A Method for the Determination of the Current Distribution in Tokamaks

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1. Introduction

The current density distribution in the plasma determines important characteristics of a tokamak discharge, like the ohmic heating distribution and the safety factor profile. It was shown by Christiansen and Taylor that the current distribution can be determined from purely geometric information about the shape of the magnetic surfaces [1] . Hence, assuming that in a tokamak the flux and constant temperature surfaces coincide, the current distribution can be determined experimentally using soft X-ray imaging methods [2] .

In a recent paper concerning the calculation of the external inductance in tokamaks [3] it was argued that the current distribution can be determined, in principle, from the Cauchy boundary conditions at the plasma edge, that is, from the geometric information about the shape of the last flux surface and the distribution, on the edge, of the magnetic field due to external sources. The link between the internal current distribution and the boundary conditions is given by an integral consistency condition that must be satisfied by the current density for any solution of the homogeneous Grad-Schlüter-Shafranov equation.

In this paper a general method is derived to determine the current distribution by using a truncated set of integral consistency conditions defined for a finite number of homogeneous solutions of the Grad-Schlüter-Shafranov equation. The current density is given in terms of parametrized representations of the flux surfaces and of the total toroidal current through the cross-section of each flux surface. The free parameters in these representations are calculated in order to satisfy the consistency conditions, one condition for each parameter. In this way, the current distribution is reconstructed, according to a given parametrized representation, in terms of the boundary conditions at the plasma edge only. As pointed out by Christiansen and Taylor [1] , these methods of reconstruction based on geometrical information are more suitable for small aspect ratio tokamak equilibria with strongly non-circular cross-sections.

2. Integral consistency condition

The flux function Φ in a system with axial symmetry gives the magnetic flux between a given flux surface and the symmetry axis. It is related to the toroidal current density j_T inside a volume V by Ampère's law

$$\Delta^2\Phi = R^2\nabla\cdot(R^{-2}\nabla\Phi) = -2\pi\mu_0 R j_T, \tag{1}$$

where R is the distance to the symmetry axis. When j_T is given in terms of Φ by magnetohydrodynamics considerations, equation [1] leads to the Grad-Schlüter-Shafranov equation for Φ. At the plasma boundary for an ideal tokamak equilibrium the flux function satisfies the Dirichlet condition

$$\Phi = \Phi_0 - \Phi_P(a), \tag{2}$$

where Φ_0 is the flux enclosed by the magnetic axis and $\Phi_P(a)$ is the poloidal flux between the magnetic axis and the plasma boundary, which is denoted by the constant topological radius a. The gradient of Φ, normal to the plasma boundary, is related to the toroidal component of the equivalent surface current density K_T by the Neumann condition

$$\hat{n} \cdot \nabla\Phi = -2\pi\mu_0 R K_T. \tag{3}$$

Now, consider a general solution ψ of the homogeneous equation $\Delta^2\psi = 0$ in V. Multiplying equation [1] by ψ/R^2, the homogeneous equation for ψ by Φ/R^2, subtracting and integrating over the total volume $V(a)$ of the plasma, one obtains the integral consistency condition

$$\frac{1}{2\pi}\iiint_{V(a)} \left(\frac{j_T}{R}\right)\psi\, d^3r = \frac{1}{2\pi}\iint_{S_P(a)} \left[\left(\frac{K_T}{R}\right)\psi + \left(\frac{\Phi_0 - \Phi_P(a)}{2\pi\mu_0 R^2}\right)\hat{n}\cdot\nabla\psi\right]_a d^2r(a), \tag{4}$$

where $d^2r(a)$ is the differential area element on the boundary surface $S_P(a)$. This consistency condition, which must be satisfied for any solution ψ, can be used to determine the internal current distribution in the plasma for the over specified Cauchy boundary conditions.

3. Integral boundary conditions

For ψ=constant, equation [4] leads to the consistency condition of the Neumann problem, which is simply the integral equivalence of the volume and surface current densities in terms of the total toroidal current $I_T(a)$,

$$\iint_{S_T(a)} j_T d^2r(\zeta) = \oint_{\ell_P(a)} K_T d\ell(\theta) = I_T(a), \tag{5}$$

with $d^2r(\zeta)$ denoting the differential area element in the coordinate surface ζ = constant and $d\ell(\theta)$ the differential arc length along the coordinate curve θ in a flux coordinate system (ρ, θ, ζ). The area of the cross-section of the plasma for ζ = constant is denoted by $S_T(a)$ and the poloidal perimeter of the plasma by $\ell_P(a)$.

Now, introducing the Green's function G for a toroidal ring current,

$$\Delta^2 G(\vec{r}, \vec{r}\,') = -2\pi R \delta^2(\vec{r} - \vec{r}\,'), \tag{6}$$

the vector analogue of Green's theorem [4] allows to write the flux function Φ_{int}, due to sources enclosed by a given flux surface of radius ρ, in terms of the equivalent surface current density

$$\Phi_{int}(\vec{r}) = \mu_0 \oint K_T(\vec{r}\,')G(\vec{r}, \vec{r}\,')d\ell'(\theta). \tag{7}$$

Inside the plasma the equivalent surface current K_T at the plasma boundary, taken with the opposite sign, produces a magnetic field that coincides with the field produced by the external sources. This is a consequence of the vector analogue of Green's theorem and is equivalent to the principle of virtual casing [5] . Therefore, the flux produced by the external sources inside the plasma can be written as

$$\Phi_{ext}(\vec{r}) = \Phi_0 - \Phi_P(a) - \mu_0 \oint_{\ell_P(a)} K_T(\vec{r}\,'(a))G(\vec{r}, \vec{r}\,'(a))d\ell'(\theta). \tag{8}$$

The constant of integration was chosen so that, at the plasma boundary, the sum of equations [7] and [8] satisfies the Dirichlet condition [2].

Finally, the integral form of Ampère's law relates the toroidal plasma current contained by the flux surface ρ to the radial derivative of the poloidal flux function

$$I_T(\rho) = \frac{1}{\mu_0}\oint \vec{B_P}\cdot d\vec{r}(\theta) = \frac{K(\rho)}{\mu_0}\frac{d\Phi_P}{d\rho}, \tag{9}$$

where $K(\rho)$ stands for the geometrical factor

$$K(\rho) = \frac{1}{2\pi}\oint \frac{|\nabla\rho|}{R} d\ell(\theta). \tag{10}$$

Hence, the poloidal flux and the surface current at the plasma edge are given respectively by

$$\Phi_P(a) = \mu_0 \int_0^a \frac{I_T(\rho)}{K(\rho)} d\rho, \qquad K_T(a,\theta) = \left(\frac{|\nabla\rho|}{2\pi R}\right)_a \frac{I_T(a)}{K(a)}. \tag{11}$$

4. Reconstruction of the current density

In the flux coordinate system the current density distribution for a scalar plasma pressure equilibrium is given in terms of the toroidal current $I_T(\rho)$ and pressure $p(\rho)$ profiles by

$$j_T(\rho,\theta) = \frac{\mu_0 dI_T/d\rho}{2\pi R dL/d\rho} - \frac{K(\rho)}{\mu_0 I_T(\rho)}\left(2\pi R - \frac{\mu_0 dV/d\rho}{2\pi R dL/d\rho}\right)\frac{dp}{d\rho}, \tag{12}$$

where the volume $V(\rho)$ enclosed by a magnetic surface ρ and the inductance $L(\rho)$ of the toroidal solenoid corresponding to this surface are geometry-dependent quantities defined by

$$V(\rho) = 2\pi \iint_{S_T(\rho)} R d^2r(\zeta), \qquad L(\rho) = \frac{\mu_0}{2\pi}\iint_{S_T(\rho)} \frac{d^2r(\zeta)}{R}. \tag{13}$$

The evaluation of $V(\rho)$, $L(\rho)$ and $K(\rho)$ involves poloidal angle averages over θ that can be calculated analytically using a spectral representation $[R(\rho,\theta), Z(\rho,\theta)]$ for the flux surfaces in cylindrical coordinates [6] . In this way the dependence of $j_T(\rho,\theta)$ on the poloidal angle θ is given explicitly through the distance $R(\rho,\theta)$ only. The quantities $V(\rho)$ and $L(\rho)$ depend on ρ through the Fourier coefficients in the spectral representation, whereas $K(\rho)$ depends on the values of these Fourier coefficients as well as their radial derivatives due to the dependence on $|\nabla\rho|$. The strong variations of R and $|\nabla\rho|$ make the method particularly suited for small aspect ratio tokamaks. Truncated Taylor series around the magnetic axis provide convenient polynomial approximations for the Fourier coefficients, with an arbitrary number of parameters corresponding to the degree of the polynomials. For the pressure and toroidal current profiles any convenient parametrized functions of ρ^2 can be utilized. In the present problem one assumes that the pressure profile is specified, while the parametrized toroidal current profile is to be determined, from the external information, in conjunction with the geometry of the internal flux surfaces. Notice that the Neumann condition [5] is automatically satisfied by the representation [12] for the current density.

Now, the flux produced by the actual external sources is given, in general, by the superposition of the fluxes due to an ideal magnetizing transformer, a uniform vertical equilibrium field and an arbitrary number of circular current loops

$$\Phi_{ext}(\vec{r}) = \Phi_M + \pi R^2 B_{vert} + \mu_0 \sum_k I_k G(\vec{r}, \vec{r_k}). \tag{14}$$

Comparing the expression of the flux inside the plasma due to the ideal [8] and actual [14] external sources, one obtains the general condition of the Dirichlet problem

$$\begin{aligned}&\Phi_0 - \Phi_P(a) - \frac{\mu_0 I_T(a)}{2\pi K(a)} \oint_{\ell_P(a)} \left(\frac{|\nabla\rho|}{R}\right)'_a G\left[\vec{r}(a), \vec{r}'(a)\right] d\ell'(\theta) = \\ &\Phi_M + \pi R(a)^2 B_{vert} + \mu_0 \textstyle\sum_k I_k G\left[\vec{r}(a), \vec{r_k}\right]\end{aligned} \tag{15}$$

Introducing the major radius $R_0(a)$ of the plasma torus and taking the poloidal angle average $\langle\ldots\rangle_\theta$ of [15] the flux balance equation follows

$$\Phi_0 - \Phi_M = \Phi_P(a) + L_e I_T(a) + M\pi R_0^2(a) B_{vert} + \sum_k M_k I_k, \tag{16}$$

where the external inductance L_e of the plasma is defined by

$$L_e = \frac{\langle \Phi_{int}\left[\vec{r}(a)\right]\rangle_\theta}{I_T(a)} = \frac{\mu_0}{2\pi K(a)} \oint_{\ell_P(a)} \left(\frac{|\nabla\rho|}{R}\right)'_a \langle G\left[\vec{r}(a), \vec{r}'(a)\right]\rangle_\theta d\ell'(\theta), \tag{17}$$

and the mutual inductance coefficients M between the plasma and the vertical field, and M_k between the plasma and the current loops are defined respectively by

$$M = \frac{\langle R^2(a)\rangle_\theta}{R_0^2(a)}, \qquad M_k = \mu_0 \langle G\left[\vec{r}(a), \vec{r_k}\right]\rangle_\theta. \tag{18}$$

As pointed out in the Introduction, one assumes that the shape of the plasma boundary and the applied external field are known quantities. The first step in the reconstruction of the internal current distribution is to determine the free parameters at the plasma edge that satisfy the Dirichlet condition [15]. These free parameters are the values, at the plasma edge, of the radial derivatives of the Fourier

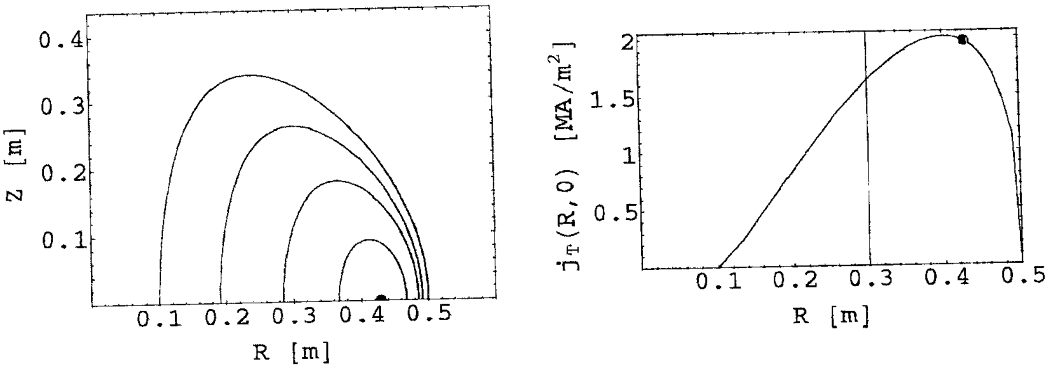

Figure 1: Reconstructed flux surface countours and current density in the equatorial plane.

coefficients in the spectral representation of the flux surfaces. Furthermore, the flux at the plasma boundary $\Phi_0 - \Phi_P(a)$ is assumed to be a free parameter that must be adjusted for a given value of the plasma current $I_T(a)$. In the calculations involving the Dirichlet condition the minor radius a of the plasma and the plasma current $I_T(a)$, as well as μ_0, can be taken as normalization variables. Possibly, the simplest method to adjust the parameters is by matching moments in a multipolar expansion of the condition [15], with the introduction of as many moment matching equations as the number of free parameters [7]. Then, the values of the external and mutual inductance coefficients in equations [17] and [18], which depend only on the geometrical parameters at the plasma edge, can be calculated. The value of $\Phi_0 - \Phi_P(a)$, obtained by matching moments, should be consistent with the required value of $I_T(a)$ substituted in the flux balance equation [16].

The next step in the reconstruction procedure is to substitute the geometrical parameters at the plasma boundary in the integral consistency condition [4], in order to determine the free internal parameters in the current density distribution. At least two Fourier coefficients remain to be determined at this point, which are the position of the magnetic axis and the elongation on the axis. Additional free parameters are used to adjust the flux surface averaged toroidal current profile $I_T(\rho)$. Again, as many consistency equations as dictated by the number of free parameters can be set up using homogeneous solutions ψ_m of increasing order m of the Grad-Schlüter-Shafranov equation. In the present paper homogeneous solutions in toroidal coordinates were used to implement the method.

Finally, the poloidal flux at the plasma edge $\Phi_P(a)$ can be calculated using the first of equations [11], allowing the evaluation of the poloidal flux Φ_0 enclosed by the magnetic axis. The difference between Φ_0 and Φ_M corresponds to the flux necessary to increase the plasma current from zero to its final equilibrium value (not including resistive losses). At this point the reconstruction of the current distribution is completed and all the equilibrium quantities, like the internal inductance, current beta, current diamagnetism parameter, plasma beta and safety factor profile are determined.

Figure 1 shows an example of the current distribution determined for an ETE (Experimento Tokamak Esférico) plasma. The given equilibrium parameters are: major radius=0.30m, minor radius=0.20m, elongation=1.7, triangularity=0.3, plasma current=200kA, pressure on the magnetic axis=6kPa and vertical equilibrium field=0.113T. The safety factor is 1.00 at the magnetic axis and 11.3 at the plasma edge for a toroidal magnetic field of 0.6T.

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