) = \frac{p_o}{R^2}(R^2 - r^2) \tag{4.42}$$

where

$$p_o = \frac{\pi J_z^2 R^2}{c^2} = \frac{B_\theta^2(R)}{4\pi}$$

Thus, for a given central pressure p_o, the radius R of the fluid is determined from

$$R^2 = \frac{p_o c^2}{\pi J_z^2} \tag{4.43}$$

which shows the pinching effect produced by the current density J_z: the higher the current density J_z, the smaller the radius R.

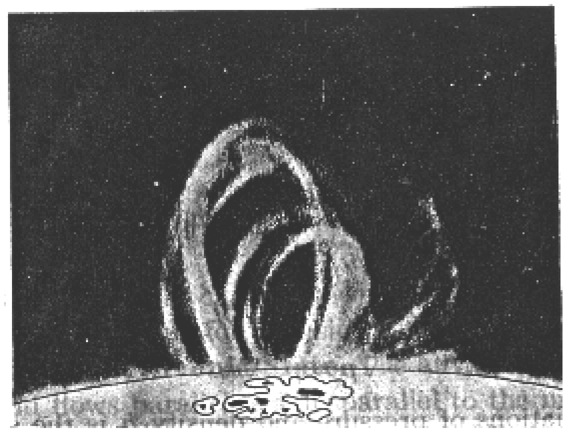

Figure 4.4. Solar Coronal Loops (Bray et. al. 1991)

The pressure balance condition Equation (4.46) also ensures that a fluid of pressure p_1 with magnetic field B_1 can be in equilibrium with a fluid of pressure p_2 without magnetic field, provided

$$p_1 + \frac{B_1^2}{8\pi} = p_2 \qquad (4.44)$$

which, for an isothermal equation of state states that the density ρ_2 of the magnetic field-free fluid must be larger than the density ρ_1 of the magnetized fluid. Such magnetized fluid elements being lighter than the surrounding heavier fluid, experience buoyant forces in a gravitational field and rise up. The appearance of discrete and strongly magnetized structures on the solar surface is attributed to this process. A high density fluid overlying a low density fluid in a gravitational field pointing downwards, can be in equilibrium, but it is an unstable equilibrium. It is the same mechanism due to which we can invert a glass full of water (I haven't tried with wine!) without the water flowing down, but how well we know that a slight carelessness or an air current can destroy the equilibrium.

4.9. Magnetohydrodynamic Equilibrium

We will now discuss MHD equilibrium including flow, i.e., $\frac{\partial \vec{U}}{\partial t}=0$ but $\vec{U}\neq 0$ .Neglecting viscous effects, the equilibrium condition becomes:

$$\rho_m\left[(\vec{U}.\vec{\nabla})\vec{U}\right]=-\vec{\nabla}p+\frac{\vec{J}\times\vec{B}}{c}-\rho_m\vec{\nabla}\varphi_g \qquad (4.45)$$

The inertial force can be balanced either singly or jointly by the pressure gradient, the Lorentz force and the gravitational force. Either in the absence of magnetic field or for a force free magnetic field, the equilibrium (4.45) can be expressed as

$$\vec{\nabla}\left[\frac{\vec{U}^2}{2}+h+\varphi_g\right]=\vec{U}\times(\vec{\nabla}\times\vec{U}) \qquad (4.46)$$

where we have written

$$\vec{\nabla}p=\rho_m\vec{\nabla}h \qquad (4.47)$$

For adiabatic variations of pressure and density *h* is the specific enthalpy since from thermodynamics

$$dh=Tds+\frac{1}{\rho_m}dp \qquad (4.48)$$

and the change in entropy, ds = 0 for adiabatic changes. The right hand side of Equation (4.46) vanishes either for **Irrotational Flows**, i.e., when $\vec{\nabla}\times\vec{U}=0$, or for **Aligned Helical Flows**, i.e., when $\vec{U}$ is parallel to $\vec{\nabla}\times\vec{U}=\vec{\omega}$,where $\vec{\omega}$ is known as the vorticity and $(\vec{U}\cdot\vec{\omega})$ is called the helicity. Thus, for irrotational or fully helical flows, we get the well known Bernoulli's relation:

$$\frac{U^2}{2}+h+\varphi_g=0 \qquad (4.49)$$

We will say more on this equilibrium in a later chapter on Non-conducting Fluids.

Aligned helical flows, similarly to force free magnetic fields, satisfy the following equation

$$\vec{\nabla}\times\vec{V}=\alpha_v\vec{V} \tag{4.50}$$

Such flows are called Beltrami Flows. An equilibrium in which the pressure gradients are balanced by a Beltrami flow has been shown to offer a good description of a class of solar coronal loops, since gravitational effects are negligible and magnetic fields are believed to be nearly force-free in solar corona (Krishan 1996).

We can find another equilibrium, if we rewrite Equation (4.55) as:

$$-\vec{\nabla}\left[p+\rho_m\varphi_g+\frac{B^2}{8\pi}\right]=-\frac{1}{4\pi}\left[(\vec{B}.\vec{\nabla})\vec{B}-4\pi\,\rho_m\,(\vec{U}.\vec{\nabla})\vec{U}\right] \tag{4.51}$$

This shows that

$$p+\rho_m\varphi_g+\frac{B^2}{8\pi}=constant$$

If

$$\vec{U}=\pm\frac{\vec{B}}{(4\pi\rho_m)^{1/2}}\equiv\pm\vec{V}_A \tag{4.52}$$

$\vec{V}_A$ is known as the Alfven velocity and the equilibrium, which is nothing but the magnetostatic equilibrium for a tension free magnetic field, corresponds to what is known as the Alfvenic State. The alfvenic state is one in which a conducting fluid flows parallel or anti-parallel to the magnetic field with the Alfven speed. We shall learn more about the Alfven velocity in a later section.

4.10. Magnetohydrodynamic Waves

In order to learn about a system, we must disturb or nudge it and watch how it responds.For example, when we displace a pendulum, a little, from its equilibrium vertical position and release it, the pendulum begins to oscillate. For small displacements the oscillations are harmonic. The pendulum will oscillate forever, if there are no retarding forces due to the surrounding environment. For large displacements, the oscillations are nonlinear, i.e., the amplitude of the oscillations is no longer a constant. The period of oscillations gives us a relation between the characteristics of the system; here, for example, the length of the pendulum, and the forces trying to restore equilibrium, here, for example, the gravitational force. In the same way, when a conducting fluid is disturbed form its equilibrium configuration we see it set into oscillations. The period of the oscillations is related to the characteristics of the conducting fluid such as mass density, pressure, temperature and the restoring forces, which may include pressure gradient, Lorentz and gravitational. These oscillations, also called **Waves** since they propagate in the fluid, have a great diagnostic potential. We can estimate the fluid properties through the detection of these waves. Further, in the presence of dissipative effects like viscous and resistive forces, the amplitude of these waves decreases with time. The energy carried by waves is deposited in the fluid as a result of which it may heat up. Magnetohydrodynamic waves have been considered very favourably for heating the solar corona, which at a temperature of ~ 10^6 K, lies outside the solar photosphere with temperature ~ 6000 K, and therefore needs sources of heat and mechanisms to maintain its temperature.

In the next section, we shall study different types of waves that a conducting fluid exhibits, when disturbed by a small amount from its equilibrium. These waves are called linear waves. We can introduce a small disturbance in the various parameters singly or jointly, depending upon our interest. We may wish to know the response of the conducting fluid to a perturbation in its, say, density. A change in density will produce a change in the gravitational force, a change in pressure, a change in the fluid velocity and a change in magnetic field such that the conservation laws of mass, momentum and energy as well as the Maxwell equations always remain satisfied. In order to see the restoring action of one particular force, we may ignore other forces. Of course, if we include all the forces the problem becomes quite complex, though not intractable. Anyway, it helps if we have some idea of the relative importance of the various forces. In the study of linear waves, we get a dispersion relation which contains everything on propagation characteristics: the phase and group velocities as well as the polarization characteristics. The only property we cannot determine is the amplitude of the wave, for which we must learn to do nonlinear studies. However, for the present, we limit ourselves to linear studies.

4.11. Dispersion Relation of Ideal MHD Waves

Let the equilibrium state of an ideal MHD fluid be described by the space and the time independent mass density ρ_o (for the rest of this chapter, the subscript *m* will be dropped), the fluid velocity $\vec{U}_o = 0$, the uniform and time independent magnetic field B_o, the uniform pressure p_o , the current density $\vec{J}_o = 0$ and the inductive electric field $\vec{E}_o = 0$. We now perturb this equilibrium such that

$$
\begin{aligned}
\rho &= \rho_o + \rho_1 \\
\vec{U} &= \vec{U}_1 \\
\vec{B} &= \vec{B}_o + \vec{B}_1 \\
p &= p_o + p_1 \\
\vec{E} &= \vec{E}_1 \\
\vec{J} &= \vec{J}_1
\end{aligned}
\tag{4.53}
$$

where all the quantities with subscript 1 are much smaller than the corresponding equilibrium values (except $\vec{E}_1$, $\vec{J}_1$ and $\vec{U}_1$). The linearized ideal MHD equations of mass and momentum conservation, neglecting shear and, dissipative effects, the linearized generalized Ohm's law, and the Maxwell equations are:

$$\frac{\partial \tilde{n}_1}{\partial t} + \vec{\nabla}.[\ \tilde{n}_o \vec{U}_1\ ] = 0$$

$$\tilde{n}_o \frac{\partial \vec{U}_1}{\partial t} = -\vec{\nabla} p_1 \ + \ \frac{\vec{J}_1 \times \vec{B}_o}{c}$$

$$\vec{E}_1 = -\frac{\vec{U}_1 \times \vec{B}_o}{c} \tag{4.54}$$

$$\vec{\nabla} \times \vec{B}_1 = \frac{4ð}{c} \vec{J}_1$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times (\vec{U}_1 \times \vec{B}_o)$$

We still have to use the energy conservation law to relate perturbations in density ρ1 to perturbations in pressure p1. We recall that p is the sum of pressures due to each species of fluid, i.e.

$$p = \sum_s p_s \tag{4.55}$$

For an adiabatic energy equation:

$$p \propto \rho^{\gamma} \tag{4.56}$$

we find, to the first order,

$$\vec{\nabla} p = \vec{\nabla} p_1 = \sum_s \gamma \frac{p_{os}}{\rho_{os}} \vec{\nabla} \rho_{1s} \tag{4.57}$$

we then assume that the fractional change in density for all species is the same, i.e.:

$$\frac{\rho_{1s}}{\rho_{os}} = \frac{\rho_1}{\rho_o} \tag{4.58}$$

so that

$$\vec{\nabla} p_1 = \sum_s \gamma\, p_{os} \frac{\vec{\nabla} \rho_1}{\rho_o} = \frac{\gamma\, p_o}{\rho_o} \vec{\nabla} \rho_1 = C_s^2 \vec{\nabla} \rho_1 \tag{4.59}$$

where C_s^2 is the adiabatic sound speed. We can use the general relation,

$$\vec{\nabla} p = C_s^2\, \vec{\nabla} \rho \tag{4.60}$$

for adiabatic or isothermal cases and identity C_s with the corresponding sound speed.

What remains to be done is to eliminate all the first order quantities except one among Equations (4.54) and (4.59). This is easy as we have six first order quantities ($\rho_1, \vec{U}_1, p_1, \vec{E}_1, \vec{B}_1, \vec{J}_1$) and six linearized equations. The elimination procedure becomes simple when we assume a plane wave type variation for all the first order quantities. We write:

$$\vec{U}_1(\vec{r}, t) = \vec{U}_1{}'\exp\left[i\vec{k}\cdot\vec{r} - i\omega t\right] \tag{4.61}$$

and similarly for the other five quantities. $\vec{U}_1'$ is the space and time independent amplitude of the oscillating velocity $\vec{U}_1(\vec{r}, t)$. On the completion of the elimination exercise, we find an equation of the form:

$$\text{D(First order quantity)} = 0 \tag{4.62}$$

Since the first order quantity $\neq 0$, we obtain the dispersion relation D = 0, which is a relation between the wave frequency ω and the wave vector $\vec{k}$. For the present case, we find:

$$\left[-\omega^2 + (\vec{k}\cdot\vec{V}_A)^2\right]\vec{U}_1{}' + \left[(C_s^2 + V_A^2)(\vec{k}\cdot\vec{U}_1{}') - (\vec{k}\cdot\vec{V}_A)(\vec{V}_A\cdot\vec{U}_1{}')\right]\vec{k} - \\ - (\vec{k}\cdot\vec{V}_A)(\vec{k}\cdot\vec{U}_1{}')\vec{V}_A = 0 \tag{4.63}$$

Here, $\vec{V}_A = \dfrac{\vec{B}_o}{(4\pi\rho_o)^{1/2}}$ is the Alfven velocity. We can find different wave motions corresponding to the roots of the dispersion relation.

Let θ be the angle between the zeroth order magnetic field $\vec{B}_o$ (which we have taken to be in the z direction), and the wave vector $\vec{k}$, so that $k_x = k\ sin\theta$, $k_z = k\ cos\theta$ and $k_y = 0$.We find that with this choice the motion in the y direction is decoupled from the motion in the (x , z) plane. And we get a root

$$\omega^2 = k^2 V_A^2 \cos^2\theta$$

or (4.64)

$$\omega = \pm k_z V_A = \pm \vec{k}.\vec{V}_A$$

for $U'_{1y} \neq 0$. Equation (4.64) is the dispersion relation of the Alfven Wave propagating at an angle θ to the zeroth order magnetic field $\vec{B}_o$. For this wave

$$\vec{\nabla}.\vec{U}'_1 = 0,\ \rho_1 = 0 \qquad (4.65)$$

i.e., this wave does not produce any density and therefore pressure changes. Such a wave is called Transverse and Non-compressional. The phase velocity V_{ph} of the Alfven wave is

$$V_{ph} = \frac{\omega}{k} = \pm V_A \cos\theta \qquad (4.66)$$

and the group velocity $V_g = d\,\omega/d\,k = \pm V_A \cos\theta$. There is no Alfven wave for $\theta = \pi/2$. The Alfven wave has the maximum phase and group velocity parallel and antiparallel to the field $\vec{B}_o$. The polarization of the Alfven wave, i.e., the relative orientations of the electric field $\vec{E}_1$, the magnetic field $\vec{B}_1$, can be determined from Equations (4.54). We find $\vec{B}_1$ is in the y direction and $\vec{E}_1$ lies in the (x, z) plane, the plane containing the wave vector $\vec{k}$ and the magnetic field $\vec{B}_o$, as shown in Figure (4.5). It is clear that in the linear study of waves, we cannot estimate the absolute value of the amplitudes, $\vec{U}_1, \vec{E}_1, \vec{B}_1$ etc. But we can estimate their relative values. Thus we find that the electric energy density $E_1^2/8\pi$, the magnetic energy density $B_1^2/8\pi$ and the kinetic energy density $\rho_o U_1^2/2$ are in the ratio 1: $\frac{c^2}{V_A^2} : \frac{c^2}{V_A^2}$.

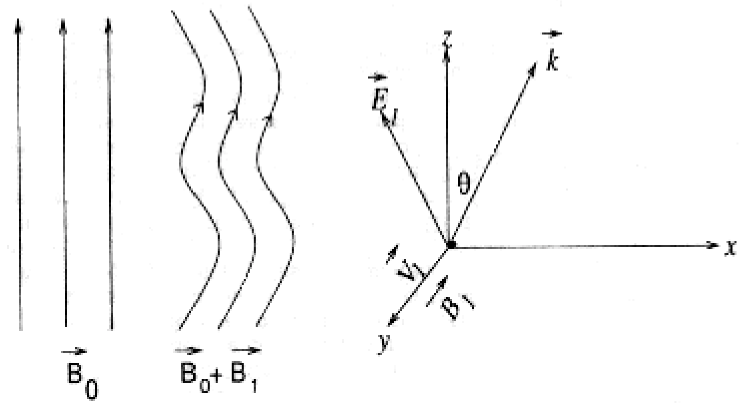

Figure 4.5. The Alfven Wave

The electric energy density is much smaller than the kinetic and magnetic energy densities. Further the kinetic and magnetic energy densities are equal, this is again a reminder of the field frozen to the fluid in the absence of dissipative effects.

The physical mechanism underlying the excitation of Alfven waves is identical to that of the transverse oscillations of a plucked stretched string. A wavy disturbance $\vec{B}_1$ curves the magnetic field lines, but the tension in the curved field tries to straighten the field lines, and the Alfven oscillations set in.

Thus, we find that the velocity $\overline{U_1}'$ of the conducting fluid is given by

$$\vec{U}'_1 = \frac{c\ \vec{E}_1 \times \vec{B}_o}{B_o^2} \tag{4.67}$$

This, combined with Faraday's law of induction in a moving medium, again leads to the conclusion that the fluid and the field remain together until dissipation parts them.

We find two more waves with dispersion relations:

$$\omega_F^2 = \frac{k_F^2}{2}(V_A^2 + C_S^2) + \frac{k_F^2}{2}\left[(V_A^2 + C_S^2)^2 - 4C_S^2 V_A^2 \cos^2\theta\right]^{1/2} \tag{4.68}$$

and

$$\omega_S^2 = \frac{k_S^2}{2}(V_A^2 + C_S^2) - \frac{k_S^2}{2}\left[(V_A^2 + C_S^2)^2 - 4C_S^2 V_A^2 \cos^2\theta\right]^{1/2} \tag{4.69}$$

Here, the wave with frequency ω_F and wave vector k_F is known as the **Fast Magnetosonic Wave** and the wave (ω_S , k_S) is known as the **Slow Magnetosonic Wave**. The fast and slow refer to the phase velocities of these waves. We notice that fast wave has a phase velocity which is larger than both the Alfven speed V_A and sound speed C_S . We can determine other properties of these waves now. First, since

$$\vec{k}_F . \vec{U}_1' \neq 0$$
$$\vec{k}_S . \vec{U}_1' \neq 0 \tag{4.70}$$

Both these waves affect density variations, i.e., $\rho_1 \neq 0$. The restoring force for both the waves is provided jointly by the gradient of kinetic and magnetic pressures; that is why they are called Magnetosonic waves. These waves are neither transverse nor longitudinal. They have mixed polarizations. For $\theta = \pi/2$, $\omega_S = 0$, i.e., the slow wave does not exist, whereas the fast wave has the maximum frequency and phase speed and becomes purely longitudinal, i.e., $\vec{k}_F \,//\, \vec{U}_1$. In this case the directions of the various fields are $\vec{B}_o = B_z$, $\vec{B}_1 = B_{1z}$, $\vec{E}_1 = E_{1y}$, $\vec{U}_1 = U_{1x}$ and $\vec{k} = k_x$. The fast wave produces density as well as magnetic field condensations and rarefactions as shown in Figure (4.6). For $\theta = 0$, we find that there are two types of waves: (i) a transverse wave with $\omega = \pm k.V_A$, $\vec{k} \cdot \vec{U}_1 = 0$; this is the Alfven wave, we have already studied and

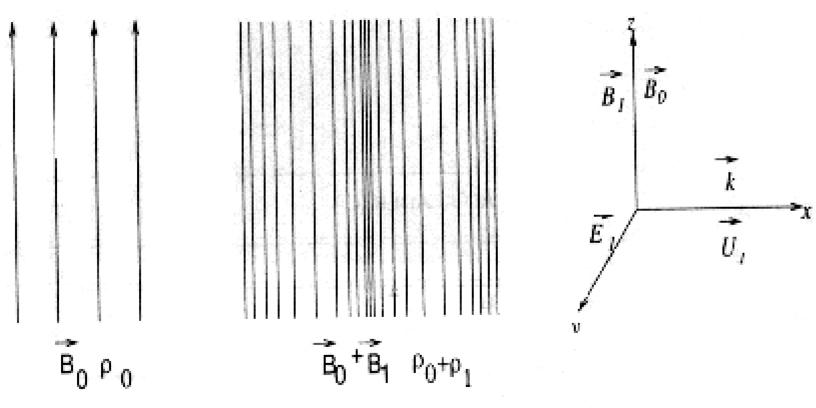

Figure 4. 6. Magnetosonic Waves Produce Condensations and Rarefactions.

(ii) a longitudinal wave with $\omega = \pm k.C_S$, $\vec{k} \,//\, \vec{U}_1$; this is the ordinary sound wave. Thus, we see that only for oblique propagation, i.e. at an angle to the ambient magnetic field $\vec{B}_o$, do all three waves, the Alfven and fast and slow magnetosonic waves exist.

The fluid and the magnetic field, in reality, do not keep oscillating forever, for there are resistive forces: the fluid is viscous and the magnetic field decays due to the electrical resistivity of the fluid. The magnetohydrodynamic waves suffer damping due to finite viscosity and electrical resistivity. We can study the MHD waves in non-ideal fluids by including the viscous force in the momentum equation, and the resistivity term $\eta\vec{J}$ in Ohm's law. We write the linearized momentum equation sans gravitational force as:

$$\rho_o \frac{\partial \vec{U}_1}{\partial t} = -C_S^2 \vec{\nabla}\rho_1 + \frac{\vec{J}_1 \times \vec{B}_o}{c} + \mu \nabla^2 \vec{U}_1 + \left(\zeta + \frac{1}{3}\mu\right)\vec{\nabla}(\vec{\nabla}.\vec{U}_1) \qquad (4.71)$$

Using Ohm's law with conductivity σ, the linearized Faraday's law becomes

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla}\times(\vec{U}_1 \times \vec{B}_o) + \nu_m \nabla^2 \vec{B}_1 \qquad (4.72)$$

We can carry out the elimination procedure as before and determine the dispersion relations of the three MHD waves. For Alfven waves, $\nabla.\vec{U}_1 = 0$ and the dispersion relation including dissipative effects becomes for θ = 0;

$$\omega^2 + i\omega\left(\nu_m k^2 + \frac{\mu k^2}{\rho_o}\right) - \left(\frac{\nu_m \mu k^4}{\rho_o} + k^2 V_A^2\right) = 0 \qquad (4.73)$$

This equation has complex roots. Treating the dissipative effects as small, the roots of equation (4.73) are:

$$\omega \cong \pm kV_A - \frac{i}{2}k^2\left(\nu_m + \frac{\mu}{\rho_o}\right) \qquad (4.74)$$

Recalling that all the first order quantities have a time dependence $e^{-i\omega t}$, we see that dissipative effects produce an exponential damping of the wave amplitudes ($\vec{U}_1$, $\vec{B}_1$, $\vec{E}_1$). The damping rate ω_{IA} equal to the imaginary part of the complex frequency ω, is :

$$\omega_{IA} = -\frac{k^2}{2}\left(\nu_m + \frac{\mu}{\rho_o}\right) \cong -\frac{\omega^2}{2V_A^2}\left(\nu_m + \frac{\mu}{\rho_o}\right) \qquad (4.75)$$

We see that high frequency or short wavelength waves suffer more damping than do the low frequency waves, for constant values of the magnetic field $\vec{B}_o$ and the mass density ρ_o. The wave intensity decays to e^{-1} of its initial value in a time $(2\,\omega_{IA})^{-1}$, which is the same as the diffusion time t_d of the magnetic field $\vec{B}_o$, in the absence of the fluid viscosity. The distance L_d traveled by the wave in the time $(2\,\omega_{IA})^{-1}$ is:

$$L_d = V_A(2\omega_{IA})^{-1} = \frac{V_A^3}{\omega^2\left(\nu_m + \frac{\mu}{\rho_o}\right)} \qquad (4.76)$$

So, the high frequency waves have short damping lengths. Similarly, we can determine the damping rates for other waves too.

The damping rate of the fast MHD mode propagating perpendicular to the magnetic field $\vec{B}_o$ ($\theta = \pi/2$) is found to be:

$$\omega_{IF} \cong -\frac{k^2}{2}\left[\nu_m \frac{V_A^2}{(V_A^2 + C_S^2)} + \frac{\mu}{\rho_o} + \frac{(\xi + \mu/3)}{\rho_o}\right] \qquad (4.77)$$

with the real part

$$\omega_{RF}^2 \cong k^2\,(V_A^2 + C_S^2)$$

in which corrections due to the dissipative effects have been ignored. Again, the damping rate increases with frequency. The fast wave, being compressional, has an additional contribution to its damping rate from compressibility of the fluid.

It is quite easy to excite MHD waves. All it takes is to shake the magnetized plasma like one shakes a string. One of the most favourable astrophysical sites for excitation of MHD waves is the solar atmosphere. The outer layers of the solar atmosphere- the chromosphere and the corona- are at a much higher temperature than is the photosphere to which we owe our existence. Further, the chromosphere and the corona are highly inhomogeneous media supporting a variety of filamentary structures in the form of arches and loops Figure (4.4). A coronal loop is a bipolar structure whose foot points are anchored in the poles of the subphotospheric magnetic field. The foot points undergo a continuous turning and twisting due to convective motions in the subphotospheric layers of the sun. This turning and twisting is enough to excite MHD waves in coronal loops. These waves then dissipate and spend their energy in heating the corona. Typically, waves of periods of a few seconds are believed to be excited in the corona. These waves can be detected through the periodic variations in the intensities of the continuum and line radiation as well as through the Doppler shifts of the line radiation. The Alfven waves, which are purely velocity and magnetic field oscillations without any accompanying mass density oscillations, do not produce any changes in the intensity and are observed through the Doppler effect. The magnetosonic waves, which are compressional, show up both as intensity and velocity oscillations. Although the MHD waves have received a lot of attention from theoreticians for a long time, their unambiguous detection in the solar atmosphere is still awaited.

At high frequency, the displacement current begins to contribute and we must use the Ampere law as modified by Maxwell.

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{4.78}$$

with

$$\vec{E}=-\frac{\vec{U}\times\vec{B}}{c} \tag{4.79}$$

The inclusion of the displacement current modifies the dispersion relation of the Alfven wave to

$$\grave{u} = \frac{\vec{k} \cdot \vec{V}_A}{\left(1 + \frac{V_A^2}{c^2} \right)^{1/2}} \tag{4.80}$$

In low density, astrophysical plasmas, the Alfven speed V_A can approach the speed of light c. We can also define the refractive index n_R for Alfven waves as:

$$n_R = \frac{kc}{\grave{u}} = \left(1 + \frac{c^2}{V_A^2} \right)^{1/2} \tag{4.81}$$

A conducting fluid has an index of refraction n_R for electromagnetic waves of frequency smaller than the electron-ion collision frequency when charge separation effects are negligible.

4.12. Magnetohydrodynamic Instabilities

We have seen one type of response of a system to a small stimulus. The system begins to oscillate about its equilibrium and never ventures too far from it. It could happen, however, that the restoring forces are not strong enough to bring the system back towards its equilibrium. On the other hand, there are forces which drive the system farther and farther from its equilibrium. This is a circumstance of an instability. The energy for driving a system unstable could be in any of the spatial and/or temporal gradients of the fluid characteristics such as magnetic field, current density, pressure, mass density, gravity or rotation of a magnetofluid. Thus the inhomogeneous distribution of fluid parameters can lead to macroscopic or **MHD Instabilities**. The system, in response, reconfigures itself to new equilibrium by shedding the excess energy stored in inhomogeneities. There are two methods for exploring whether a system is stable or unstable.

(i) **The Normal Mode Method**, where we perturb the system by a small extent, linearize the mass, momentum and energy equations, find a plane wave solution for them and explore conditions for complex roots of the frequency; since the complex roots occur in pairs, one of the roots corresponds to the exponential growth with ime of the amplitude of the perturbation and we have a case of an instability. It is of the utmost importance to identify the source of energy responsible for the excitation of the instability.

(ii) **The Energy Principle**, where we perturb the system by a small extent, linearize the relevant equations, calculate the potential energy of the perturbed system; if the potential energy of the perturbed system is larger than that of the unperturbed system, it is said to be stable against this perturbation and vice-versa, since a system always likes to acquire a state o minimum potential energy. The change in the potential energy can then be related to the frequency of oscillations of the perturbed quantities. The normal mode method is used when we wish to include more complex physical processes like the finite Larmor radius effect or the Hall current, whereas the energy principle is used when the complexity lies in the geometrical configuration of fluids.

4.13. The Rayleigh-Taylor Instabilities

The simplest example of the class of Rayleigh-Taylor (R-T) instabilities is the inverted glass full of water where the heavy fluid, water, is supported by the light fluid, air, at least for a few uncertain moments. This is a case of an unstable equilibrium since it is easily lost by a small air current. Thus, a fluid with an inverted density gradient, i.e., where the mass density increases in the direction of decreasing gravity ($\vec{g}$) is Rayleigh-Taylor unstable.

The frequency ω becomes purely imaginary if the density scale height $H_\rho = \frac{1}{\rho_o}\frac{d\rho_o}{dr}$ is smaller than the pressure scale height $\gamma\, H_P = \frac{1}{p_o}\frac{dp_o}{dr}$. The perturbations therefore grow with time as exp[$\omega_I\, t$] . The imaginary part ω_I of the frequency ω is called the **Growth Rate** of the instability. This is a case of Rayleigh-Taylor Instability for compressible perturbations for which $\nabla.\vec{U}_1 \neq 0$. Expressing in terms of the entropy $s \equiv \frac{p_o}{\rho_o^\gamma}$, the growth rate ω_I becomes

$$\omega_I = \left[\frac{g}{\gamma}\frac{d}{dr}\ln\left(\frac{1}{s}\right)\right]^{1/2} \tag{4.82}$$

and the Instability is driven for negative entropy gradient (ds/ dr) < 0 even when ($d\rho_o/dr$) < 0.

For $\gamma H_p >> H_\rho$, i.e., for the incompressible case for which $\vec{\nabla}.\vec{U}_1 = 0$, we get

$$\omega_I^2 = \frac{g}{H_\rho} \tag{4.83}$$

We again have a R-T Instability if the density scale height H_ρ is positive, i.e., when the density increases in a direction opposite to that of the acceleration due to gravity. The growth rate is, now, $\left(g/H_\rho\right)^{1/2}$.

For an isothermal equation of state, $\gamma = 1$ and $p_o = \rho_o k_B T_o / M$. The scale height H_T for temperature is related to H_ρ and H_p as

$$\frac{1}{H_\rho} = \frac{1}{H_p} - \frac{1}{H_T} \tag{4.84}$$

and we find

$$\omega_I^2 = -\frac{g}{H_T} \tag{4.85}$$

We, again, have R-T instability for $H_T < 0$,i.e., if the temperature decreases in a direction opposite to $\vec{g}$. The growth rate is, now, $\left(g/H_T\right)^{1/2}$.

In conclusion, in the absence of the Lorentz force, either due to the absence of the magnetic field or due to the magnetic field being force free, the Rayleigh-Taylor instability is excited (i) when a heavy fluid lies at the top of a light fluid, (ii) when a cold fluid lies at the top of a hot fluid and (iii) when the upper fluid has lower entropy than the lower fluid. The result of R-T instability is the mixing of either different fluids or different parts of a fluid. The internal gravity waves in their stable and unstable form (R-T instability) are believed to play an important role in the mixing of elements and distribution of angular momentum in the radiative zones of stars. The passage of a shock wave during a supernova explosion also creates circumstances of R-T instability with a typical exponentiation time of ~10^4 sec.

4.14. Rayleigh-Taylor Instability in Magnetized Fluid

Astrophysical fluids are, more often than not, magnetized. We study the effect of magnetic field on the growth rate of the R-T instability for two cases: (i) when the magnetic field $\vec{B}$ is parallel to the acceleration due to gravity $\vec{g}$ and (ii) when $\vec{B}$ is perpendicular to $\vec{g}$. We shall neglect all non-ideal effects . We shall use what is known as the Boussinesq Approximation to deal with density variations. Under this approximation, we neglect all changes in density except where they are coupled with external forces like gravity. It is valid when small changes in temperature lead to small changes in density due to smallness of the coefficient of volume expansion. So, we have $\vec{\nabla}.\vec{U}_1 = 0$ along with $\rho_1\vec{g} \neq 0$.

(i) $\vec{g} = -g\hat{z}$; $\vec{B} = B\hat{z}$ and $\rho_o = \rho_o(z)$

Assuming a spatial and time dependence of the perturbed quantities of the form:

$$\vec{U}_1 = \vec{U}_1(z)e^{i k_x x + i k_y y + i \omega t}$$

We can find the linearized MHD equations including the gravitational forces.

We, now, need boundary conditions to solve these equations. Let the fluid be confined between two boundaries at z = z_1, and z = z_2 (Figure 4.7).

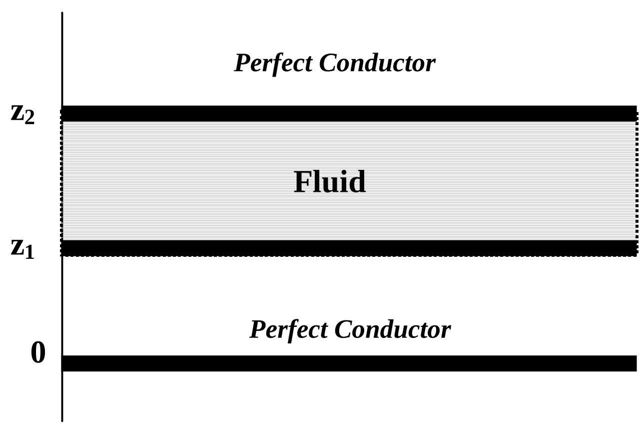

Figure 4.7. Fluid Confined in the Region $z_1 \leq z \leq z_2$

If the boundaries are rigid, there can be no motion across them and $U_{1z}(z_1) = U_{1z}(z_2) = 0$. If the medium (at $z < z_1$ and $z > z_2$) adjacent to the fluid is a perfect conductor, then no magnetic field can cross the boundary (the fluid itself has been assumed to be a perfect conductor) and $B_{1z} = 0$ and $E_{1x} = E_{1y} = 0$ on the plane boundary. If the medium adjacent to the fluid is non-conducting , then no current can cross the boundary and $J_z = 0$; the magnetic field at $z < z_1$ and $z > z_2$ must correspond to a vaccum field. The continuity of the tangential stresses requires that B_{1x} and B_{1y} are continuous which implies the continuity of U'_{1z} and U''_{1z} .

The dispersion relation is found to be:

$$\begin{aligned}&-i\omega^3-\omega^2\left[2\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)kV_{AT}\right]+i\omega\left[2k^2V_{AT}^2-gk(\alpha_2-\alpha_1)\right]+\\&+2gk^2V_{AT}\left[\sqrt{\alpha_1}-\sqrt{\alpha_2}\right]=0\end{aligned}\quad, \quad (4.86)$$

We can look at the asymptotic solutions of the dispersion relation for $k \to 0$. Then we find

$$(i\omega)^2 \to gk(\alpha_2-\alpha_1) \equiv \omega_I^2 , \quad (4.87)$$

which corresponds to the hydrodynamic R-T instability for $\alpha_2 > \alpha_1$. This shows that the large wavelength perturbations are unaffected by magnetic field.

For $k \to \infty$

$$i\omega \to \frac{g}{V_{AT}}\left(\sqrt{\alpha_2}-\sqrt{\alpha_1}\right) \equiv \omega_I \quad (4.88)$$

i.e., the growth rate tends to a fixed value independent of *k*.

Thus the growth rate ω_I increases the equation linearly with *k* for small values of *k* and becomes independent of *k* for large values of *k*.

The case $\alpha_2 < \alpha_1$ represents a stable configuration. The system responds to small perturbations by exciting Alfen oscillations. The growth rate ω_I for this case is found to be:

$$(i\omega)^2 \equiv \omega_I^2 = \frac{1}{(\rho_1+\rho_2)}\left[kg(\rho_2-\rho_1)-\frac{2k_x^2B_0^2}{4\pi}\right] \quad (4.89)$$

We, first, notice that the growth rate ω_I is reduced by the presence of the horizontal magnetic field B_0. As the heavier fluid of density ρ_2 tries to sink into fluid of lower density ρ_1, it carries the magnetic field also with it, bending it in the process. The tension in the magnetic field however tries to straighten the field lines and inhibits the sinking tendency of the fluid (Figure 4.8). Thus, the horizontal magnetic field can provide a support to a fluid with inverted density gradient in a gravitational field, against perturbations propagating in the horizontal direction ($k_x \neq 0$). This is one of the mechanisms proposed for the existence of high density and low temperature structures, called **Prominences**, in the solar corona. The density of the prominence is about 100-1000 times and the temperature is about 0.01 that of the solar corona and they are embedded in a magnetic field of the order of 10 Gauss.

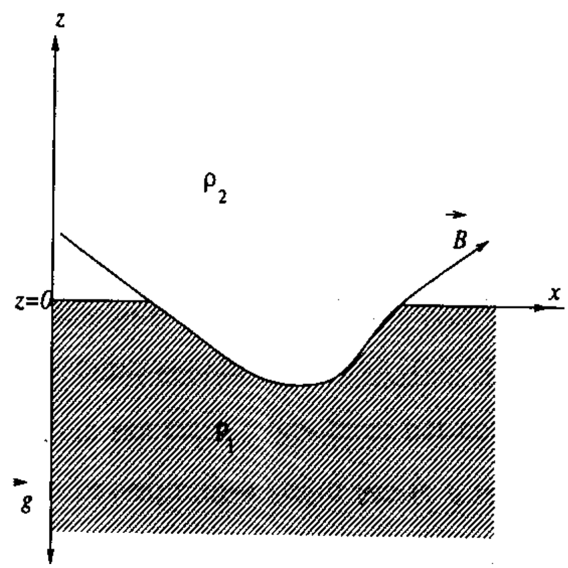

Figure 4.8. The Curved Field $\vec{B}$ Develops Tension Which Inhibits the Instability.

4.15. The Kelvin-Helmholtz Instability

So far, we have considered fluids of different densities lying over each other in the presence of gravitational and magnetic forces. We can give several examples where fluids of varying densities coexist in relative motion. Wind flowing over oceans, cometary tails whizzing against the solar wind, accreting flows around compact objects, propagating extragalactic jets and exploding supernovae ejecta are a few familiar sites. Such configurations of streaming fluids may have discontinuities in their flow speeds. This excess kinetic energy could drive the system unstable. The resulting instability is known as the **Kelvin-Helmholtz (K-H) Instability**. We shall study the development of K-H instability including the magnetic field. The direction of the magnetic field is specified in relation to the direction of streaming. We shall consider two cases (i) when the magnetic field is parallel to the flow velocity $\vec{U}_0$ and (ii) when it is perpendicular to the flow velocity.

(i) $\vec{B}_0 = B_0\hat{x}$; $\vec{U}_0 = U_0(z)\hat{x}$; $\rho_0 = \rho_0(z)$; $\vec{g} = -g\hat{z}$ (Figure 4.9)

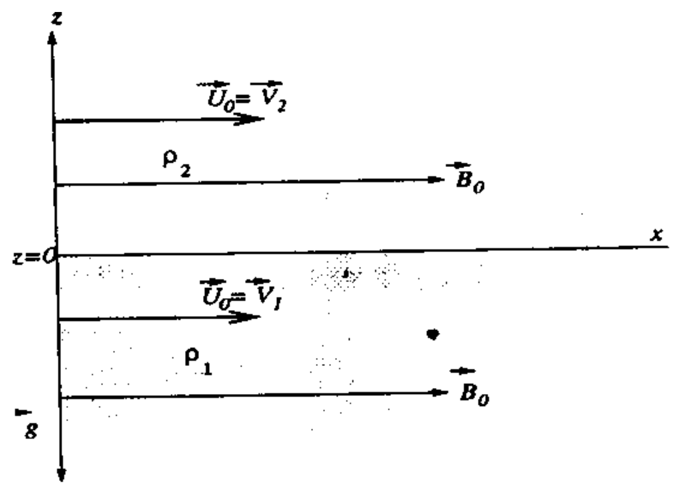

Figure 4.9. The Kelvin-Helmholtz Instability for $\vec{U} \parallel \vec{B}_0$

The dispersion relation is found to be

$$\omega = -k_x[\alpha_1 V_1 + \alpha_2 V_2] \pm \left[gk(\alpha_1 - \alpha_2) + 2k_x^2 V_A^2 - k_x^2 \alpha_1 \alpha_2 (V_1 - V_2)^2\right]^{1/2} \quad (4.90)$$

We notice that:

For $V_1 = V_2 = V_A = 0$, we recover the growth rate of the hydrodynamic R-T instability for $\alpha_2 > \alpha_1$;

For $V_1 = V_2 = 0$, we recover the growth rate of the hydrodynamic R-T instability for $\vec{B}_0 \perp \vec{g}$. For excitation of the K-H instability, in the absence of the magnetic field, i.e., for $V_A = 0$, we must have

$$k_x^2 \alpha_1 \alpha_2 (V_1 - V_2)^2 > gk(\alpha_1 - \alpha_2), \quad (4.91)$$

for $\alpha_2 < \alpha_1$. For $\alpha_2 > \alpha_1$, the expression inside the square root is negative and we have an imaginary part of the frequency ω. There is also a real part of ω. Thus this is an oscillatory instability – the amplitude of the perturbed quantity oscillates as well as grows in time. The real part ω_R is

$$\omega_R = -k_x(\alpha_1 V_1 + \alpha_2 V_2), \tag{4.92}$$

4.16. Current Driven Instabilities

Util now, we have considered the effect of a uniform magnetic field. There is another class of instabilities driven by an electric current flowing through a conducting fluid. The presence of current is associated with inhomogeneous magnetic field. The stability or otherwise of the fluid depends upon the magnitude and apatial variation of the current density.

A pinch with sharp boundaries develops ‘waists’ or ‘necks’ at the surface as shown in figure (4.10). This is known as **The Sausage Instability**. For a uniform surface current density J_{0z} for a total current I, the azimuthal magnetic field $B_{0\theta} = 2I/cr$ for $r \geq R$. An axisymmetric perturbation that causes a reduction of the radius enhances the magnetic field $B_{0\theta}$, which further enhances the magnetic pressure and kinetic pressure p_0. As a result, the fluid is forced to move out of this region into a region of lower magnetic pressure. The whole column of fluid acquires bulges and waists. The presence of an axial magnetic field B_{0z} has a stabilizing effect. Thus, depending upon the magnitude of B_{0z}, the Sausage instability can either saturate to finite values of the perturbed quantities like $\vec{B}_1$, p_1 or totally quench to vanishing values of $\vec{B}_1$ and p_1.

The action of the sausage instability has been seen in cometary tails, extragalactic jets and at other astrophysical sites showing filamentary structures. For non-axisymetric perturbations the mode m = 1 is unstable for all $\beta > 2/5$. This unstable m = 1 mode is refered to as the **Kink Instability** (Figure 4.11a). The m = 1 perturbation bends the magnetic field lines to produce convex and concave curvatures. The lines of force experience a compression at the concave side (region A) and an expansion at the convex side (region B). The magnetic pressure is, therefore, stronger at A than at B. The resulting force from A to B pushes this region up to further enhance the bending and the perturbation grows. Again an axial magnetic field B_{0z} will try to straighten the bend, and the kink instability can either be entirely quenched or it can attain a final saturation level. The perturbation with high *m* values attribute a multistranded form to the cylindrical fluid (Figure 4.11b).

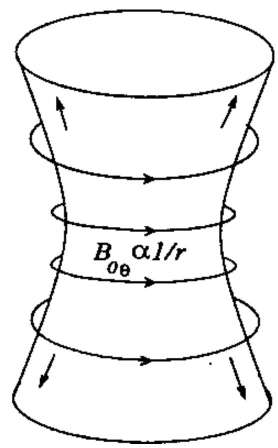

Figure 4.10. The Sausage Instability, the Arrow Represent the Flow Into the Low Magnetic Pressure Regions.

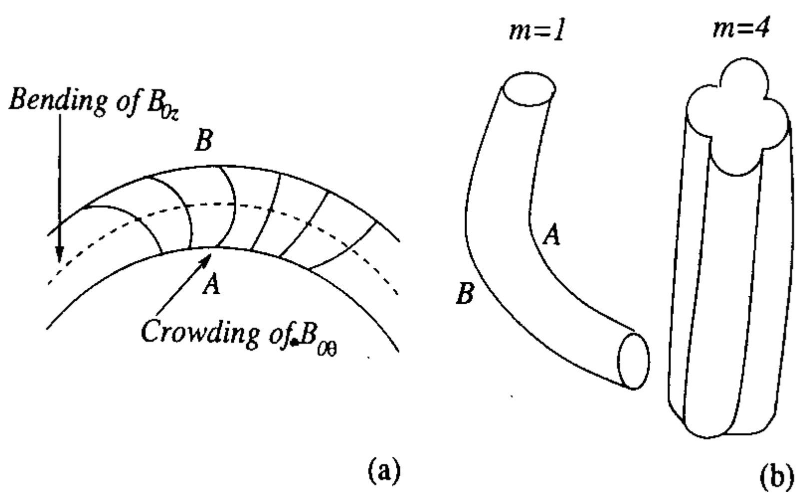

Figure 4.11. (a) The Kink Instability (m = 1 mode); (b) Multistranded Fluid Column for m = 4.

4.17. Resistive Instabilities

The presence of electrical resistivity allows a certain degree of freedom to the magnetic field to depart from the flow of the conducting fluid. The field and the fluid no longer remain frozen. The advantage is that the energy contained in complex fluid flow and fluid configurations can now be dissipated in the system, as a result of which the system becomes hot and may begin to radiate electromagnetic radiation. This is believed to be what happens, for example, during a solar flare. Enormous amounts of energy are released, in the form of mass motions and electromagnetic radiation over a wide spectral range, in an explosive manner. The magnetic field and the conducting fluid on the solar surface are continuously subjected to stresses caused by a variety of convective and wave-like motions. Beyond its endurance limit, the fluid-field configuration becomes unstable, and then relaxes to a lower state of energy, throwing out the excess energy in various forms. The genaric name for this class of instability is the **Tearing Mode Instabilities**. They can occur due to any non-ideal MHD circumstances such as electron inertia, charge separation or displacement current in addition to the resistivity.

Chapter 5

TWO-FLUID DESCRIPTION OF PLASMA

5.1. Electron and Proton Plasmas

We have learnt in Chapter 1 that an electrically quasi-neutral system of negative and positive charges qualifies to be a plasma and that a plasma exhibits cooperative phenomena on certain spatial and temporal scales. Some of the consequences of the quasi-neutral nature of a plasma can be studied by treating each of its constituent components as a fluid. Thus, at this level of description, an electron-proton plasma consists of two fluids – the electron fluid and the proton fluid. Each fluid is allowed to have charge density fluctuations about the overall mean density. This is the most significant deviation from the MHD description. The charge density fluctuations produce current density fluctuations. The associated electric and magnetic fields can be determined from Maxwell's equations. The space and time dependences of these fields can manifest themselves in the form of longitudinal and transverse waves.

In the presence of free sources of energy, such as a relative streaming motion between the electron and the proton fluids or a temperature inequality and/or an anisotropy, the electric and magnetic fields may begin to grow exponentially with time or distance. Such circumstances produce instabilities. The energy contained in the growing fields could either leave the system as radiation, or be damped within the system. The periods of the waves and the growth rates of the instabilities carry information on the plasma parameters such as density, temperature and electric and magnetic fields. So, the observations of waves and instabilities help us to diagnose plasmas. We may wonder, if the quasineutrality, which exists over short space and time scales, has any role to play in the huge expanse of typically long-lived astrophysical plasmas. The very fact that most of the high energy astrophysical sources emit more radiation than their temperatures would permit is a pointer to the cooperative plasma phenomena. Further, the extremely short temporal variability of radiation (with or without polarization changes) can sometimes only be accounted for by plasma processes occurring over short time scales. So, if we wish to look for plasma phenomena in astrophysical sources, we must study the spectral, temporal and polarization characteristics of their radiation.

In this chapter, we will first study the static and dynamic equilibria of the electron and the proton fluids in the presence of electric and magnetic fields and then their stability under small departures from these equilibria. The mathematical tools needed for this investigation have already been developed in Chapter 2.

5.2. Static Equilibria of Electron and Proton Fluids

We begin with the two fluid equations derived in Chapter 2. The mass and momentum conservation laws of the electron fluid are:

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot \left(\rho_e \vec{U}_e\right) = 0 \tag{5.1}$$

and

$$\rho_e \left[\frac{\partial \vec{U}_e}{\partial t} + \left(\vec{U}_e \cdot \vec{\nabla}\right) \vec{U}_e\right] = -\frac{e\rho_e}{m_e}\left[\vec{E} + \frac{\vec{U}_e \times \vec{B}}{c}\right] - \rho_e \vec{\nabla}\varphi_g - \vec{\nabla} \cdot \vec{\Pi}_e + \vec{\Gamma}^{ei} \tag{5.2}$$

The static equilibrium of the electron fluid ($\vec{U}_e = 0$) is described by:

$$\frac{\partial \rho_e}{\partial t} = 0, \tag{5.3}$$

and

$$-\frac{e\rho_e}{m_e}\vec{E} - \rho_e \vec{\nabla}\varphi_g - \vec{\nabla} \cdot \vec{\Pi}_e + \vec{\Gamma}^{ei} = 0 \tag{5.4}$$

Similarly, the static equilibrium of the proton fluid ($\vec{U}_i = 0$) is described by

$$\frac{\partial \rho_i}{\partial t} = 0, \tag{5.5}$$

and

$$\frac{e\rho_i}{m_i}\vec{E} - \rho_i \vec{\nabla}\varphi_g - \vec{\nabla}\cdot\vec{\Pi}_i + \vec{\Gamma}^{ie} = 0 \tag{5.6}$$

In Chapter 2, we discussed one model of the collision term $\vec{\Gamma}^{ie}$. According to this model $\vec{\Gamma}^{ie} = 0$ for a zero relative velocity of the two fluids. In static equilibrium the stress tensors Π_e and Π_i have only their diagonal parts non-zero representing the pressures. Now expressing the electric field $\vec{E}$ in terms of the electric potential φ, we find, by integrating equations (5.4) and (5.6), that

$$n_e = \frac{\rho_e}{m_e} = n_0 \exp[(e\varphi - m_e\varphi_g / K_B T_e)], \tag{5.7}$$

and

$$n_i = \frac{\rho_i}{m_i} = n_0 \exp[-(e\varphi + m_i\varphi_g)/k_B T_i]. \tag{5.8}$$

Thus, in static equilibrium, the electron and proton densities n_e and n_i follow the **Maxwell-Boltzmann Distribution**, where n_0 is the particle density in the absence of forces. We find that the electron density n_e increases with an increase of electric potential φ whereas the proton density n_i decreases. Under the circumstances that the two fluids have equal temperatures (T_e=T_i) – though in the absence of collisions, unless there is some other plasma mechanism acting, the two temperatures can remain unequal – we see that the two fluids can sustain a net charge density given by:

$$\frac{e(n_i - n_e)}{n_0} = \frac{e\Delta n}{n_0} = \frac{2e^2\varphi}{K_B T} \tag{5.9}$$

for weak potential $\varphi << K_B T / e$. Here, we have used the isothermal equation of state $p = nK_B T$ and the temperature T is assumed to be space and time independent. This is essentially the content of the energy conservation law of each fluid.

From Poisson's equation, we can estimate the electric field $\vec{E}$ produced by the net charge density ($e\Delta n$) over a region of length x as

$$E = 4\pi(e\Delta n)x \tag{5.10}$$

For a 1% change in electron density over a length x = 1 cm in a plasma of density n_0, we get

$$E = 4\pi e n_0 \times 10^{-2} \qquad \text{C.G.S. units} \tag{5.11}$$

In a solar coronal plasma with $n_0 \approx 10^{10}$ cm^{-3}, $E \approx 0.6$ C.G.S units or 1.8 x 10^4 volt/m; but in high density plasmas such as in the accretion disk of the X-ray binary source Cygnus X-1, where $n_0 \approx 10^{20}$ cm^{-3}, the electric field could be as large as ~ 9 x 10^{13} volt/m. This exercise shows us that in dense plasmas, the charge separation must be extremely small and that plasmas are quasi-neutral.

Now, in the absence of electric potential ($\varphi = 0$), we see from Equations (5.7) and (5.8) that the gravitational potential can also create a charge imbalance due to the different scale heights of protons and electrons. However, while studying plasma phenomena, the relevant spatial scales are generally much less than/both the scale heights, and therefore the difference between them may not be of much significance.

5.3. Wave Motions of Electron and Proton Fluids

Analogous to the excitation of waves in a single conducting fluid, there are a variety of wave motions exhibited by two conducting fluids. We shall follow the standard procedure for the study of waves, i.e., we shall study the response of the fluids under small departures form their equilibria. There are essentially two major types of wave motions – high frequency waves governed by the response of the electron fluid and the low frequency waves governed by the response of the proton fluid. The presence of the magnetic field $\vec{B}_0$ introduces two more characteristic periods – the gyroperiods of the electrons and protons. The collisions between the two fluids result in the dissipation of these waves. We shall now consider the following cases.

Electron-Plasma Oscillations

In the absence of magnetic and gravitational fields, the static equilibria of the electron and the proton fluids are described by:

$n_{e0} = n_{i0} = n_0 = cte.$

$\vec{E}_0 = 0$

$p_{e0} = p_{i0} = p_0 = cte.$

$T_{e0} = T_{i0} = T_0 = cte.$

$\vec{U}_{e0} = \vec{U}_{i0} = \vec{U}_0 = cte.$

$\vec{B} = 0$

The perturbations n_{e1} in the electron density and $\vec{U}_{e1}$ in the electron velocity satisfy the linearized mass and momentum conservation laws as

$$\frac{\partial n_{e1}}{\partial t} + \vec{\nabla} \cdot \left[n_0 \vec{U}_{e1} \right] = 0 \tag{5.16}$$

and

$$\frac{\partial \vec{U}_{e1}}{\partial t} = -\frac{e}{m_e}\vec{E}_1 - \frac{K_B T}{n_0 m_e}\vec{\nabla} n_{e1} - \nu_{ei}\left(\vec{U}_{e1} - \vec{U}_{i1}\right), \tag{5.17}$$

where we have substituted for Γ^{ei} . The perturbation in electric field $\vec{E}_1$ is related to the perturbation in charge density through Poisson's Equation:

$$\vec{\nabla} \cdot \vec{E} = -4\pi e\left(n_{e1} - n_{i1}\right) \tag{5.18}$$

The perturbations in the proton density n_{i1} and the proton velocity $\vec{U}_{i1}$ satisfy the linearized mass and momentum conservation laws as:

$$\frac{\partial n_{i1}}{\partial t} + \vec{\nabla} \cdot \left[n_0 \vec{U}_{i1} \right] = 0 \tag{5.19}$$

and

$$\frac{\partial \vec{U}_{i1}}{\partial t} = \frac{e}{m_i}\vec{E}_1 - \frac{K_B T}{n_0 m_i}\vec{\nabla} n_{i1} + \nu_{ie}\left(\vec{U}_{e1} - \vec{U}_{i1}\right) \tag{5.20}$$

We now assume a plane wave type of variation for all the first order quantities and write:

$$n_{e1}(\vec{r},t)=n'_{e1}\exp\left[i\vec{k}\cdot\vec{r}-i\omega t\right] \tag{5.21}$$

In order to determine the dispersion relation $\omega(k)$ of these waves, we substitute the solution, Equation (5.21), in Equations (5.16) – (5.20), subtract Equation (5.20) from Equation (5.17), take a dot product with $\vec{k}$, subtract Equation (5.19) from Equation (5.16) and use Equation (5.18) to get

$$\omega(\omega+i\nu_{ei})(n_{e1}-n_{i1})=4\pi n_0 e^2\left(\frac{1}{m_e}-\frac{1}{m_i}\right)(n_{e1}-n_{i1})+K_B T\left(\frac{n_{e1}}{m_e}-\frac{n_{i1}}{m_i}\right)k^2 \tag{5.22}$$

If we assume $n'_{i1}=0$ and use $m_i>>m_e$, we find:

$$\omega(\omega+i\nu_{ei})=\omega_{pe}^2+\frac{K_B T}{m_e}k^2 \tag{5.23}$$

where

$$\omega_{pe}=\left[\frac{4\pi n_0 e^2}{m_e}\right]^{1/2}$$

is known as the electron-plasma frequency. Equation (5.23) is the dispersion relation of the **Electron Plasma Waves** also called Langmuir waves. These waves represent oscillations of the net charge density n_{e1}.

The physics of these oscillations can be understood by referring to Figure (5.1). In a quasi-neutral plasma, local charge density fluctuations can arise. If there is an excess of, say, positive charge at some place, the negative charges would rush to that place and try to cancel it. However, in this attempt, the negative charges, due to their kinetic energy, may overshoot

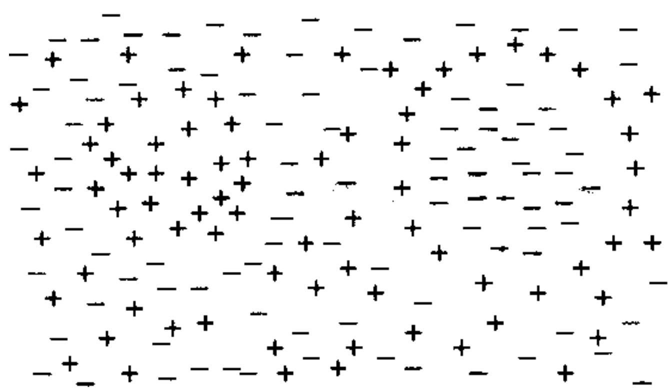

Figure 5.1. Oscillations Set Up Due to Localized Regions of Excess Charge Density

the place of excess positive charge and create an excess of negative charge elsewhere, from where they will be pulled back by the positive charge. Thus, in an attempt to maintain quasi-neutrality, charge density oscillations set in. The protons form a static positively charged background. We see from the dispersion relation that the frequency ω of the electron plasma waves is a function of the ambient electron density n_0 and the temperature T. Further ω is a complex number due to the presence of collisions. In the limit $\nu_{ei} << \omega$, we find the real part ω_R of ω is given by:

$$\frac{\omega_R^2}{\omega_{pe}^2} = 1 + \frac{k^2}{k_D^2} \tag{5.24}$$

where $k_D^2 = \left(4\pi n_0 e^2 / K_B T\right)$ and the imaginary part ω_I of ω is given by:

$$\omega_I = -\nu_{ei} \tag{5.25}$$

The collisions, therefore damp the wave amplitude. The variation of ω_R with wave vector k is shown in Figure (5.2). The damping rate ω_I is nearly directly proportional to the electron density n_0 and inversely proportional to the cube-root of the temperature T, a behavior resulting from of the Coulombic binary collisions. The electron plasma waves are longitudinal in their polarization – the displacement, and the electric field $\vec{E}_1$ are both parallel to the direction of the wave propagation vector $\vec{k}$.

The cooperative behavior of electrons is contained in the term ω_{pe}^2. The temperature dependent term is responsible for dispersion. We learnt in Chapter 1 that the condition for the collective behavior is that the characteristic spatial scale must be larger than the Debye

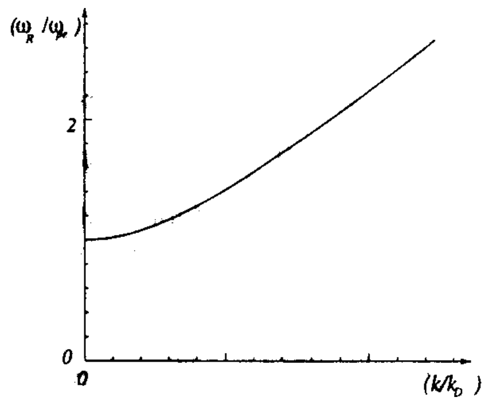

Figure 5.2. Dispersion Relation of the Electron-Plasma Waves (Equation 5.24)

wavelength. This implies that in Equation (5.23)

$$\omega_{pe}^2 > \frac{K_B T}{m_e} k^2, \tag{5.26}$$

from which we see that the largest wavevector allowed for the electron plasma waves is the Debye wavevector k_D.

The electron-plasma waves have been observed in laboratory plasmas. It is impossible to see these waves in astrophysical plasmas since they are localized oscillations and can only be picked up by in-situ probes. However, their presence has been inferred in otherwise inaccessible regions by indirect methods. One way, for example, is through the conversion of electron-plasma waves into electromagnetic waves which can leave the heavenly plasma and impinge on our telescopes. This is how some of the radio radiation from the sun is believed to originate.

The first simultaneous detections of the electron-plasma waves as well as the attendant radio emission were done by the solar orbiting Helios1 and Helios2 spacecrafts.

Ion-Plasma Oscillations

We now study low frequency oscillations in which electrons and protons both participate. In the absence of magnetic and gravitational fields the static equilibria of the two fluids are described by:

$n_{e0} = n_{i0} = n_0 = cte.$

$\vec{E}_0 = 0$.

$p_{e0} = n_0 K_B T_{e0}$.

$p_{i0} = n_0 K_B T_{i0}$.

$\vec{U}_{e0} = \vec{U}_{i0} = 0$.

We assume that the mass m_e of an electron is vanishingly small, i.e., $m_e \to 0$. In this limit, Equation (5.17) for the linearized motion of the electron fluid becomes:

$$-e\vec{E}_1 - \frac{K_B T_{e0}}{n_0}\vec{\nabla} n_{e1} = 0 \tag{5.27}$$

where we have ignored the collisional forces. The solution of Equation (5.27) gives:

$$n_{e1} = \frac{n_0 e \varphi_1}{K_B T_{e0}} \tag{5.28}$$

where we have expressed $\vec{E}_1 = -\vec{\nabla}\varphi_1$. For a plane-wave variation of the perturbed quantities, the mass conservation laws of the two fluids give:

$$n_{e1} = n_0 \frac{\vec{k} \cdot \vec{U}_{e1}}{\omega} \tag{5.29}$$

and

$$n_{i1} = n_0 \frac{\vec{k} \cdot \vec{U}_{i1}}{\omega} \tag{5.30}$$

The ion equation of motion dotted with the wave vector $\vec{k}$ gives:

$$\omega \vec{k} \cdot \vec{U}_{i1} = \frac{e}{m_i} k^2 \varphi_1 + \frac{K_B T_{i0}}{n_0 m_i} k^2 n_{i1} \tag{5.31}$$

Finally, Poisson's equation relating the perturbations in charge density with the potential φ_1 is:

$$k^2 \varphi_1 = -4\pi e (n_{e1} - n_{i1}) \tag{5.32}$$

Substituting for $\vec{U}_{e1}$, $\vec{U}_{i1}$ and n_{e1} in Equation (5.31), we find the dispersion relation for $n_{i1} \neq 0$ as:

$$\left(\frac{\omega^2}{\omega_{pi}^2}\right) = \left[1 - \left(1 + \frac{k^2}{k_D^2}\right)^{-1}\right] + \frac{k^2 T_{i0}}{k_D^2 T_{e0}} \tag{5.33}$$

where $\omega_{pi} = \left(\frac{4\pi n_0 e^2}{m_i}\right)^{1/2}$ is the ion-plasma frequency and

$$k_D^2 = \frac{4\pi n_0 e^2}{K_B T_{e0}} \tag{5.34}$$

is the Debye wave number. Equation (5.33) is the dispersion relation of the **Ion-Plasma Waves.** In the short wavelength limit, i.e., for $(k^2 / k_D^2) >> 1$, the dispersion relation of the ion-plasma waves becomes:

$$\omega^2 = \omega_{pi}^2 + \frac{K_B T_{io}}{m_i} k^2 \tag{5.35}$$

which looks very much like the dispersion relation of the electron-plasma waves. In the large wavelength limit, i.e., for $(k^2 / k_D^2) << 1$, the dispersion relation of the ion-plasma waves becomes:

$$\omega^2 = \frac{k_B}{m_i}(T_{i0} + T_{e0}) k^2 = c_s^2 k^2 \tag{5.36}$$

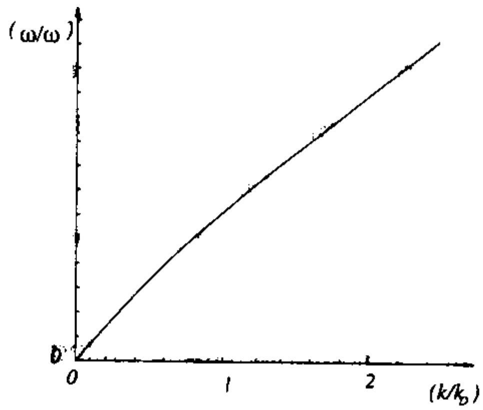

Figure 5.3. Dispersion Relation of the Ion-Plasma Waves (Equation 5.33) for $T_{i0} = T_{e0}$

which looks like the dispersion relation of the sound waves. Here, c_s, is the isothermal sound speed. For this reason, these waves are also known as the Ion-Acoustic Waves.

In contrast to the case of the electron-plasma waves where ions from a static and uniform background, during ion-plasma wave excitations, the electrons and ions both play a dynamic role. Electrons are pulled by a bunch of ions and they screen the electric field produced by the bunching ions. As for sound waves, here too, the ions form regions of high and low density. The ion thermal motion produces a spreading of the condensation. Due to the thermal motion of electrons, only a partial screening of the electric field is achieved. These two effects are contained in the temperature dependence of the dispersion relation (Equation 5.36). The full dispersion relation of the ion-plasma waves is illustrated in Figure (5.3).

Electron-Plasma Waves in Magnetized Fluids

We now investigate the effect of a uniform ambient magnetic field $\vec{B}_0$ on the characteristics of the electron plasma waves. The static equilibria of the electron and proton fluids are the same as before except that, now, $\vec{B}_0 = (0,0,B_0)$. We, further, take for electron plasma oscillations, $n_{i1} = U_{i1} = 0$ and $m_e / m_i \to 0$. The wave vector $\vec{k} \parallel \vec{E}_1$ makes an angle θ with the magnetic field as shown in Figure (5.4). The mass conservation for electron fluid gives:

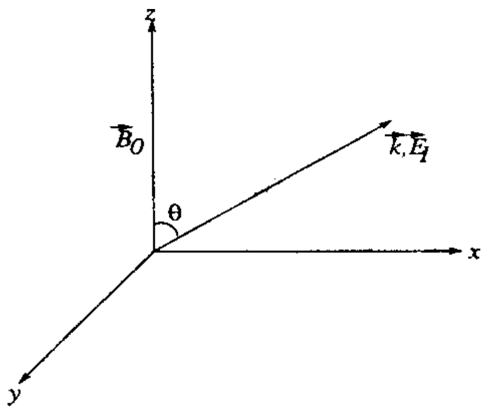

Figure 5.4. Electron-Plasma Waves Propagating at an Angle θ with $\vec{B}_0$.

$$\frac{n_{e1}}{n_0} = \frac{\vec{k} \cdot \vec{U}_{e1}}{\omega} \tag{5.37}$$

Poisson's equation gives:

$$\vec{k} \cdot \vec{E}_1 = i4\pi e n_{e1} \tag{5.38}$$

The addition of x and z components of the momentum conservation laws for electron and ion fluids gives (Here U_e stands for U_{e1}):

$$(-i\omega + 2\nu_{ei})\vec{k} \cdot \vec{U}_e = -\frac{e}{m_e}\vec{k} \cdot \vec{E}_1 - \omega_{ce} U_{ey} k_x - \frac{iK_B T}{n_0 m_e} k^2 n_{e1} \tag{5.39}$$

The addition of the y component of the momentum conservation laws for the electron and ion fluids gives:

$$(-i\omega + \nu_{ei})\vec{U}_{ey} = \omega_{ce} U_{ex} \tag{5.40}$$

From equations (5.37) – (5.40), by eliminating the various first order quantities, we get the dispersion relation:

$$\begin{aligned} & sin^2\theta\left[-(\omega+i\nu_{ei})^2+\omega_{ce}^2+\frac{\omega_{pe}^2}{\omega}(\omega+i\nu_{ei})+\frac{K_BTk^2}{m_e\omega}(\omega+i\nu_{ei})\right]+ \\ & +\cos^2\theta\left[-(i\omega+i\nu_{ei})^2+\frac{\omega_{pe}^2}{\omega}(\omega+2i\nu_{ei})+\frac{K_BTk^2}{m_e\omega}(\omega+i\nu_{ei})\right] \end{aligned} \tag{5.41}$$

Here, $\omega_{ce}=(eB_0/m_ec)$ is the electron cyclotron frequency. We see that for $\theta=0$, i.e., for propagation along the magnetic field, we recover the dispersion relation of the electron-plasma waves (Equation 5.23) in the absence of a magnetic field. For oblique propagation, the dispersion relation is modified by the presence of the magnetic field. In the absence of collisions and temperature effects, Equation (5.41) simplifies to:

$$\omega^2=\omega_{pe}^2+\omega_{ce}^2 sin^2\theta \tag{5.42}$$

This wave is known as the **Upper Hybrid Wave**, since its frequency ω is higher than the electron plasma frequency ω_{pe}. This is due to the additional restoring Lorentz force. The group velocity of these waves is zero in the absence of thermal effects.

Ion-Plasma Waves in Magnetized Fluids

In the presence of a uniform zeroth order magnetic field $\vec{B}_0=0$, we write the first order mass and momentum conservation laws for the hot electrons ($T_e\neq 0$) and cold ions ($T_i=0$) assuming a plane wave for space and time dependence for the perturbations:

$$m_i(-i\omega)\vec{U}_{i1}=-e(i\vec{k})\varphi_1+\frac{e}{c}(\vec{U}_{i1}\times\vec{B}_0) \tag{5.43}$$

$$m_e(-i\omega)\vec{U}_{e1}=e(i\vec{k})\varphi_1-\frac{K_BT_e}{n_0}(i\vec{k})n_{e1}-\frac{e}{c}(\vec{U}_{e1}\times\vec{B}_0) \tag{5.44}$$

$$(-i\omega)n_{e1}+i\vec{k}\cdot\vec{U}_{e1}n_0=0 \tag{5.45}$$

and

$$(-i\omega)n_{i1}+i\vec{k}\cdot\vec{U}_{i1}n_0=0 \tag{5.46}$$

We take $\vec{k} = (k_x, 0, 0)$ (Figure 5.4) and use the plasma approximation $n_{e1} = n_{i1}$, but $\vec{E}_1 = \vec{\nabla}\varphi_1 \neq 0$. Mass conservation then demands $\vec{U}_{e1} = \vec{U}_{i1}$. From the x and y components of Equation (5.43), we find

$$\left(\vec{U}_{i1}\right)_x = \frac{ek_x}{m_i\omega}\left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1}\varphi_1 \tag{5.47}$$

where $\omega_{ci} = eB_0 / m_i c$ is the ion-cyclotron frequency. For $T_e = 0$, from the x and y components of Equation (5.44) we find:

$$\left(\vec{U}_{e1}\right)_x = -\frac{ek_x}{m_e\omega}\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1}\varphi_1 \tag{5.48}$$

Using $\left(\vec{U}_{e1}\right)_x = \left(\vec{U}_{i1}\right)_x$, we find, in the limit $(m_e / m_i) \to 0$,

$$\omega = (\omega_{ce}\omega_{ci})^{1/2} \tag{5.49}$$

This is the dispersion relation of the **Lower Hybrid Waves**. They have frequencies lower than the electron cyclotron frequency ω_{ce} but higher than the ion cyclotron frequency ω_{ci}. They propagate perpendicular to the magnetic field B_0. For a propagation vector $\vec{k}$ parallel to $\vec{B}_0$, we recover the dispersion relation of the ion acoustic wave.

We now investigate the case of oblique propagation, i.e., for $\vec{k} = (k_x, 0, k_z)$ in the limit $m_e \to 0$.

The ion Equation (5.43) furnishes:

$$(U_{i1})_x = \frac{ek_x\varphi_1}{m_i\omega}\left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1}, \qquad U_{iz} = \frac{ek_z\varphi_1}{m_i\omega} \tag{5.50}$$

The electron Equation (5.44) furnishes:

$$(U_{e1})_x = 0\,; \qquad (U_{e1})_y = 0 \qquad \text{and} \qquad \frac{n_{e1}}{n_0} = \frac{e\varphi_1}{K_B T_e} \tag{5.51}$$

The continuity equations for electrons and ions under the plasma approximation give:

$$\frac{n_{i1}}{n_0} = \frac{\vec{k}\cdot\vec{U}_{i1}}{\omega} = \frac{n_{e1}}{n_0} \tag{5.52}$$

Eliminating $(U_{i1})_x$ between Equations (5.50) and (5.52) using Equation (5.51), we find:

$$\omega^2 = \omega_{ci}^2 + \frac{k_x^2 c_s^2}{\left(1 - \dfrac{k_z^2 c_s^2}{\omega^2}\right)} \tag{5.53}$$

This is the dispersion relation of the electrostatic **Ion-Cyclotron Waves**. In the limit

$$\left(\vec{k}\cdot\vec{U}_{i1}\right) \cong k_x\left(\vec{U}_{i1}\right)_x, \tag{5.54}$$

the dispersion relation for ion-cyclotron waves resembles the dispersion relation of the upper hybrid waves and predictably so, as the ‘acoustic’ motion of the ions is now modified by their cyclotron motion. We must, here, appreciate the need for $k_z \neq 0$. In order to preserve charge neutrality $n_{e1} = n_{i1}$, the electrons must move along the magnetic field, since their motion across the magnetic field is highly restricted. Thus, during the ion-cyclotron wave motion, the motion of the ions is predominantly perpendicular to the magnetic field while that of the electrons is essentially parallel to the magnetic field.

Electromagnetic Waves in Electron-Proton Fluids

So far, we have studied two examples of longitudinal waves which propagate only in a matter medium. We will now study the excitation of transverse electromagnetic waves which, though, they can propagate in vacuum, are modified in the presence of a medium. The static equilibria of the two fluids are described by:

$n_{e0} = n_{i0} = n_0 = cte.$

$\vec{E}_0 = 0$.

$p_{e0} = n_0 K_B T_{e0}$.

$p_{i0} = n_0 K_B T_{i0}$.

$\vec{U}_{e0} = \vec{U}_{i0} = 0$.

Using Maxwell's equations, the wave equation for the electric field is found to be:

$$-\nabla^2 \vec{E} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = -\frac{4\pi}{c^2}\frac{\partial \vec{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \tag{5.55}$$

where $\vec{J}$ is the current density.

The linearized current density is given by

$$\vec{J}_1 = en_0\left[\vec{U}_{i1} - \vec{U}_{e1}\right] \tag{5.56}$$

We wish to study transverse waves, for which

$$\vec{\nabla} \cdot \vec{E}_1 = i\vec{k} \cdot \vec{E}_1 = 0 \tag{5.57}$$

therefore we must put $n_{e1} = n_{i1}$. The mass conservation equations, then, could be satisfied with $\vec{U}_{e1} = \vec{U}_{i1}$. The wave equation (5.55) then describes propagation of electromagnetic waves in vacuum ($\vec{J} = 0$) with dispersion relation

$$\omega^2 = k^2 c^2 \tag{5.58}$$

The linearized forms of the momentum conservation laws of the two fluids, describe a Boltzmann distribution of the density perturbations as

$$n_{e1} = n_{i1} = \frac{en_0\left(\frac{1}{m_e} + \frac{1}{m_i}\right)\varphi_1}{K_B\left(\frac{T_{e0}}{m_e} - \frac{T_{i0}}{m_i}\right)} \tag{5.59}$$

The other way of satisfying Equation (5.57) is by putting $n_{e1} = n_{i1} = 0$. Mass conservation then gives:

$$\vec{\nabla} \cdot \vec{U}_{e1} = \vec{\nabla} \cdot \vec{U}_{i1} = 0 \tag{5.60}$$

i.e., the motion of the particles is transverse to the direction of the propagation vector $\vec{k}$.

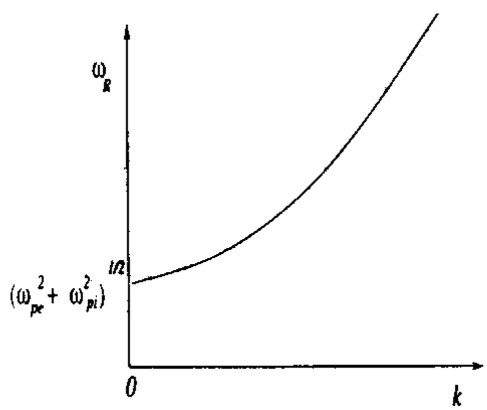

Figure 5.5. Dispersion Relation of the Electromagnetic Waves in an Electron-Proton Fluid.

From Equations (5.17) and (5.20), we find for the current density

$$\vec{J}_1 = \frac{n_0 e^2 (1/m_e + 1/m_i)\vec{E}_1}{(-i\omega + \nu_{ei})} \tag{5.61}$$

On substituting for $\vec{J}_1$ in the wave equation (5.55), we get the dispersion relation for the transverse electromagnetic waves as:

$$\omega^2 = \frac{\omega_{pe}^2 + \omega_{pi}^2}{\left(1 + \frac{2i\nu_{ei}}{\omega}\right)} + k^2c^2 \tag{5.62}$$

Again assuming that $\nu_{ei} << \omega$, we find the real part ω_R from:

$$\omega_R^2 \cong \omega_{pe}^2 + \omega_{pi}^2 + k^2c^2 \tag{5.63}$$

and the imaginary part

$$\omega_I \cong -\frac{\nu_{ei}}{2\omega_R^2}\left(\omega_{pe}^2 + \omega_{pi}^2\right) \tag{5.64}$$

We see that the phase and the group velocities of the electromagnetic waves become different in a plasma as there is a minimum value of $\omega_R = (\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$ below which the waves cannot propagate in a plasma (the wave vector $\vec{k}$ becomes imaginary). The waves suffer damping due to collisions between electrons and ions. The dispersion relation of the electromagnetic waves is plotted in Figure (5.5).

We can define the refractive index *n* of a plasma for electromagnetic waves from the dispersion relation as:

$$n^2 = \left(\frac{kc}{\omega}\right)^2 = 1 - \frac{\omega_{pe}^2}{\omega^2\left(1 + \frac{i\nu_{ei}}{\omega}\right)} \tag{5.65}$$

where we have neglected ω_{pi}^2 as it is much less than ω_{pe}^2. Equation (5.65) provides the basis for reflection of short wavelength radio waves in the earth's ionosphere facilitating communication around the earth. The ionosphere, itself, has been studied through the reflection of the radio pulses. The reflection occurs at a place, where the frequency of the radio pulse equals the electron plasma frequency. By this technique, the electron density in, as well as the distance to, the reflection region can be estimated. Electron densities of 10^5-10^6 cm^{-3} have been inferred at an altitude of 500 km in the ionosphere. These densities corresponds to electron plasma frequencies of 17-54 MHz.

A note of caution is in order here. The reflection of electromagnetic waves with frequencies near the electron-plasma frequency is true only for low intensity radiation. High intensity radiation can change the properties processes or novel conditions for reflection or transmission may set in.

The dispersive properties of the interstellar medium have been put to good use for determining the distances of pulsars.

Since, a plasma has an index of refraction which is less than unity, electromagnetic waves diverge while passing through it. However, by tailoring the density, a part of the plasma can be made to work as a focusing device. The self-focusing of Laser beams results from such processes which fall in the category of nonlinear processes.

Electromagnetic Waves in Magnetized Fluids

For plane-wave-type space and time variations of all the first order quantities, the wave Equation (5.55) becomes:

$$\left(\frac{\omega^2}{c^2} - k^2\right)\vec{E}_1 + \vec{k}\left(\vec{k} \cdot \vec{E}_1\right) = \frac{4\pi i \omega n_0 e}{c^2}\vec{U}_{e1}, \tag{5.75}$$

where we have assumed the ions to form a static positively charged back-ground so that $\vec{U}_{i1} = n_{i1} = 0$ and the current density

$$\vec{J}_1 = -en_0\vec{U}_{e1}, \tag{5.76}$$

is provided only by electrons.

All we have to do now is to determine the electron velocity $\vec{U}_{e1}$ in the presence of a uniform magnetic field, substitute it in the wave equation and

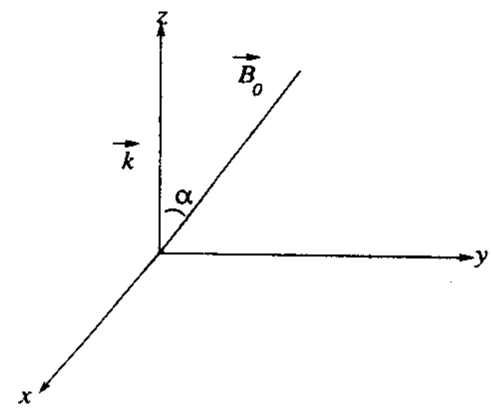

Figure 5.6. Oblique Propagation of Electromagnetic Waves.

we get the dispersion relation for electromagnetic waves in a magnetized plasma.

Let us take the propagation vector $\vec{k} = (0,0,k)$ and the magnetic field $\vec{B}_0 = (0, B_0 sin\alpha, B_0 \cos\alpha)$ where α is the angle between $\vec{k}$ and $\vec{B}_0$ (Figure 5.6). The electron velocity in then found to be:

$$U_{1x} = \frac{e}{m_e i\omega}\left[E_{1x} + \frac{U_{1y} B_o \cos\alpha}{c} + \frac{U_{1z} B_o sin\alpha}{c}\right], \tag{5.77}$$

$$U_{1y} = \frac{e}{m_e i\omega}\left[E_{1y} - \frac{U_{1x} B_o \cos\alpha}{c}\right], \tag{5.78}$$

and

$$U_{1z} = \frac{e}{m_e i\omega}\left[E_{1z} + \frac{U_{1x} B_o sin\alpha}{c}\right], \tag{5.79}$$

where we have removed the subscript *e* from $\vec{U}_1$. We can, now, solve for $\vec{U}_1$ in terms of $\vec{E}_1$ and substitute in Equation (5.75). We get three homogeneous equations in $\vec{E}_{1x}$, $\vec{E}_{1y}$ and $\vec{E}_{1z}$:

$$\left[1 - n^2 - \frac{X}{1-Y}\right]E_{1x} + \left[\frac{iX\sqrt{Y}}{1-Y}\cos\alpha\right]E_{1y} - \left[\frac{iX\sqrt{Y}}{1-Y} sin\alpha\right]E_{1z} = 0 \tag{5.80}$$

$$-\left[\frac{iX\sqrt{Y}}{1-Y}\cos\alpha\right]E_{1x}+\left[1-n^2-\frac{X\left(1-Ysin^2\alpha\right)}{1-Y}\right]E_{1y}+\left[\frac{XYsin\alpha\cos\alpha}{1-Y}\right]E_{1z}=0 \qquad (5.81)$$

$$\left[\frac{iX\sqrt{Y}}{1-Y}sin\alpha\right]E_{1x}+\left[\frac{XYsin\alpha\cos\alpha}{1-Y}\right]E_{1y}+\left[1-\frac{X\left(1-Y\cos^2\alpha\right)}{1-Y}\right]E_{1z}=0 \qquad (5.82)$$

By putting the determinant of these equations to zero, we get the dispersion relation:

$$\begin{vmatrix} 1-n^2-\dfrac{X}{1-Y} & \dfrac{iX\sqrt{Y}}{1-Y}\cos\alpha & -\dfrac{iX\sqrt{Y}}{1-Y}sin\alpha \\ -\dfrac{iX\sqrt{Y}}{1-Y}\cos\alpha & 1-n^2-\dfrac{X\left(1-Ysin^2\alpha\right)}{1-Y} & \dfrac{XYsin\alpha\cos\alpha}{1-Y} \\ \dfrac{iX\sqrt{Y}}{1-Y}sin\alpha & \dfrac{XYsin\alpha\cos\alpha}{1-Y} & 1-\dfrac{X\left(1-Y\cos^2\alpha\right)}{1-Y} \end{vmatrix}=0 \qquad (5.83)$$

where we have followed the notations usually used while studying wave propagation in the earth's ionosphere, i.e.,

$$X=\frac{\omega_{pe}^2}{\omega^2}, \qquad Y=\frac{\omega_{ce}^2}{\omega^2} \text{ and } \quad n^2=\frac{k^2c^2}{\omega^2} \qquad (5.84)$$

We shall study a few special cases using Equation (5.83). First,, notice that for $Y=0$, the dispersion relation (Equation 5.63) for electromagnetic waves in the absence of magnetic fields and collisions is recovered.

For waves propagating perpendicular to the magnetic field i.e., for $\alpha=\pi/2$, we get:

$$\left(1-n^2-X\right)\left[\left(1-n^2-\frac{X}{1-Y}\right)\left(1-\frac{X}{1-Y}\right)-\frac{X^2Y}{(1-Y)^2}\right]=0 \qquad (5.85)$$

The two roots of n^2 given by Equation (5.84) describe two types of waves. The root

$$n^2=1-X=1-\frac{\omega_{pe}^2}{\omega^2}, \qquad (5.86)$$

describes what is known as the **Ordinary Wave**, since it remains unaffected by the presence of the magnetic field. By substituting for n^2 in Equation (5.80 – 5.82), we find $E_{1x} = E_{1z} = 0$ and $E_{1y} \neq 0$. Thus the ordinary wave is linearly polarized with its electric field parallel to the ambient magnetic field.

The other root of n^2 is:

$$n^2 = \frac{(1 - X - Y) - X^2 Y (1 - X - Y)^{-1}}{1 - Y} = 1 - \frac{\omega_{pe}^2 (\omega^2 - \omega_{pe}^2)}{\omega^2 (\omega^2 - \omega_{pe}^2 - \omega_{ce}^2)} \qquad (5.87)$$

This is the dispersion relation of what is known as the Extraordinary Wave. From Equations (5.80) and (5.82), we find:

$$\frac{E_{1x}}{E_{1z}} = \frac{iX\sqrt{Y}}{(1-Y)\left(1 - n^2 - \frac{X}{1-Y}\right)} = -\left(1 - \frac{X}{1-Y}\right)\frac{1-Y}{iX\sqrt{Y}}, \qquad (5.88)$$

from which, we, again recover the dispersion relation of the extraordinary wave, Equation (5.87). Thus, the extraordinary wave is elliptically polarized with its electric field ($E_{1x}, 0, E_{1z}$) perpendicular to the magnetic field $B_0 = (0, B_0, 0)$. We must also acknowledge that this wave is not purely transverse as it has an electric field (E_{1z}) in the direction of propagation vector $\vec{k}$ (Figure 5.7).

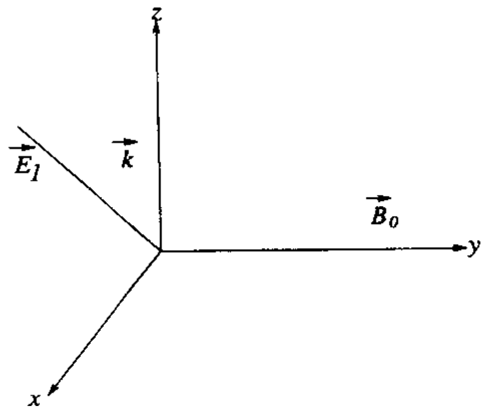

Figure 5.7. Polarization of the Extraordinary Wave.

Form the dispersion relation of the extraordinary wave, we notice that the refractive index n becomes infinite for

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 \tag{5.89}$$

which, we reckon, is the dispersion relation of the **Upper Hybrid Wave**. The frequency at which $n = \infty$ is known as the **Resonance Frequency**. At this frequency, the wavelength becomes zero. Had we included collisions, we would have found that the wavevector $\vec{k}$ is purely imaginary at the resonance. This implies that the wave is completely absorbed within the plasma and its group and phase velocities are zero. We further see a transformation of the nature of wave. The electromagnetic extraordinary wave has become an electrostatic upper hybrid wave. When we substitute Equation (5.89) in Equation (5.88), we find that $E_{1x} = 0$ and the extraordinary wave has become purely longitudinal with only $E_{1z} \neq 0$.

The extraordinary wave also has a Cutoff Frequency. This is the frequency at which the refractive index vanishes, so that the wavelength, the group and the phase velocities all become infinite. The wave at the cutoff frequency suffers a reflection. Although Equation (5.87) for $n^2 = 0$ gives four roots, we retain only the two positive frequency roots given by:

$$\omega_{RP} = \frac{1}{2}\left[\omega_{ce} + \left(\omega_{ce}^2 + 4\omega_{pe}^2\right)^{1/2}\right], \tag{5.90}$$

and

$$\omega_{LP} = \frac{1}{2}\left[-\omega_{ce} + \left(\omega_{ce}^2 + 4\omega_{pe}^2\right)^{1/2}\right]. \tag{5.91}$$

At the cutoff frequency ω_{RP}, the polarization of the extraordinary wave is found to be (Equation 5.88):

$$\frac{E_{1x}}{E_{1z}} = i \tag{5.92}$$

and at the cutoff frequency ω_{LP}, the polarization of the extraordinary wave is found to be (Equation 5.88):

$$\frac{E_{1x}}{E_{1z}} = -i \tag{5.93}$$

Obviously the subscripts R and L denote the right-handed and the left-handed circular polarizations. The pass band or the region of propagation of the extraordinary wave can be seen in a plot of n^2 vs. ω (Figure 5.8). We see that as $\omega \to \infty$, $n^2 \to 1$. As ω decreases from ∞, n^2 decreases from 1 and becomes zero at $\omega = \omega_{RP}$, the higher cutoff frequency. For $\omega < \omega_{RP}$, $n^2 < 0$ until $\omega = \omega_h$, the upper hybrid frequency at which $n^2 = -\infty$. From $\omega = \omega_h$ to ω_{pe}, n^2 increases from $-\infty$ to 1. From $\omega = \omega_{pe}$ to ω_{LP}, n^2 decreases from 1 to zero. For $\omega < \omega_{LP}$, n^2 remains negative. Thus, the regions $\omega_{LP} < \omega < \omega_h$ and $\omega > \omega_{RP}$ for which $n^2 > 0$ are the pass bands of the extraordinary wave. It is circularly polarized at $\omega = \omega_{LP}$ and ω_{RP}; elliptically polarized at $\omega = \omega_{pe}$ and $\omega > \omega_h$ and longitudinal at $\omega = \omega_h$. So, we, now, know all about the extraordinary wave except its amplitude.

Let us now consider the case $\alpha = 0$ for the propagation of wave along the magnetic field, so that $\vec{k} = (0,0k)$ and $\vec{B}_0 = (0,0,B_0)$. We find from Equation (5.83):

$$\left(1-n^2-\frac{X}{1-Y}\right)\left[\left(1-n^2-\frac{X}{1-Y}\right)(1-X)\right]-\frac{iX\sqrt{Y}}{1-Y}\left[\frac{-iX\sqrt{Y}}{1-Y}(1-X)\right]=0$$

or

$$n^2=1-\frac{X\left(1\pm\sqrt{Y}\right)}{1-Y} \tag{5.94}$$

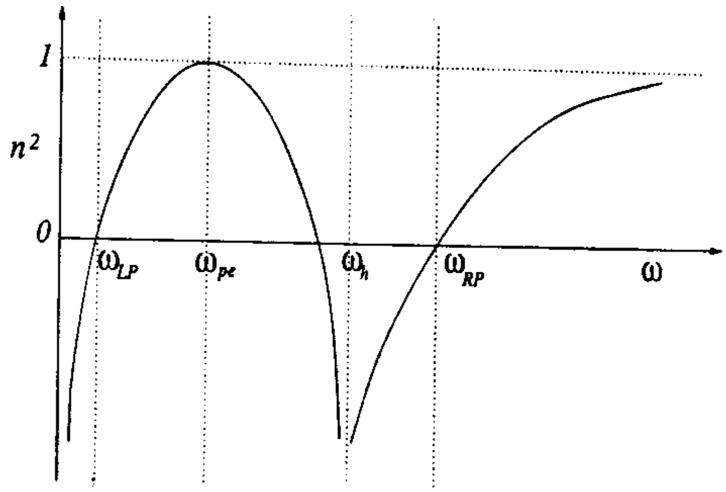

Figure 5.8. Pass-Band of the Extraordinary Wave

We write

$$n_R^2=1-\frac{X}{1-\sqrt{Y}}=1-\frac{\omega_{pe}^2}{\omega\left(1-\dfrac{\omega_{ce}}{\omega}\right)} \tag{5.95}$$

and

$$n_L^2=1-\frac{X}{1+\sqrt{Y}}=1-\frac{\omega_{pe}^2}{\omega\left(1+\dfrac{\omega_{ce}}{\omega}\right)} \tag{5.96}$$

for the two roots of n^2 from Equation (5.94). These are the dispersion relations of the two waves propagating parallel to the magnetic field. The polarization of these waves found (from Equations 5.80 – 5.82) is

$$\frac{E_{1x}}{E_{1y}} = \frac{-iX\sqrt{Y}}{(1-Y)(1-n^2)-X} \tag{5.97}$$

and $E_{1z} = 0$.

By substituting for $n^2 = n_R^2$ in Equation (5.97) we get

$$\frac{E_{1x}}{E_{1y}} = -i \tag{5.98}$$

Referring to the coordinate system shown in Figure (5.6) we see that Equation (5.98) represents a right-handed or an anticlockwise circular polarization which is also the sense of polarization of the extraordinary wave at the cutoff frequency ω_{RP}.

By substituting for $n^2 = n_L^2$ in Equation (5.97) we get:

$$\frac{E_{1x}}{E_{1y}} = i \tag{5.99}$$

which represents a left-handed or a clockwise circular polarization which is also the sense of polarization of the extraordinary wave at the cutoff frequency ω_{LP}.

We, now investigate the pass-bands of the R-wave (Equation 5.95) and the L-wave (Equation 5.96), the way we did for the extraordinary wave.

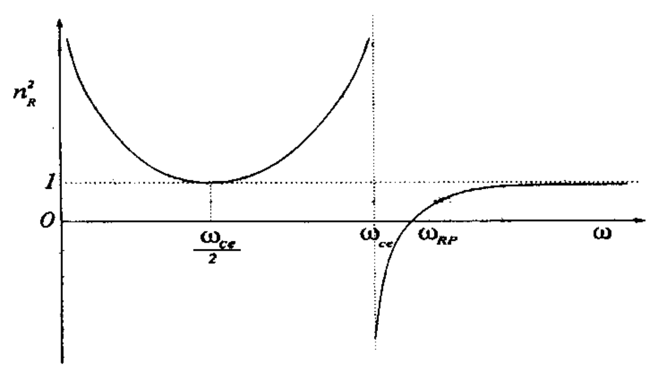

Figure 5.9. Pass-Band of R-Wave.

The cutoff frequency of the R-wave is given by $n_R = 0$ and is found to be ω_{RP} defined in Equation (5.90). The resonance frequency of the R-wave is given by $n_R^2 = \infty$ and is found to be at $\omega = \omega_{ce}$. A plot of n_R^2 vs. ω is shown in Figure (5.10). We find that n_R^2 has a minimum at $\omega = \omega_{ce}/2$. There is a low frequency pass band for $0 < \omega < \omega_{ce}/2$ in which n_R^2 decreases with an increase in ω and therefore the phase velocity is an increasing function of the frequency. It can be easily checked that the group velocity in this region is also an increasing function of ω. The waves in this pass-band have been named **Whistler Waves**. These waves propagate along the earth's magnetic field between the Northern and the Southern hemispheres and were detected in the ionosphere as radio waves in the audible range, producing a whistling sound. Due to the increase of the group and the phase velocities with ω, the low frequencies arrive later giving rise to descending tones. Thus, the pass band for the R-wave is $0 < \omega < \omega_{ce}$ and $\omega > \omega_{RP}$.

The cutoff frequency of the L-wave is given by $n_L = 0$ and is found to be ω_{LP} defined in Equation (5.91). The resonance frequency of the L-wave is zero. A plot of n_L^2 vs. ω (Figure 5.10) shows that L-waves propagates only for $\omega > \omega_{LP}$.

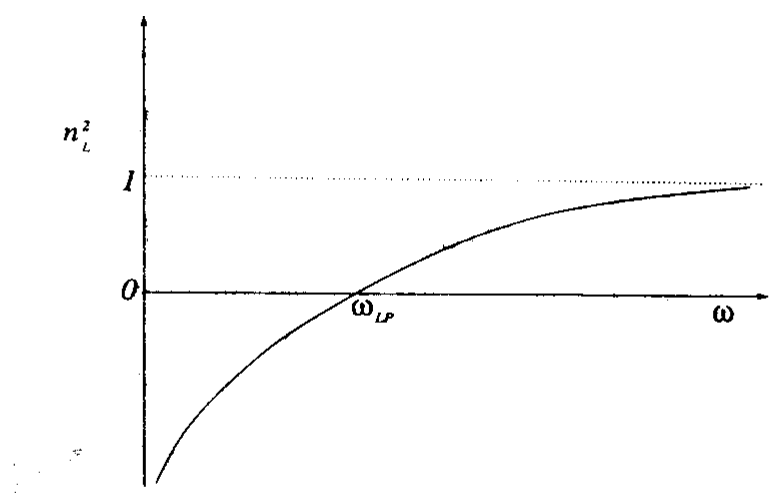

Figure 5.10. Pass-Band of L-Wave

The propagation of electromagnetic waves at values of α other than zero and $\pi/2$ can be studied by using the general dispersion relation given by Equation (5.83). In real situations, such as in stellar atmospheres, the plasma density and the magnetic field continuously vary with distance. In such a medium, an electromagnetic wave undergoes continuous refraction and the angle α itself becomes position dependent. Under such circumstances, we could either divide the medium into small homogeneous regions and use the results obtained above in each region or go to full-throttle numerical computations.

Another important consequence of the propagation of electromagnetic waves in a magnetized plasma is the attendant variations of the polarization. We have seen that parallel to the magnetic field, waves with right and left-hand circular polarization propagate with different phase speeds since $n_R \neq n_L$. Due to this effect, a plane polarized wave propagating parallel to the magnetic field suffers rotation in its plane of polarization. Let us represent the electric field of a plane wave polarized in the $\hat{x}$ direction as:

$$\vec{E} = \hat{x} E_x e^{i(kz-\omega t)} = \frac{1}{2}\left[\vec{E}_R + \vec{E}_L\right], \tag{5.100}$$

where $\vec{E}_R = \left(E_x + iE_y\right)\exp i\left(kz - \omega t\right)$ and $\vec{E}_L = \left(E_x - iE_y\right)\exp i\left(kz - \omega t\right)$ as the superposition of a right E_R and a left E_L circularly polarized waves. The wave vector of the R-wave becomes k_R and that of the L-wave becomes k_L, in the magnetized medium. After propagating a distance s in the medium, the electric field of the R-wave is given by

$$\vec{E}_R = \left(E_x + iE_y\right)\exp i\left(k_R s - \omega t\right), \tag{5.101}$$

and of the L-wave by

$$\vec{E}_L = \left(E_x - iE_y\right)\exp i\left(k_L s - \omega t\right) \tag{5.102}$$

If, we now superimpose the two waves, we find

$$\vec{E}_R + \vec{E}_L = \left[E_x \cos\left(k_L - k_R\right)\frac{s}{2} + E_y sin\left(k_L - k_R\right)\frac{s}{2}\right] \times \exp\left[i\left(k_L + k_R\right)\frac{s}{2} - \omega t\right] \tag{5.103}$$

which represents a plane-polarized wave with its electric field at an angle θ to the x-axis where

$$\theta = \left(k_L - k_R\right)\frac{s}{2} = \frac{s}{2c}\frac{\omega_{pe}^2\omega_{ce}}{\omega^2}, \tag{5.104}$$

where k_L and k_R have been determined from Equations (5.96) and (5.95). Thus, the propagation through a magnetized medium of size s rotates the electric vector of the electromagnetic wave by the angle θ. This effect is known as the **Faraday Rotation** of the plane of polarization. This is an observable effect. We can estimate that $\theta \cong 1$ radian when radiation at $\omega = 6\times10^8$ sec^{-1} passes through the interstellar medium of dimensions $\approx 10^{19}$ cm, the electron density $n \cong 10^{-2}$ cm^{-3} and the magnetic field $B_0 \cong 3\times10^{-6}$ Gauss. The observations of polarization of radiation from a single source, for example, the Crab Nebula, at different frequencies can confirm the presence of the Faraday effect in addition to providing the parameters of the intervening medium.

With this we end our discussion of waves in a magnetized medium.

5.4. Instabilities of Electron and Proton Fluids

If the electron and or the proton fluids contain free energy in the form of density, temperature and pressure gradients or a relative streaming motion between them, the equilibrium of such a system could become unstable against small perturbations. The excess energy is released through the growth of electric and magnetic fields, leading to macroscopic configurational changes or heating of plasma with or without emission of radiation. We illustrate the excitation of instabilities through a few simple examples.

Instabilities in Unmagnetized Fluids

Relative streaming between the electron and the proton fluids is the most common occurrence, especially in space and astrophysical environs, where the electrons and protons subjected to common acceleration mechanisms, end up with unequal velocities. The excess streaming energy is consumed by waves with their amplitudes growing at an exponential rate with time. Let us assume that in the equilibrium the proton fluid streams with a uniform velocity $\vec{V}_i$ and the electron fluid with a uniform velocity $\vec{V}_e$.

We neglect the random component of motion and take $T_e = T_i = 0$. For a plane wave space-time variation of all the perturbed quantities, we obtain the linearized equations of mass and momentum conservation for the electron and the proton fluid as:

$$\left(-i\omega + i\vec{k}\cdot\vec{V}_e\right)n_{e1} + in_0\vec{k}\cdot\vec{U}_{e1} = 0 \tag{5.105}$$

$$n_0 m_e\left(-i\omega + i\vec{k}\cdot\vec{V}_e\right)\vec{U}_{e1} = -en_o\vec{E}_1 \tag{5.106}$$

$$\left(-i\omega + i\vec{k}\cdot\vec{V}_i\right)n_{i1} + in_0\vec{k}\cdot\vec{U}_{i1} = 0 \tag{5.107}$$

and

$$n_0 m_i\left(-i\omega + i\vec{k}\cdot\vec{V}_i\right)\vec{U}_{i1} = en_0\vec{E}_1 \tag{5.108}$$

Poisson's equation becomes

$$i\vec{k}\cdot\vec{E}_1 = 4\pi e\left(n_{i1} - n_{e1}\right) \tag{5.109}$$

Carrying out the usual elimination exercise, we find the dispersion relation:

$$1 - \frac{\omega_{pe}^2}{\left(\omega - \vec{k}\cdot\vec{V}_e\right)^2} - \frac{\omega_{pi}^2}{\left(\omega - \vec{k}\cdot\vec{V}_i\right)^2} = 0 \tag{5.110}$$

We can solve this polynomial, look for complex roots of ω; since they occur in pairs, one of them has a positive imaginary part. This root represents the instability as all the perturbed quantities grow exponentially with time in this case. We shall, here, illustrate an approximate way of solving. Equation (5.110). We know that if one of the terms in Equation (5.110) becomes very large, the equation will have complex roots. Let us take $\vec{V}_i = 0$, so that $\vec{V}_e$ stands for the relative velocity between electrons and ions. Let us further assume that

$$\left(\omega - \vec{k}\cdot\vec{V}_e\right) \cong \pm\omega_{pe}\,; \qquad \omega << \vec{k}\cdot\vec{V}_e \tag{5.111}$$

Equation (5.110) then gives:

$$\omega^3 = -\frac{m_e}{2m_i}\left(\vec{k}\cdot\vec{V}_e\right)^3 \tag{5.112}$$

from which, we find the real part

$$\omega_R = \left(\frac{m_e}{2^4 m_i}\right)^{1/3}\omega_{pe} \tag{5.113}$$

and the imaginary part

$$\omega_I = \sqrt{3}\left(\frac{m_e}{2^4 m_i}\right)^{1/3}\omega_{pe} \tag{5.114}$$

The growth rate of the instability is ω_I. This is the **Two-Stream Instability** also called a **Buneman Type Instability**. We must remember that Equation (5.114) is valid only if $\vec{k}\cdot\vec{V}_e \cong \omega_{pe}$. The source of energy for this instability is the kinetic energy density ($m_e n_0 V_e^2/2$) of the electrons. Thus, the growth rate $\omega_I = 0$ if $V_e = 0$.

There is another approximate way of solving Equation (5.110). We solve Equation (5.110) in the limit $(m_e/m_i)\rightarrow 0$ to find $\left(\omega - \vec{k}\cdot\vec{V}_e\right) \cong \pm\omega_{pe}$, and substitute this in the term proportional to (m_e/m_i). We get

$$1 - \frac{\omega_{pe}^2}{\left(\omega - \vec{k}\cdot\vec{V}_e\right)^2} - \frac{(m_e/m_i)\omega_{pe}^2}{\left(\vec{k}\cdot\vec{V}_e \pm \omega_{pe}\right)^2} = 0 \tag{5.115}$$

The complex root with positive imaginary part is now given by

$$\omega = \vec{k}\cdot\vec{V}_e + i\omega_{pe}\left[\frac{\omega_{pi}^2}{\left(\vec{k}\cdot\vec{V}_e - \omega_{pe}\right)^2 - 1}\right]^{-1/2} \tag{5.116}$$

Thus, depending upon the approximations used, we get different values of the growth rate.

We see that in the approximate methods used above to determine the growth rate, we have used a matching of the Doppler shifted frequency $\left(\omega - \vec{k} \cdot \vec{V}_e\right)$ with the frequency ω_{pe} of the normal mode – the electron plasma wave. Therefore, it appears that it is this resonance that drives the instability. There is the electron plasma wave associated with the motion of the electrons and there is the ion-plasma wave associated with the motion of the ions. The Doppler shift of the proper sign brings these otherwise well separated frequencies to be nearly equal. It can be shown that in the presence of streaming, the electrons support what is known as a **Negative Energy Wave** i.e., the average energy density of the system in the presence of the wave is less than that in its absence or

$$\frac{1}{2} m_e n_0 V_e^2 > \frac{1}{2} m_e \overline{\left(n_0 + n_{e1}\right)\left(\vec{V}_e + \vec{U}_{e1}\right)^2}, \tag{5.117}$$

where the bar represents the average over space and time. This results due to the phase relation between the perturbed density n_{e1} and the perturbed velocity $\vec{U}_{e1}$ given by the mass conservation requirements. In the same way, the ions are associated with a positive energy wave. During the growth of the two stream instability, both the negative energy as well as the positive energy waves grow maintaining the constancy of the total energy.

The presence of finite amplitude low frequency waves plays an important role in modifying the electrical resistivity of the plasma. The usual Coulomb collisions among electrons and protons are replaced by the scattering of electrons by the low frequency waves which are manifestations of the collective behavior of the ions. The resistivity in these circumstances could be larger by several orders of magnitude than for normal Coulombic interactions. An actual estimate of the resistivity would require a knowledge of the amplitudes of these low frequency waves. A large resistivity facilitates a fast release of magnetic energy through an ohmic dissipation type of mechanism. This kind of energy release, also known as flaring has been proposed to take place in situations as diverse as the Sun and accretion disks around compact objects.

The **Beam-Plasma Instability** is another instability of great importance for different astrophysical objects. This is excited when a beam of electrons propagates through a non-streaming two-fluid plasma of electrons and protons. The equilibrium of this system consists of a beam of electron density n_b beaming with a velocity $\vec{V}_b$ through a plasma of density n_0. We take the massive protons to only provide the positively charged uniform background. In order to determine the dispersion relation for this case, we can use Equations (5.105) and (5.106) with $\vec{V}_e = 0$ for the electron fluid. The linearized equations for the electron beam are:

$$\left(-i\omega + i\vec{k} \cdot \vec{V}_b\right) n_{b1} + i n_b \vec{k} \cdot \vec{U}_{b1} = 0\,, \tag{5.118}$$

and

$$n_b m_e \left(-i\omega + i\vec{k} \cdot \vec{V}_b\right) \vec{U}_{b1} = -e n_b \vec{E}_1 \tag{5.119}$$

The Poisson equation is

$$i\vec{k} \cdot \vec{E}_1 = -4\pi e\left(n_{b1} + n_{e1}\right) \tag{5.120}$$

It is a simple task to find that the dispersion relation of the beam-plasma instability is given by:

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_b^2}{\left(\omega - \vec{k} \cdot \vec{V}_b\right)^2} = 0 \tag{5.121}$$

Where $\omega_b^2 = \dfrac{4\pi n_b e^2}{m_e}$.

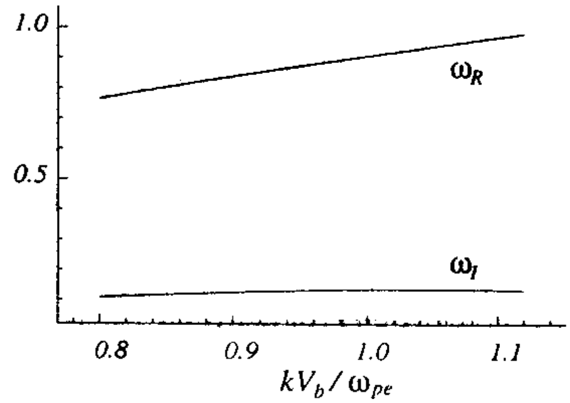

Figure 5.11. Variation of Oscillation Frequency ω_R and Growth Rate ω_I with $\left(kV_b / \omega_{pe}\right)$ for Beam Plasma Instability for $\left(\omega_b / \omega_{pe}\right) = 0.1$.

We can use the cues described during the discussion of the two-stream instability to approximately solve Equation (5.121) for complex roots. Thus, for $\omega << \vec{k} \cdot \vec{V}_b$ and $\left(\vec{k} \cdot \vec{V}_b\right)^2 \cong \left(n_b / n_0\right)\omega_{pe}^2$, we find:

$$\omega_R \cong \frac{1}{2^{4/3}}\left(\frac{n_b}{n_0}\right)^{1/6}\omega_{pe}$$

and (5.122)

$$\omega_I \cong \frac{\sqrt{3}}{2^{4/3}}\left(\frac{n_b}{n_0}\right)^{1/6}\omega_{pe}$$

For $\omega = \vec{k} \cdot V_b + i\omega_I < \omega_{pe}$ and $\omega_I << \omega_{pe}$, we find

$$\omega_R \cong \vec{k} \cdot \vec{V}_b \tag{5.123}$$

and

$$\omega_I \cong \left(\frac{n_b}{n_0}\right)^{1/2}\omega_{pe}\left(\frac{\omega_{pe}^2}{\omega^2} - 1\right)^{-1/2}$$

The physical mechanism described for the excitation of the two-stream instability also holds for the beam-plasma instability except that, presently there is relative streaming between the two species of electrons instead of between electrons and protons.

Variations of ω_R and ω_I with the ratio $(\vec{k}\cdot\vec{V}_b)\omega_{pe}^{-1}$ for the beam-plasma instability are shown in Figure (5.11). The beam-plasma instability has the maximum growth rate ω_I for $\omega_R \cong \omega_{pe}$. This means that electrostatic waves at the electron plasma frequency are produced. These Langmuir waves can be converted into electromagnetic waves through nonlinear scattering on the plasma particles, specifically the protons. The frequency of the electromagnetic waves so produced is again near the electron-plasma frequency ω_{pe}. This is believed to be the mechanism for the generation of type III radio bursts form the Sun. An electron-beam accelerated during a solar flare propagates outwards in the solar corona (density n_0) with typical values of $(n_b/n_0) \approx 10^{-4}$ and $V_b \cong 0.2c$. As the electron beam passes through the corona with continuously decreasing density n_0, electromagnetic waves of lower and lower frequency are excited. This gives rise to a drift rate of the frequency of radio emission. Drifting radio emission is taken as the signatures of the beam-plasma instability. The type III radio bursts have also been inferred to be emitted at twice the electron-plasma frequency. The emission at the second harmonic is believed to be generated by nonlinear interactions among the Langmuir waves.

In high energy sources, such as pulsars and quasars, relativistic electrons are expected to exist along with an ambient non-relativistic plasma. Such a system gives rise to the **Relativistic Version** of the beam-plasma instability. We can determine the dispersion relation by using the relativistic equation of motion for the beam electrons. The linearized form of the relativistic equation of motion is found to be:

$$\left(-i\omega + i\vec{k}c\cdot\vec{\beta}_0\right)\gamma_0\left(1+\gamma_0^2\beta_0^2\right)\vec{U}_{b1} = -\frac{e}{m_e}\vec{E}_1 \tag{5.124}$$

The other equations [(5.105) and (5.106) with $\vec{V}_e = 0$ and (5.119) and (5.120)] remain unaltered. The dispersion relation (Equation 5.121) is modified to:

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_b^2}{\gamma_0^3 \left(\omega - c\vec{k} \cdot \vec{\beta}_0\right)^2} = 0 \qquad (5.125)$$

where $\beta_0 = V_b / c$ and $\gamma_0 = \left(1 - \beta_0^2\right)^{-1/2}$.

We have already learnt how to solve this equation.

Chapter 6
KINETIC DESCRIPTION OF PLASMAS

6.1. Back to the Vlasov-Maxwell Way

We have come back full circle! In Chapter 2, we started with the phase-space description of *N* discrete particles and then transformed it into a continuum two-fluid and finally one fluid description using several averaging processes. After investigating some characteristics of the one-fluid and two-fluid descriptions, we now deal head-on with *N* discrete particles, electrons and protons, using the Vlasov equation. In this description, we work with particle distribution functions in the phase space of velocities and positions. The time evolution of the distribution function defines the stability or otherwise of the system. Plasmas are particularly interesting because they often submit to, or support, or generate, nonthermal (non-Maxwellian) and non-equilibrium distributions for finite durations of time. In other words, different species of particles can have unequal temperatures. Even a single species of particles can have different temperatures corresponding to different degrees of freedom. The free energy contained in these non-equilibrium distribution functions is then released in the form of heat and radiation. Plasmas are valued for their intrinsic cooperative nature due to which the transport, dissipate and radiative processes proceed at anomalously large rates as compared to single particle processes. Several astrophysical sources with extremely high luminosities with spectral energy distribution far from that of the blackbody, often showing variability on extremely short time scales, warrant the operation of coherent plasma processes. In this chapter, we shall study what is known as the kinetic or microscopic equilibrium and stability of an electron – proton plasma.

6.2. Kinetic-Equilibrium of an Electron-Proton Plasma

The equilibrium is now determined from the Vlasov equation, one each for electron and proton species, and Maxwell's equations. Neglecting collisional processes, we write the Vlasov equation for electrons as (Chapter 2):

$$\frac{\partial f_e}{\partial t}\left(\vec{r},\vec{V},t\right)+\vec{V}\cdot\frac{\partial f_e\left(\vec{r},\vec{V},t\right)}{\partial \vec{r}}-\left[\frac{e}{m_e}\vec{E}+e\frac{\vec{V}\times\vec{B}}{m_e c}\right]\cdot\frac{\partial f_e\left(\vec{r},\vec{V},t\right)}{\partial \vec{V}}=0 \qquad (6.1)$$

In the absence of electric and magnetic fields, in equilibrium, Equation (6.1) reduces to:

$$\vec{V}\cdot\frac{\partial f_e}{\partial \vec{r}}=0 \qquad (6.2)$$

which implies that the equilibrium single particle electron distribution function, f_e, must be independent of the space and the time coordinates and is a function only of velocity. For example, the **Maxwell-Boltzmann Distribution** of velocities expressed as:

$$f_e\left(\vec{V}\right)=n_0\left(\frac{m_e}{2\pi K_B T_e}\right)^{3/2}\exp\left(-\frac{m_e V^2}{2K_B T_e}\right), \qquad (6.3)$$

is a solution of Equation (6.2). As a matter of fact, we can choose for f_e, any function which depends only on the constants a_i, of motion of a particle, since

$$\frac{d}{dt}f_e\left(a_1,a_2,...\right)=\sum_i\frac{\partial f_e}{\partial a_i}\frac{da_i}{dt}=0 \qquad (6.4)$$

We must remember that constants of motion are functions of $\left(\vec{r},\vec{V}\right)$ and are independent of time only for each single particle's motion. In general, a_i are functions of $\left(\vec{r},\vec{V},t\right)$. When a_i are independent of time, so is f_e – the equilibrium distribution function.

The first example, perhaps, of a constant of motion independent of time is the total energy. For a free particle, the total energy is $\left(mv^2/2\right)$ and the Maxwell-Boltzmann distribution function (Equation 6.3) is realized. For electrons executing circular motion in a uniform magnetic field, the total energy, the energy associated with motion perpendicular to the magnetic field and the angular momentum are all constants of motion. Thus, the electron distribution function in the presence of magnetic field could be represented as:

$$f_e\left(\vec{V}\right)=\frac{n_0 m_e^{3/2}}{\left(2\pi K_B T_{//}\right)^{1/2}\left(2\pi K_B T_{\perp}\right)}\exp\left(-\frac{m_e V_{//}^2}{2K_B T_{//}}-\frac{mV_{\perp}^2}{2K_B T_{\perp}}\right), \tag{6.5}$$

where $//$ and $\perp$ are with respect to the direction of the magnetic field.

In slowly varying fields, the adiabatic invariants play the role of the constants of motion. We have studied in Chapter 3 that in a magnetic mirror, charged particles with velocities inclined at small angles to the magnetic field escape from the system; the resulting phase space distribution of the particles is known as the **Loss-Cone Distribution**.

The condition for the escape of particles from a magnetic mirror has been derived in Chapter 3. It says that for a given value of V_z, particles with $V_{\perp} < pV_z$ are absent from the system, and the loss cone angle θ_M is given by $\theta_M = \tan^{-1} p$.

One representation of the Loss-cone distribution function is (Figure 6.1):

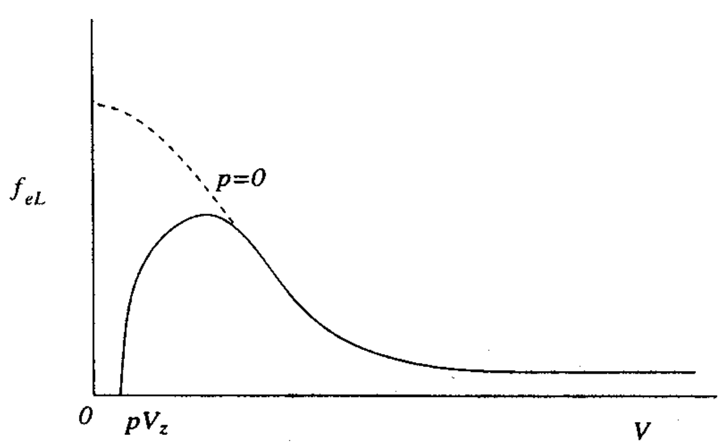

Figure 6.1. Representation of the Loss-Cone Distribution Function f_{eL}.

$$f_{eL} = \left(\frac{m_e}{2\pi K_B T_e}\right)^{3/2} \left(p^2 + 1\right)^{1/2} \exp\left[-\frac{m_e V^2}{2K_B T_e}\right] \Theta\left(V_\perp^2 - V_z^2 p^2\right) \qquad (6.6)$$

which reduces to a Maxwellian for $p = 0$. Here Θ is the unit step function which is unity if its argument is greater than zero and zero otherwise.

6.3. Kinetic Description of Electron-Plasma Waves and Instabilities

After determining the kinetic equilibrium, we would like to find out if this equilibrium is stable or not. For this purpose, we perturb the plasma so that its two distribution functions, one for electron $\left(f_e\right)$ and the other for protons $\left(f_i\right)$, are given by

$$f_s\left(\vec{r},\vec{V},t\right) = f_{s0}\left(\vec{r},\vec{V}\right) + f_{s1}\left(\vec{r},\vec{V},t\right), \qquad (6.7)$$

where the species index s stands for e (electrons) and p (protons). The linearized Vlasov equation

$$\frac{\partial f_{s1}\left(\vec{r},\vec{V},t\right)}{\partial t} + \vec{V}\cdot\frac{\partial f_{s1}\left(\vec{r},\vec{V},t\right)}{\partial \vec{r}} + \frac{e_s}{m_s}\left[\vec{E}_1 + \frac{\vec{V}\times\vec{B}_1}{c}\right]\cdot\frac{\partial f_{s0}\left(\vec{r},\vec{V}\right)}{\partial \vec{V}} = 0 \qquad (6.8)$$

is then obtained by assuming $\left|f_{s1}\right| << \left|f_{s0}\right|$. Here, $\vec{E}_1$ and $\vec{B}_1$ are the first order electric and magnetic fields to be determined from Maxwell's equations. There are no zeroth order electric and magnetic fields.

In order to get familiarity with the kinetic approach, we first consider the simplest and the most instructive case of electrostatic oscillations $\left(\vec{B}_1 = 0\right)$. We treat protons as a positively charged background providing charge neutrality to the plasma. The Vlasov equation for the electronic component of the plasma, therefore, becomes:

$$\frac{\partial f_{e1}}{\partial t} + \vec{V}\cdot\frac{\partial f_{e1}}{\partial \vec{r}} - \frac{e}{m_e}\vec{E}_1\cdot\frac{\partial f_{e0}}{\partial \vec{V}} = 0\,. \qquad (6.9)$$

The electric field $\vec{E}_1$ is determined from Poisson's equation:

$$\vec{\nabla} \cdot \vec{E}_1 = -4\pi \int f_{e1} d\vec{V} \, . \tag{6.10}$$

Again assuming a plane-wave type variation for f_{e1} as

$$f_{e1}\left(\vec{r}, \vec{V}, t\right) = f_{e1}\left(\vec{V}\right)\exp\left[i\left(k \cdot r - \omega t\right)\right], \tag{6.11}$$

we find from Equations (6.9) and (6.10):

$$\frac{4\pi e^2}{m_e k}\int \frac{\hat{k} \cdot \dfrac{\partial f_{e0}}{\partial \vec{V}}}{\left(\vec{k} \cdot \vec{V} - \omega\right)} d\vec{V} = 1 \, , \tag{6.12}$$

where for electrostatic perturbations, $\vec{k} \,//\, \vec{E}_1$ and $\hat{k} = \dfrac{\vec{k}}{|k|}$ have been used.

The evaluation of the integral in Equation (6.12) is a trifle tricky since the integrand diverges for $\vec{k} \cdot \vec{V} = \omega$. It was the Russian physicist Lev Landau (Landau 1946) who realized the importance of the singularity at $\vec{k} \cdot \vec{V} = \omega$ and showed a way to handle it. He stressed that this problem must be treated as an initial value problem, which means that the perturbations can be Fourier decomposed in space but we must use the Laplace transform for the time coordinate.

We shall consider first some special cases.

High Phase-Velocity

The case with high phase velocity i.e., for $\omega/k >>$ the thermal velocity V_{th}, one can carry out the integration by parts to find the dispersion relation

$$1 = \frac{\omega_{\rho e}^2}{k^2}\langle\left(V_x - \omega/k\right)^{-2}\rangle_0 \tag{6.13}$$

where the $\langle\rangle_0$ is calculated using f_{e0}. In the limit $\frac{\omega}{k} \to \infty$ the dispersion relation becomes

$$\omega^2 = \omega_e^2 \left(1 + 2\langle \mathrm{V_x} \rangle_0 \frac{k}{\omega} + 3\langle \mathrm{V_x^2} \rangle_0 \frac{k^2}{\omega} + ... \right). \tag{6.14}$$

For a stationary plasma $\langle V_x \rangle_0 = 0$ and for an isotropic distribution f_{eo}, $\langle V_x^2 \rangle = \frac{k_B T}{m}$ for T as the temperature of the plasma. Thus the dispersion relation reduces to:

$$\omega^2 = \omega_{pe}^2 \left(1 + \frac{3K_B T}{m\omega^2} k^2 + ... \right) \approx \omega_{pe}^2 \left(1 + \frac{3K_B T}{m\omega^2{}_{pe}} k^2 \right). \tag{6.15}$$

This is known as the Bohm-Gross Dispersion Relation. This is identical to the dispersion relation of the electron-plasma waves obtained in the fluid description.

Landau Damping

The major difficulty in evaluating the integral in equation (6.12) is the pole at $V_x = \omega/k$. One way out is to include collisions using the krook collision model. The Boltzmann equation then reads

$$-i\omega f_{e1} + ikV_x f_{e1} - \frac{e}{m} E \frac{\partial f_{e1}}{\partial V_x} = -\nu f_{e1} \quad , \tag{6.16}$$

and the dispersion relation becomes

$$1 = \frac{\omega_{pe}^2}{n_0 k^2} \int \frac{\partial f_{e0}}{\partial V_x} \frac{d^3 V}{V_x - (\omega + i\nu)/k} \qquad . \tag{6.17}$$

The Landau prescription for handling this integral after taking the Laplace transform in the time coordinate and using the causality condition $f(t)=0$ for t < 0, consists of using Plemelj formula

$$\frac{1}{\omega - kV_x + i\nu} = P\frac{1}{\varpi - kV_x} - i\pi\delta(\varpi - kV_x), \tag{6.18}$$

where P stands for the Cauchy's Principal value integral. We find a new contribution to the dispersion relation due to the Dirac-delta term and it is

$$-\frac{\omega_{pe}^2}{n_0 k}\int\frac{\partial f_{e0}}{\partial V_x}(-i\pi)\delta(\omega - kV_x)dV_x = \frac{i\pi\omega_{pe}^2}{n_0 k^2}\left(\frac{\partial f_{e0}}{\partial V_x}\right)_{\omega=kV\,x}, \tag{6.19}$$

and the complete dispersion relation becomes:

$$1+\frac{\omega_{pe}^2}{n_0 k}P\int\frac{\partial f_{e0}}{\partial V_x}\frac{d^3V}{\omega - kV_x} - \frac{i\pi\varpi_{pe}^2}{n_0 k^2}\left(\frac{\partial f_{e0}}{\partial V_x}\right)_{\omega=kV_x} = 0 \quad . \tag{6.20}$$

We find the real part ω_R of the frequency ω to be:

$$\omega_R^2 = \omega_{pe}^2 + \frac{3}{2}k^2 V_{T_e}^2 \quad , \tag{6.21}$$

under the approximation that the thermal term $\left(3k^2V_{T_e}^2/2\right) << \omega_{pe}^2$ and the imaginary part ω_1 to be:

$$\omega_1 = -\sqrt{\pi}\,\frac{\omega_R^4}{k^3 V_{T_e}^3}\exp\left[-\frac{\omega_R^2}{k^2 V_{T_e}^2}\right]. \tag{6.22}$$

The value of ω_R is identical to that obtained with the two-fluid description of plasmas. But we now have an imaginary part ω_I with a negative value even in the absence of collisions. This is a major outcome of the kinetic approach. The electron-plasma waves suffer damping, known as **Collisionless** or **Landau Damping**. It has originated from the presence of electrons with velocity equal to the phase velocity of the wave. Such electrons are called **Resonant Electrons**. They move with the phase velocity of the wave and therefore see an almost static electric field $\vec{E}_1$. Under such conditions, electrons and the wave can exchange energy between themselves: the electrons can gain energy from the wave, resulting in the damping of the wave, or the electrons can lose energy to the wave, resulting in the amplification of the wave. Which of the two processes occurs is decided by the electron distribution function $f_{e0}(\vec{V})$. We have found that the wave damps for the Maxwellian distribution function because it has

$$\frac{\partial f_{e0}(V_x)}{\partial V_x} < 0 . \tag{6.23}$$

A velocity distribution with

$$\frac{\partial f_{e0}(V_x)}{\partial V_x} > 0 . \tag{6.24}$$

gives a positive value of ω_1 and the wave amplitude $\vec{E}_1$ grows as $\exp(\omega_I t)$. This produces the circumstances of an **Instability**. We have seen earlier that an electron beam passing through an electron-proton plasma gives rise to an instability. The one dimensional velocity distribution function of an electron beam of velocity V_0 in the *x* direction and temperature T_b can be represented by the drifted Maxwellian as:

$$f_{b0}(V_x) = n_b \left(\frac{m_e}{2\pi K_B T_b} \right)^{1/2} \exp\left[- \frac{m_e (V_x - V_0)^2}{2 K_B T_b} \right] . \tag{6.25}$$

The total equilibrium electron distribution function f_{e0} is, therefore, given by (Figure 6.2):

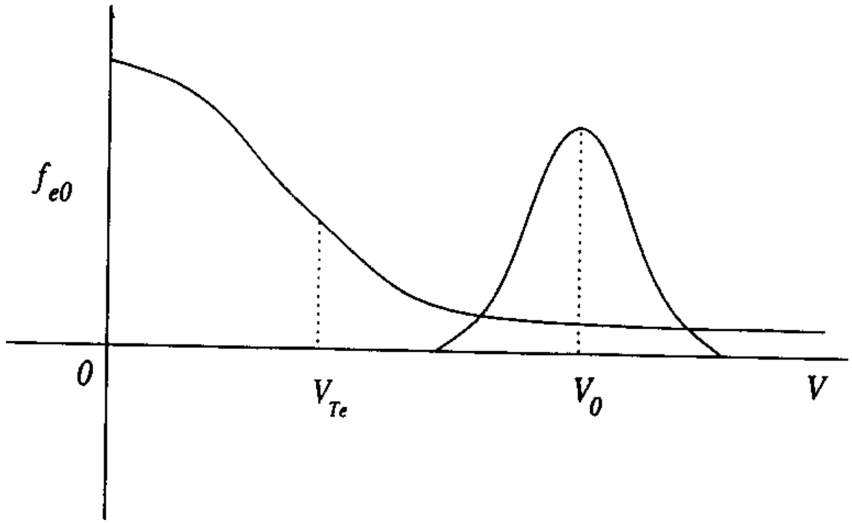

Figure 6.2. The Electron Distribution Function Consisting of a Superposition of Two Maxwellian Velocity Distributions has a Region of Positive Velocity Gradient (e.g. Equation 6.26).

$$f_{e0}(\vec{V}) = \frac{n_0}{\pi^{3/2} V_{T_e}^3} \exp\left[-\frac{V^2}{V_{T_e}^2}\right] + \frac{n_b}{\sqrt{\pi} V_{T_b}} \exp\left[-\frac{(V_x - V_0)^2}{V_{T_b}^2}\right], \tag{6.26}$$

Following the procedure outlined alone, we can find the dispersion relation and the imaginary part of the frequency is the growth rate of the beam-plasma instability. We can also study kinetic instabilities in magnetized plasmas, but at a price and the price is the hard work that one has to do to deal with this much more complex system.

Additional Reading

1. Benz, A. O. *Plasma astrophysics*. Ed. Kluwer Academic Publishers, 1993, 299p.

2. Melrose, D. B. *Instabilities in space and laboratory plasmas*. Ed. Cambridge University Press, 1986, 280p.

3. Nicholson, D. R. *Introduction to plasma theory*. Ed. John Wiley and Sons, Nova York, 1983, 292p.

4. Tanenbaum, B. S. *Plasma physics*. Ed. McGraw-Hill Book Company, 1967, 361p.

5. Vinod, K. *Astrophysical plasmas and fluids*. Ed. Kluwer Academic Publishers, 1999, 358p.