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EFFECTIVE DYNAMICS FOR CHAOS SYNCHRONIZATION IN NETWORKS WITH TIME-VARYING TOPOLOGIES

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Abstract: A coupled map lattice, whose topology changes at each time step, is considered. We show that the transversal dynamics of the synchronized state can be studied by the introduction of effective dynamical quantities. We show that an ensemble of short time observations can be used to predict the long time behavior of the lattice. Finally, we point out that is possible to obtain a lattice with constant topology whose dynamical behavior is identical to one of the time-varying topologies.

keywords: Synchronization, chaotic dynamics, complex networks, time-varying coupling.

1. INTRODUCTION

There are, in the physical world, sets whose elements interact with each other. Neurons, individuals, Josephson junctions and computers are examples of elements that compound these sets, which we call networks. For each situation abovementioned, the topology is different, namely: regular networks, random networks, small-world networks, and scale-free networks. However, there is a common feature of these systems: the capability of achieving the synchronized state.

2. PURPOSE

In this work, we deal with networks whose topology varies with time. Based on the stability of the synchronized state we present exact results in order to predict the long time behavior of such networks. We hypothesized that the original time-varying network, for an observation period long enough, can be replaced by an effective system. Such a system is obtained by calculating a weighted average over a set of possible states.

3. METHODS

We examine a network called *coupled map lattice*, which is defined in the following way: let $g : \omega \to \omega$ be a chaotic onedimensional map defined in $\omega \subset \mathbb{R}$. Let $\mathbf{F} : \Omega \to \Omega$ be a lattice with N coupled maps in the form g, with $\Omega = \omega^N$. This lattice reads

$$\mathbf{y}_{n+1} = \mathbf{F}(\mathbf{y}_n; n) = \mathbf{G}_n \mathbf{f}(\mathbf{y}_n), \tag{1}$$

where $\mathbf{y}_n \doteq \left[\begin{array}{ccc} x_n^{(0)} & \cdots & x_n^{(N-1)} \end{array}\right]^T$, $x_n^{(m)}$ defines the state of the m-th site, $f^{(m)}(\mathbf{y}_n) = g(x_n^{(m)})$, and $\mathbf{G}_n$ is a $N \times N$ matrix that depends on discrete time n, in an explicit way [1]. This is the *coupling matrix*.

We shall restrict the lattice to the linear couplings and consider the lattice topology varying stochastically with time. However, the following conditions must be observed: (i) the lattice is a periodic one; and (ii) the topology in each time interval is invariant under translation in the lattice.

4. RESULTS AND DISCUSSION

From the conditions (i) and (ii) imposed over the lattice topology we can show that all but the first eigenvalues (η_m) of the time-n product of the coupling matrices approaches to zero as $n \to \infty$. We can define an effective time-1 eigenvalue to each eigendirection m as $\hat{\theta}_m = (\eta_m)^{1/n}$, *i. e.*, $\hat{\theta}_m$ as the mean decay rate of η_m. If we observe that the dynamics in the synchronized state is ruled by $g(s)$, and characterized by the Lyapunov exponent λ_U of the uncoupled map, the Lyapunov exponents' spectrum is then given by

$$\hat{\lambda}_m = \lim_{n\to\infty} \frac{1}{n} \ln \| \left(e^{\lambda_U} \hat{\theta}_m \right)^n \| = \lambda_U + \ln |\hat{\theta}_m|. \tag{2}$$

For the lattices we are considering, an exact expression for $\hat{\theta}_m$ can be written:

$$\hat{\theta}_m \quad = \quad \prod_{k=1}^{K_N} \left(\Gamma_m^{(k)} \right)^{\pi_k}, \tag{3}$$

in which the product extends over all K_N possible coupling matrices, $\Gamma_m^{(k)}$ is the m-th eigenvalue of the k-th coupling matrix and π_k is probability of such matrix, which depends only on the topology represented by the respective matrix.

Once the dynamics of the non-autonomous system (1) is ruled by effective quantities, it is natural to ask for an effective (or equivalent) system which retains the average behavior of the first one. Under the assumptions (i) and (ii) over the lattice topology, such a system is characterized by an effective coupling matrix given by

$$[\hat{\mathbf{G}}]_{qw} = \frac{1}{N}\left\{\hat{\theta}_0 + 2\sum_{s=1}^{N'}\hat{\theta}_s \cos\left(\frac{2\pi s(w-q)}{N}\right)\right\}, \quad (4)$$

which also satisfies the constraints (i) and (ii).

As an example, we consider a lattice of coupled tent maps, $g(s) = 1 - 2|s - 1/2|$, which are connected at each time step with probability $p_r = r^{-\alpha}$, being r the smallest inter-site distance between two maps. The coupling is assumed to be

$$[\mathbf{G}_n(\mathbf{T}_n)]_{qw} = \left([\mathbf{T}_n]_{qw} - \left(1+(1-\varepsilon^{-1})\nu_n\right)\delta_{qw}\right)\frac{\varepsilon}{\nu_n}, \quad (5)$$

in which $\nu_n = 2\sum_{r=1}^{N'}[\mathbf{T}_n]_{0r}$ is normalization factor according to condition $\Omega = \omega^N$ and $[\mathbf{T}_n]_{qw}$ is 1 or 0 if the sites q and w are linked or not at time n. Here, ε and α quantifies the coupling strength and probability range, respectively.

We study the synchronization by considering the temporal evolution of Euclidean distance of typical trajectories to synchronization subspace, d_n. We illustrate how trajectories reach the synchronized state in Figure 1 for the lattice with time-varying coupling (dashed lines) and its equivalent version (solid lines), obtained by Eq. (4). The convergence to the synchronized state is not monotonic in first case, because, at a given instant n, it is possible that each site receives only the influence of a few distant sites, which are in very different states, causing a departure from such state. On the other hand, the same mechanism may explain the decrease in the synchronization time for the case in which the states are very close together.

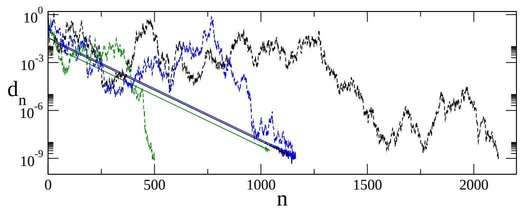

Figure 1 – Temporal evolution of the distance to synchronization subspace for typical trajectories for the time-varying topologies (dashed lines) and for the equivalent system (solid lines), with $N = 31$, $\varepsilon = 0.7$ and $\alpha = 0.6$.

We must emphasize that typical trajectories of the system, whose topology is time-varying, presents an average behavior that is determined by the equivalent static system, for any parameter set and lattice size. Such a result is shown in Fig. 2, where we analyze the average time required for synchronization of a set of typical trajectories. The average synchronization time, t_s, increases with α and becomes divergent at $\alpha_c(\varepsilon, N)$ – value in which the synchronization state loses its transversal stability. The α_c, obtained imposing $|\hat{\theta}_1| = e^{-\lambda_U}$ in Eq. (3) [2], is indicated by the dashed vertical lines. To evidence that the equivalence between both systems is attained only for the average behavior of several trajectories, we also analyze the synchronization times dispersion, given by the standard deviation of simulations and indicated in Figure 2 by the vertical (horizontal) bars for the time-varying (equivalent static) coupling.

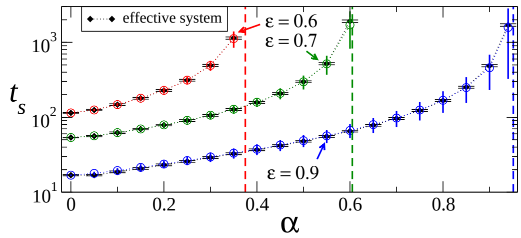

Figure 2 – The dependence on the range of coupling probability of the average synchronization times, for the time-varying system (circles) and the effective static one (diamonds), and its standard deviation (vertical and horizontal bars, respectively), for $N = 31$. The vertical dashed lines are the critical values α_c for which $\mathcal{S}$ loses its transversal stability.

5. CONCLUSIONS

We studied the chaos synchronization in a lattice of coupled chaotic maps whose connectivity between sites changes stochastically in time. Under the assumptions of periodicity and translational invariance of lattice topology we derive an exact analytical expression for the Lyapunov exponents' spectrum of the synchronized state. Such exponents are obtained by the introduction of effective quantities defined as weighted averages over all possible topologies. We show that the effective Lyapunov numbers of synchronized state can be found by the average decay rate to zero of the product of coupling matrix eigenvalues. From such quantities we construct an *deterministic autonomous* system which presents exactly the same average behavior of the time-varying one. This agreement is illustrated by analyzing the transition to the synchronization, characterized by the stability of the synchronized state, as well as the average time necessary for typical trajectories to reach the synchronization subspace.

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