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HUB SYNCHRONIZATION IN LARGE SCALE-FREE NETWORKS

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Recent results reveal that disparate real-world networked systems share universal structural features such as the scale-free property [1–3]. A scale-free network is characterized by a high heterogeneity in the node's degree – the degree of a node is the number of connections it receives. A few high-degree nodes, termed hubs, are present in the network while most nodes have only a few connections. The hubs are thought to serve specific purposes on their networks, such as information flow and resilience under attacks. They severely affect the dynamical processes taking place over scale-free networks [2, 3], particularly the emergence of a collective synchronized motion [4, 5].

Scale-free networks are more difficult to synchronize than random homogeneous networks [5]. The main reason is that scale-free networks are strong heterogeneous in the degree distribution. This suggests that, although structurally advantageous, the scale-free property would be dynamically detrimental. This poses a paradox, since many real-world networks whose functioning requires precise timing have evolved to scale-free-like topologies [1, 2].

In this letter, we show the existence of a partially synchronized state in large scale-free networks – hubs may undergo a transition to synchronization while most nodes behave in an unsynchronized matter. Global synchronization in large scale-free networks may not be a possible stable state and, in this scenario, only partially synchronized states is possible.

We consider a network compose of n nodes, and we label the nodes according to their degrees $k_1 \leq k_2 \leq \cdots \leq k_n$, where k_1 and k_n denote the minimal and maximal node degree, respectively. Hence, the ith node has degree k_i. A scale-free network is characterized by the degree distribution $P(k)$, the probability that a randomly chosen node within the network has degree k, that follows a power-law

$$P(k) = ck^{-\gamma},$$

for $k_1 \leq k_i \leq k_n$, where c is the normalization factor. The degree distribution is normalizable for $\gamma > 1$, and for large k_n we have $c \approx (\gamma - 1)k_1^{\gamma-1}$. The mean degree $\langle k \rangle$ attains a finite limit for large k_n provided $\gamma > 2$. We consider only networks with well defined mean degree, that is, $\gamma > 2$.

The dynamics of a general network of n identically coupled elements is described by

$$\dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{n} A_{ij}[E(x_j) - E(x_i)], \qquad (1)$$

here $x_i \in \mathbb{R}^m$ is the m-dimensional vector describing the state of the ith node (node with degree k_i), $F : \mathbb{R}^m \to \mathbb{R}^m$ governs the dynamics of the individual oscillator, $E : \mathbb{R}^m \to \mathbb{R}^m$ is the coupling function (without loss of generality assumed to be a constant matrix), σ is the overall coupling strength, and A is the adjacency matrix. A encodes the topological information of the network, defined as $A_{ij} = 1$ if nodes i and j are connected and $A_{ij} = 0$ otherwise, note that by definition $k_i = \sum_j A_{ij}$. We consider A to be symmetric.

The synchronized state defined as $x_1 = x_2 = \ldots = x_n = s$ is an invariant state of the system for all σ, its linear stability can be studied by analyzing the perturbations $\xi_i = x_i - s$. In the regime $\xi_i \ll 1$ the variational equations governing the perturbations read

$$\dot{\xi}_i = K_i(s)\xi_i + \eta_i, \qquad (2)$$

where the matrix $K_i(t; \alpha) = [DF(x_n(t)) - \alpha\mu_i E]$ depends continuously on t, DF stands for the Jacobian matrix of F, $\mu_i = k_i/k_n$ is the normalized degree, and $\eta_i = \frac{1}{k_n}\sum_j(A_{ij} - A_{nj})E(\xi_j)$ is the coupling term.

Neglecting the coupling term η_i the equations governing the evolution of the perturbations ξ_i and are decoupled. The linear stability of the homogeneous variational equation is given by its maximum Lyapunov exponent $\Lambda(\alpha\mu_i)$, which can be regarded as the *master stability function* of the system [4]. The perturbation ξ_i is damped out if $\Lambda(\alpha\mu_i) < 0$.

For many widely studied oscillatory systems the master stability function $\Lambda(\alpha\mu_i)$ is negative in an interval $\alpha_1 < \alpha\mu_i < \alpha_2$ for general coupling function E [4, 7]. The perturbation ξ_i is damped out if $\alpha_1 < \alpha\mu_i < \alpha_2$. Moreover, normalization imposes $\mu_n = 1$ and $\mu_1 \propto k_n^{-1}$, hence, as k_n increases, μ_1 converges to zero. Not only μ_1, but most of the normalized degrees μ_i will converge to zero. As a consequence, it will be impossible, for large k_n, to have $\alpha_1 < \alpha\mu_i < \alpha_2$ for all $i = 1, 2, \cdots, n$. Hence, in the thermodynamic limit no stable global synchronization is possible in scale-free networks.

Now take α so that $\alpha\mu_{n-1}$ falls into the stability region. Then, the state $x_n = x_{n-1}$ is linearly stable. This is true as long as we can neglect the coupling term η_i. Under the effect of η_i local mean field arguments show that $x_n \approx x_{n-1}$ is stable. The argument goes as follows. If $\Lambda(\alpha\mu_{n-1}) < 0$, we guarantee the linear stability of ξ_{n-1}. Moreover, if the remaining oscillators are not synchronized, the coupling term η_{n-1} can be viewed as a small noise, as long as the signals x_i are uncorrelated, with α fixed and and k_n large [6]. Basic results from ordinary differential equations state that the linear stability is maintained under small perturbations.

These arguments cannot be applied to low-degree nodes. The reason is that to set the low-degree nodes into the stability region we must have $\alpha\mu_1 \approx \alpha_1$, implying $\alpha \approx \alpha_1 k_n$. This requires large values for α. Hence, the coupling term cannot be made small for low degree nodes.

To illustrate this phenomenon we have generated a Barabási-Albert (BA) scale-free network with 3×10^3 nodes and $m = 3$ [3]. The network has largest degrees $k_n = k_{n-1} = 165$. Each node x_i is modeled as a Rössler oscillator, for $x_i = (x_{1i}, x_{2i}, x_{3i})^T$ we have $F(x_i) = (x_{2i} - x_{3i}, x_{1i} + 0.2x_{2i}, 0.2 + x_{3i}(x_{1i} - 7))^T$. We consider E to be a projector in the first component, i.e., $E(x, y, z)^T = (x, 0, 0)^T$. The master stability function $\Lambda(\alpha)$ has a stability region for $\alpha \in (\alpha_1, \alpha_2)$ with $\alpha_1 \approx 0.13$ and $\alpha_2 \approx 4.55$.

For $\alpha = 0.30$ we have observed the hub synchronization $x_n \approx x_{n-1}$. In Fig. 1(a) the time series x_n is depicted in full line while x_{n-1} is depicted in light gray line. As one can see in the times series $x_{n-1} \approx x_n$. In Fig. 1(b), we depict x_n in bold line while x_{2000} in light gray line.

In summary, we have analyzed the stability of a partially synchronized state – hubs exhibit a stable collective motion while the remaining nodes behave in an unsynchronized fashion. In our case, the stability of the hub synchronization is tailored into the analysis of the master stability function of the oscillator, and fact that in large scale-free networks coupling term acts as a small noise-like term on the hubs. We believe that these findings may serve as a paradigm to address issues regarding collective dynamics of realistic networked systems.

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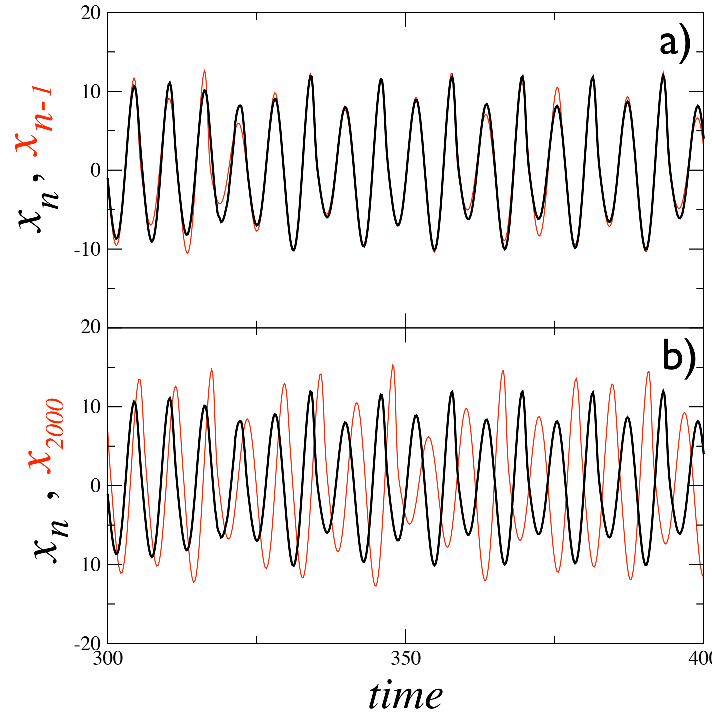

Figure 1 – [Color online] Hub synchronization for a BA scale-free network of 3000 coupled Rössler oscillators. (a) Time series of x_n (full line) x_{n-1} (light gray line). (b) ime series of x_n (full line) x_{2000}. The corresponding node degrees are $k_n = k_{n-1} = 165$ and $k_{2000} = 3$.

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[6] Without attempt at rigor, the local mean field argument is the following. First remember that $\eta_{n-1} = k_n^{-1}\sum_j (A_{(n-1)j} - A_{nj})E(\xi_j)$, since the oscillators are identical, chaotic and unsynchronized (at least for small values of α) once can think of ξ_j as identically distributed random numbers. For $k_n \gg 1$, by the center limit theorem $\eta_i = O(k_n^{-1/2})$.

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