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Nontwist Bouncing Ball Map

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The bouncing ball problem consists of a particle under action of a uniform gravitational field which suffers repeated inelastic impacts on a sinusoidally vibrating plate at the floor as shown in **figure 1**. This problem was first studied by Holmes [1] and investigated by many other authors, theoretically (e.g. [2]) and experimentally (e.g. [3]). The particle motion between consecutive impacts is integrable, however the instant of the impacts can not be obtained analytically. The resulting map presents a transcendental equation that must be solved at each iteration in order to find the instant of impacts [4].

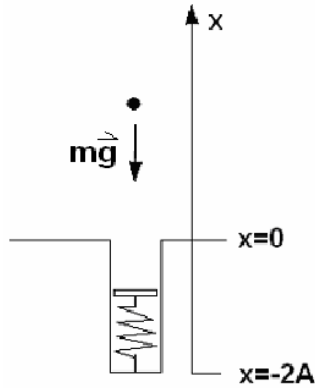

Figure 1 – Schematic view of the bouncing ball system

A simplified map could be obtained assuming the approximation proposed by Lieberman and Lichtenberg [5] and consists in considering that the wall is fixed but that, after the collision, the particle suffers an exchange of energy and momentum as if the wall were moving. The travel time between the k-th and the ($k+1$)-th impacts is $t_k=v_k/g$ for the simplified model, where v_k is the particle velocity just after the k-th impact and g is the gravitational acceleration. We define the phase of the moving floor at the k-th impact as $\varphi_k=\omega t_k$ where ω is the angular frequency of floor oscillation. The map for the simplified bouncing ball model is written as

$$\begin{cases} \varphi_{k+1} = \varphi_k + 2\omega v_k/\mathrm{g} \qquad \bmod 2\pi \\ v_{k+1} = r v_k - Ar\omega \,\mathrm{sen}(\varphi_{k+1}) \end{cases}, \qquad (1)$$

where A is the floor oscillation amplitude and r is the restitution coefficient. The parameter r gives the fraction of relative velocity between the particle and the moving floor just before the impact that will be restituted to the movement just after the impact. For conservative impacts $r=1$.

For A=0 (unperturbed case) and r=1 (conservative case) all the initial conditions lead to periodic movements. The dependence between the natural frequencies of movement and its velocity after each impact is monotonic and given by $\omega_0=2\pi/(2t_k)=\pi g/v_k$.

We modify the simplified bouncing ball system considering the effect of a ceiling at the height L from the floor. A investigation of the modified bouncing ball model was made by Leonel in [6]. For the unperturbed case the movements still periodic, but the dependence between the frequencies and impact velocity have a discontinuity. This discontinuity occurs for the impact velocity in which the kinetic energy on the floor is equal to the gravitational potential energy at height L. We refer to this value as “discontinuity velocity” and is given by $v_d= \sqrt{2gL}$. It separates the dynamics of the system in two regimes: the movements with impacts only at the inferior limit and with impacts at both limits.

The natural frequencies of the movement for the modified model is given by

$$\omega_0 = \frac{\pi g}{v_0 - \Theta(v_0 - \sqrt{2gL})\sqrt{v_0^2 - 2gL}}, \qquad (2)$$

where Θ is the Heaviside function. This dependence is depicted in **figure 2** for $L=1.54m$ and $g=9.8\ m/s^2$. For this parameters we have $v_d=5.5m/s^2$.

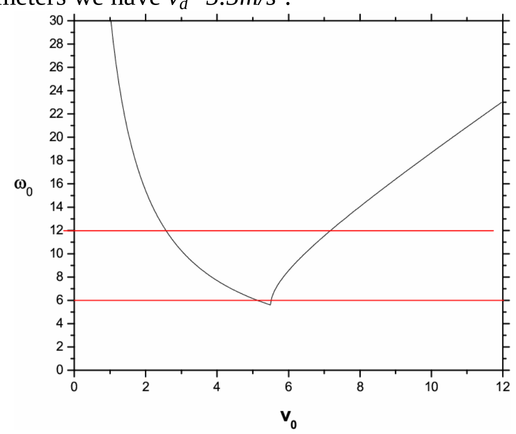

Figure 2 – Dependence of the between natural frequency of the unperturbed movement and the impact velocity. The red

lines indicate the natural frequencies ω_0=6 rads/s and 12 rad/s.

We can be observe in **figure 2** that the natural frequencies of the integrable system are degenerated. It means that exist pairs integrable torus with the same rotational number. This aspect violates the twist condition and do not satisfy the assumption of the KAM theorem. Nontwist maps and their transport properties were extensively studied for the continuous dependence on the frequencies [7], [8].

Adding the perturbing effect of the oscillating floor, the map equations for the nontwist bouncing ball map is written as

$$\begin{cases} \varphi_{k+1} = \varphi_k + \dfrac{2\omega}{g}\left(v_k - \Theta(v_k - \sqrt{2gL})\sqrt{v_k^2 - 2gL}\right) \qquad \text{mod } 2\pi \\ v_{k+1} = rv_k - Ar\omega \,\text{sen}(\varphi_{k+1}) \end{cases} \tag{3}$$

We use the resonance frequency value ω=6 rads/s. The Poincaré maps for the conservative case are depicted in **figures 3 (a)-(c)** where we vary the perturbation parameter *A=0.04m*; *0.08m* and *0.12m.*

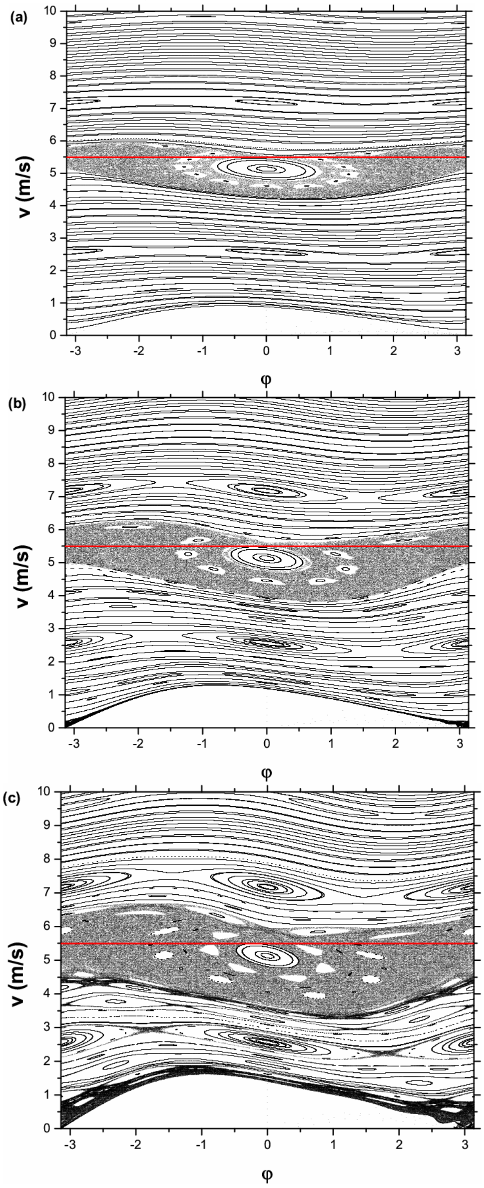

Figure 3 – Poincaré maps for the conservative nontwist bouncing ball map for the parameters ω=6 rads/s and (a) A=0.04m; (b) A=0.08m and (c) A=0.12m.

The red line in **figures 3 (a)-(c)** is for reference and indicate the value of v_d separating the two dynamical regimes.

We also consider a second restitution parameter r_2 related to the energy dissipation only on the ceiling impact. The map now becomes

For $v_k < \sqrt{2gL}$

$$\begin{cases} \varphi_{k+1} = \varphi_k + 2\omega v_k/\text{g} \qquad \text{mod}[2\pi] \\ v_{k+1} = rv_k - Ar\omega \,\text{sen}(\varphi_{k+1}) \end{cases}$$

For $v_k > \sqrt{2gL}$

$$\begin{cases} \varphi_{k+1} = \varphi_k + \dfrac{\omega}{g}\left(v_k + \sqrt{v_k^2 r_2^2 + 2gL(1-r_2^2)} - (1+r_2)\sqrt{v_k^2 - 2gL}\right) \qquad \text{mod}[2\pi] \\ v_{k+1} = r\sqrt{v_k^2 r_2^2 + 2gL(1-r_2^2)} - Ar\omega \,\text{sen}(\varphi_{k+1}) \end{cases} \tag{4}$$

With this consideration the dissipation occurs only in the second dynamical regime. There are coexistence of conservative chaos, chaotic attractor and conservative islands in the same phase space as can be observed in **figure 4.**

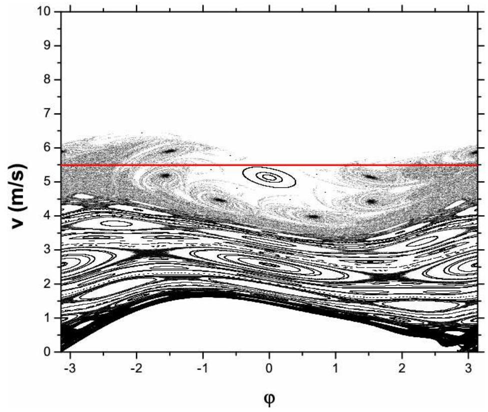

Figure 4 - Poincaré map for the nontwist bouncing ball map for the parameters r_2=0.98; ω=6 rads/s and A=0.12m.

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