



LETTERS TO THE EDITOR



A SIMPLE DERIVATION OF THE RAYLEIGH CRITERION IN INTEGRAL FORM

M. A. FERREIRA AND J. A. CARVALHO JR.

Laboratório Associado de Combustão e Propulsão, Instituto Nacional de Pesquisas Espaciais, Rod. Pres. Dutra, km 40, CEP 12630-000, Cachoeira Paulista, SP, Brazil

(Received 11 November 1996)

1. INTRODUCTION

One of the main physical conditions for the enhancement of thermally excited acoustic oscillations is given by the Rayleigh criterion [1]. The criterion states that if heat is added to an acoustically oscillatory mass of a gas, the amplitude of oscillations will increase if the heat addition takes place when the pressure is higher than the average. The same effect will occur if heat is extracted when the pressure is lower than the average.

Putnam and Dennis [2] indicated a mathematical expression for the Rayleigh criterion, given by

$$\Delta = \oint \dot{q}' P' dt > 0, \quad (1)$$

where \dot{q}' is the oscillating heat release, P' is the acoustic pressure, t is time and where the integral is calculated over a cycle of oscillation. The inequality of expression (1) means that acoustic oscillations at a determined position inside an enclosure will be amplified if the cyclic integral is larger than zero. The oscillation will have maximum amplification for the maximum value of the integral. If there is damping, which always happens, some positive value will replace zero in the right side [3].

Oran and Gardner [4] attribute to Chu [5], who first derived expression (1) based on the conservation equations, a physically consistent foundation for the mathematical formulation of Rayleigh's criterion. Chu's basic idea was to calculate if the energy associated with the perturbation grows along an oscillation cycle as a result of its interaction with the energy source. In a later paper, Chu [6] derived a generalized form of the Rayleigh criterion. Culick [7] worked on the Rayleigh criterion for non-linear acoustics.

The use of the Rayleigh criterion with restricted linear parameters is still a powerful tool to explain the fundamentals of the thermo-acoustic phenomenon in Rijke tubes. Examples of its utilization include the works of Carvalho *et al.* [8, 9] and Bai *et al.* [10].

The purpose of this note is to present a new and simple derivation, based on linear acoustics, of the Rayleigh criterion in its integral form.

2. MATHEMATICAL FORMULATION

The equations of continuity, momentum, and energy for linear one-dimensional acoustics in an inviscid non-isothermal flow in a control volume are, respectively,

$$\partial \rho' / \partial t + (\partial / \partial x)(\bar{\rho} u') = 0, \quad \bar{\rho} \frac{\partial u'}{\partial t} = -\frac{\partial P'}{\partial x}, \quad (2, 3)$$

$$\frac{\partial P'}{\partial t} = c^2 \left(\frac{\partial \rho'}{\partial t} + u' \frac{\partial \bar{\rho}}{\partial x} \right) + (\gamma - 1) \left(\dot{G}' - \frac{\partial \dot{q}'}{\partial x} \right), \quad (4)$$

where ρ is density, P is pressure, u is flow velocity, c is speed of sound, γ is the relation between specific heats, \dot{G} is the heat release rate, and \dot{q} is the heat flux through the boundaries of the control volume and where the symbols $\bar{\cdot}$ and \cdot represent average and acoustic parameters, respectively. The above equations are easily obtained by perturbing the conventional conservation equations and retaining the first order terms.

The assumptions to obtain the conservation equations in the form written here are: (1) \bar{u} depends exclusively on the axial co-ordinate x and has the same order of magnitude of the acoustic velocity amplitude, u'_{max} , both being one order of magnitude smaller than the speed of sound, (2) the mean flow is uniform and steady, (3) there is no area variation along the flow, (4) the perturbation is periodical, (5) viscosity is negligible and (6) specific heats are constant.

Figure 1 shows a tube of length L with a heat source of small longitudinal dimension located at a distance l from one of the tube ends. An acoustic perturbation is considered as propagating along the cold and hot regions, both maintained at constant mean temperatures T_1 and T_2 , and the heat flux \dot{q} is assumed to be negligible. Two control volumes are defined: one for the cold region and the other for the hot region. For constant $\bar{\rho}$, equation (3) can be multiplied by u' to give

$$(\partial/\partial t)(\frac{1}{2}\bar{\rho}u'^2) + (\partial/\partial x)(P'u') - P'(\partial u'/\partial x) = 0. \quad (5)$$

The above equation is analogous to the one-dimensional mechanical energy equation [11] and can be taken as a conservation equation. To demonstrate this fact, it is necessary to proceed with some manipulations. First, the term $\partial u'/\partial x$ is identified by its equivalent obtained from the acoustic continuity equation (2), for conditions of constant $\bar{\rho}$, i.e.,

$$\partial u'/\partial x = (1/\bar{\rho})(\partial \rho'/\partial t). \quad (6)$$

On the other hand, the acoustic energy equation (3) applied to the control volumes defined in Figure 1 results in

$$P'/\rho' = c^2. \quad (7)$$

Combination of equations (6) and (7) yields

$$(\partial/\partial t)(\frac{1}{2}\bar{\rho}u'^2 + \frac{1}{2}(P'^2/\bar{\rho}c^2)) + (\partial/\partial x)(P'u') = 0, \quad (8)$$

where the terms $\frac{1}{2}\bar{\rho}u'^2$ and $\frac{1}{2}(P'^2/\bar{\rho}c^2)$ are related to the variations of acoustic kinetic energy and acoustic potential energy, respectively. The space derivative of the product $P'u'$ represents the acoustic energy flux, also called acoustic intensity [12]. Equation (8) takes

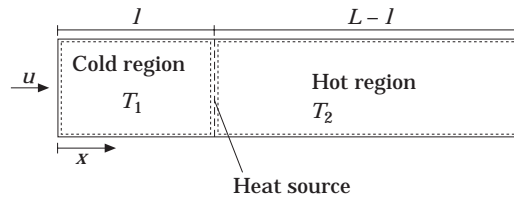


Figure 1. Schematic of tube with a heat source.

a typical form of a conservation equation if integrated over a fixed control volume V , utilizing the Gauss theorem to transform the volume integral in a surface integral:

$$\frac{\partial}{\partial t} \iiint_V \left(\frac{1}{2} \bar{\rho} u'^2 + \frac{1}{2} \frac{P'^2}{\bar{\rho} c^2} \right) dV + \iint_S P' u' dA = 0. \quad (9)$$

Up to this point, the derivation is new. In the next, Chu's concept of energy in a small disturbance [6] is followed. For the control volume of the cold region, integration of equation (9) yields

$$\partial E'_{M,1} / \partial t + A(P'_{1,L} u'_{1,L} - P'_{1,0} u'_{1,0}) = 0, \quad (10)$$

where E_M denotes mechanical energy. Analogous integration for the hot region yields

$$\partial E'_{M,2} / \partial t + A(P'_{2,L} u'_{2,L} - P'_{2,0} u'_{2,0}) = 0. \quad (11)$$

The product $P' u'$ at the sections $x = 0$ and $x = L$ will always be zero because a tube end is either a pressure node or a velocity node, depending if the section is opened or closed.

Because P' is continuous at the interface between the cold and hot regions, the time variation of the mechanical energy in the tube is obtained adding the respective time variations in each of the tube regions:

$$\partial E'_M / \partial t = (\partial / \partial t)(E'_{M,2} + E'_{M,1}) = A(P'_{2,L} u'_{2,L} - P'_{1,L} u'_{1,L}) = A P'_I (u'_{2,L} - u'_{1,L}). \quad (12)$$

Now it is necessary to evaluate the difference between the acoustic velocities at the interface between the cold and hot regions, which can be achieved with an energy balance applied to both sides of the heater element:

$$\dot{q} = \rho_2 u_2 (c_{p2} T_2 + \frac{1}{2} u_2^2) - \rho_1 u_1 (c_{p1} T_1 + \frac{1}{2} u_1^2), \quad (13)$$

where \dot{q} is the heat rate transferred to the flow per unit area of the heater element. Considering that, for low velocities, the kinetic energy term can be neglected in relation to the thermal energy term, the following results from the heat balance:

$$\dot{q} = \rho_2 u_2 c_{p2} T_2 - \rho_1 u_1 c_{p1} T_1. \quad (14)$$

The parameter T in equation (14) can be substituted by equivalent parameters using the ideal gas equation of state and the relation between specific heats for an ideal gas, leading to

$$\dot{q} = [\gamma_2 / (\gamma_2 - 1)] P_2 u_2 - [\gamma_1 / (\gamma_1 - 1)] P_1 u_1. \quad (15)$$

Replacing each flow parameter in the above equation by its corresponding mean value plus an acoustic perturbation,

$$\bar{\dot{q}} + \dot{q}' = [\gamma_2 / (\gamma_2 - 1)] (\bar{P}_2 + P'_2) (\bar{u}_2 + u'_2) - [\gamma_1 / (\gamma_1 - 1)] (\bar{P}_1 + P'_1) (\bar{u}_1 + u'_1). \quad (16)$$

Assuming that the mean variables express a particular balance, equation (16) can be rewritten as

$$\frac{\dot{q}'}{\bar{P}_2 c_2} = \frac{\gamma_2}{\gamma_2 - 1} \left(\frac{u'_2}{c_2} + \frac{P'_2 \bar{u}_2}{\bar{P}_2 c_2} + \frac{P'_2 u'_2}{\bar{P}_2 c_2} \right) - \frac{\gamma_1}{\gamma_1 - 1} \left(\frac{\bar{P}_1 u'_1}{\bar{P}_2 c_2} + \frac{P'_1 \bar{u}_1}{\bar{P}_2 c_2} + \frac{P'_1 u'_1}{\bar{P}_2 c_2} \right), \quad (17)$$

where all terms have been divided by $\bar{P}_2 c_2$ to show their relative importance. Except for flow velocity, the acoustic parameters are one order of magnitude lower than the corresponding mean parameters, and both the acoustic and mean flow velocities are one

order of magnitude lower than the speed of sound, in such a way that equation (17) is reduced to

$$\dot{q}' = [\gamma_2/(\gamma_2 - 1)]\bar{P}_2 u_2' - [\gamma_1/(\gamma_1 - 1)]\bar{P}_1 u_1'. \quad (18)$$

The relation between the mean pressures at each of the sides of the heat source is obtained by balance of momentum in the heat source section:

$$\bar{P}_1 + \bar{\rho}_1 \bar{u}_1^2 = \bar{P}_2 + \bar{\rho}_2 \bar{u}_2^2. \quad (19)$$

Because the gas is assumed to be an ideal gas, \bar{P} can be replaced by $\bar{\rho}c^2/\gamma$. Thus:

$$\bar{\rho}_1 c_1^2/\gamma_1 + \bar{\rho}_1 \bar{u}_1^2 = \bar{\rho}_2 c_2^2/\gamma_2 + \bar{\rho}_2 \bar{u}_2^2, \quad (20)$$

where the terms containing the square of flow velocity can be neglected in comparison with the terms containing the square of the speed of sound, yielding

$$\bar{P}_1 = \bar{P}_2 = \bar{P}. \quad (21)$$

Substituting the result of equation (21) in equation (18), the following is obtained for constant specific heats:

$$\dot{q}' = [\gamma/(\gamma - 1)]\bar{P}(u_2' - u_1'). \quad (22)$$

Substituting this result in equation (12) and integrating over a cycle of oscillation, the increment of acoustic energy becomes

$$\Delta E = \frac{\gamma - 1}{\gamma} \frac{A}{\bar{P}_1} \int_0^\tau \dot{q}' P' dt. \quad (23)$$

The quantities multiplying the integral on the right side of equation (23) are all positive, implying that the energy of the perturbation will grow if the integral is positive, and vice-versa. The inequality that represents Rayleigh criterion is therefore demonstrated in a simple manner.

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