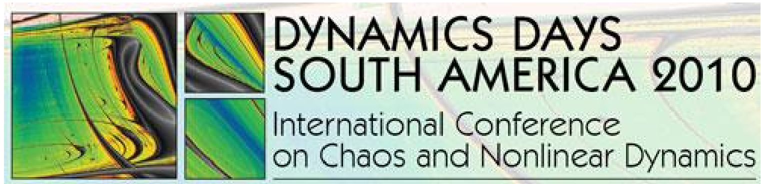

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THE METHOD OF SEPARATION OF VARIABLES FOR THE FROBENIUS-PERRON OPERATOR FOR A CLASS OF TWO DIMENSIONAL CHAOTIC MAPS

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Abstract: Analytical expressions for the invariant densities for a class of discrete two dimensional chaotic systems are given. The method of separation of variables for the associated Frobenius-Perron equation is introduced. Hence, a two dimensional chaotic system can be decoupled in two chaotic one dimensional systems. These systems are related to nonlinear difference equations which are of the type $x_{n+2} = T(x_n)$. The function T is a chaotic map of an interval whose chaotic behaviour is inherited to the two dimensional one. The efficacy of the method appears to be independent of the hyperbolicity of the map T, i.e. if the map display full chaos or intermittency. We work out in detail some examples, including some three and higher dimensional cases, in order to expose the method.

keywords: chaotic dynamics, Frobenius-Perron equation, invariant measures, nonlinear delayed difference equations.

1. INTRODUCTION

Explicit computations of invariant densities for higher dimensional maps are seldom found. In contrast, in one dimensional dynamics there are lots of exactly solvable chaotic maps, which can be generated e.g., using the conjugation property or by means of the Schröder method. The invariant density of a chaotic map is an eigenfunction of the Frobenius Perron operator induced by it. The associated eigenvalue problem is given by a functional equation but there are no known general methods to solve it. The methods used in the field of functional equations have not yet provided a general way to solve them, instead one can find a lot of operational methods adapted to each class of equations. Therefore, to find reliable methods to obtain these densities becomes a very important task. In this work a methodology is developed to solve the Frobenius-Perron functional equation, which roughly consists in introducing the method of separation of variables for it. The method presented in this work is called separation of variables, because of its resemblance with the traditional one used in differential equations. We use several examples to show how it works. Our paper contain examples of maps defined by Jacobi elliptic functions, rational functions, an example from duopoly Cournot game and from the Newton method. Also, we give a generalization of the map in example 1 (see below). Higher dimensional maps are associated to a class of delayed difference equations. Moreover, the method also works for an intermittent map, e.g. $T = 1 - 2\sqrt{|x|}$. The paper will contain about 31 references and here we only list some of them.

2. THE FROBENIUS-PERRON EQUATION

Let us consider a higher dimensional nonlinear transformations $F : U \subseteq \mathbb{R}^d \to \mathbb{R}^d$, and denote $F(X) = (f_1(X), f_2(X), \ldots, f_d(X))$ for $X = (x_1, x_2, \ldots, x_d) \in U \subseteq \mathbb{R}^d$. We are interested in the study of the discrete dynamical system $X_{n+1} = F(X_n)$. We assume that F is, at least, a C^1 function. Also, that there exists a set of functions $\phi_j : U \subseteq \mathbb{R}^d \to \mathbb{R}^d$, such that $F(\phi_j(X)) = X$, for $j = 1, 2, \ldots, d$. In other words, $\phi_j(X)$ is the $j - th$ branch of the inverse function of F. The Frobenius Perron operator associated to F is defined by:

$$\mathcal{L}_F[\rho(X)] = \sum_{F(Y)=X} \frac{\rho(Y)}{|\det D_Y F|} = \rho(X), \qquad (1)$$

where the Jacobian matrix $D_Y F = [\partial F_i / \partial x_j]$ is evaluated at Y. The scalar function supported on a given $U \subseteq \mathbb{R}^d$, denoted by ρ, which solves the functional equation is called the invariant density for F.

2.1. *Two dimensional maps*

For any $(x, y) \in \mathbb{R}^2$, the first order nonlinear system of difference equations: $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, defines a discrete dynamical system. We assume that there exists an invariant set $\Omega \subset \mathbb{R}^2$, and the dynamics in Ω being chaotic.

For the sake of clarity we begin our exposition with the simplest case. Now, for any $(x, y) \subset \mathbb{R}^2$ we consider the system: $x_{n+1} = y_n$, $y_{n+1} = T(x_n)$, where $T : I \to I$ with $I \subseteq \mathbb{R}$, is a map satisfying conditions stated in the preceding section. Also, we assume that the set $\Omega = I \times I$ is invariant. Now, the Frobenius-Perron equation associated to it is given by

$$\mathcal{L}[\rho(x, y)] = \sum_{j=1}^{r} \frac{\rho(T_j^{-1}(y), x)}{\left|T_j' \circ T_j^{-1}(y)\right|} = \rho(x, y). \qquad (2)$$

We can see that this functional equation can be solved by the *ansatz* $\rho(x,y) = A(x)B(y)$, whereby $\mathcal{L}[\rho(x,y)] = \mathcal{L}_T[A(y)]B(x) = A(x)B(y)$. The method of separation of variables for the equation (2) works as follows:

$$\frac{\mathcal{L}_T[A(y)]}{B(y)} = \frac{A(x)}{B(x)} = c, \tag{3}$$

where we denote by c the constant of separation, such that $\mathcal{L}_T[A(y)] = cB(y)$, $A(x) = cB(x)$. Note now that this is a system of two coupled functional equations. To proceed to decoupling them we make the substitution of the second equation in the first one. Hence, we obtain $\mathcal{L}_T[A(y)] = A(y)$. In other words, if the map T has an invariant density A, we must finally have $\rho(x,y) = \frac{1}{c}A(x)A(y)$.

Now we consider two different transformations, $S : J \to J$ and $T : I \to I$, both of them satisfying conditions stated above, but each one having r and s monotone pieces respectively. Therefore, T^{-1} and S^{-1} have r and s monotone branches on their respective intervals. Then, defining a discrete dynamical system on $(x,y) \in \Omega = I \times J$ by $x_{n+1} = S(y_n)$, $y_{n+1} = T(x_n)$, the associated Frobenius Perron operator is given by

$$\mathcal{L}[\rho(x,y)] = \sum_{j=1}^{r}\sum_{l=1}^{s} \frac{\rho(T_j^{-1}(y), S_l^{-1}(x))}{\left|T' \circ T_j^{-1}(y)\right| \left|S' \circ S_l^{-1}(x)\right|}. \tag{4}$$

By analogy with to the previous section we introduce the *anzats* $\rho(x,y) = A(x)B(y)$, which allows us to have $\mathcal{L}_T[A(y)] = cB(y)$, $\mathcal{L}_S[B(x)] = \frac{1}{c}A(x)$. Now, this system can be decoupled giving us $\mathcal{L}_{S\circ T}[A(x)] = A(x)$ and $\mathcal{L}_{T\circ S}[B(y)] = B(y)$. In other words, the system is decoupled in two functional equations of a single variable. More important, this time the functions $A(x)$ and $B(x)$ are (if they exists) the invariant densities of $S \circ T$ and $T \circ S$ respectively. We finally obtain $\rho(x,y) = A_{S\circ T}(x)B_{S\circ T}(y)$.

2.2. *Example 1*

An example of this last case is provided by a system studied in Gardini et al (1996):

$$\begin{aligned} x_{n+1} &= (y_n^2 - x_n - 1)^2 - (y_n^2 + x_n - 1)^2, \\ y_{n+1} &= \sqrt{(y_n^2 - x_n - 1)^2 + (y_n^2 + x_n - 1)^2}. \end{aligned} \tag{5}$$

In that reference the authors shows that this system is exactly solvable one. We compute its invariant density. After making the change of variables (this one is different from that employed by them) $x_n = (\eta_n - \xi_n)/2$ and $y_n^2 = 1 + (\eta_n + \xi_n)/2$ we obtain the system $\xi_{n+1} = 2\eta_n^2 - 1$, $\eta_{n+1} = 2\xi_n^2 - 1$, which is clearly of the form of the equation given above. Hence, the (unnormalized) density is $\rho(\xi,\eta) = 1/\sqrt{(1-\xi^2)(1-\eta^2)}$. Now, the $|\det(J)| = |4y|$, then the invariant density in the variables x, y is computed by $\rho(\xi,\eta)d\xi d\eta = \rho(x,y)\,|4y|\,dxdy$, such that

$$\rho(x,y) \propto \frac{4y}{\sqrt{(1-(y^2-x-1)^2)(1-(y^2+x-1)^2)}}. \tag{6}$$

The argument in its denominator can be rewritten as $\left(2-(y^2-x)\right)\left(y^2-x\right)\left(2-(y^2+x)\right)\left(y^2+x\right)$. After some algebra, we find that the domain of that density is given by the set $\Omega = \{(x,y) \in \mathbb{R}^2\}$ such that the following inequalities (also given in the reference cited above) $y^2 - x < 2,\ y^2 - x > 0,\ y^2 + x < 2,\ y^2 + x > 0$, are valid. Hence, this is the invariant set for the system. Since these inequalities are satisfied for $y > 0$, the absolute value in equation (6) becomes irrelevant.

3. DELAYED NONLINEAR DIFFERENCE EQUATIONS

In recent times the study of nonlinear difference equations has become an intensive area of study from the point of view of dynamical systems. In our paper we give some examples of invariant densities associated to the delayed difference equation: $x_{n+k} = T(x_n)$ where $k \in \mathbb{N}$, for a selected maps T.

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