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Marco Antonio Raupp
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AUTOR RESPONSÁVEL
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Horacio H. Yanassé

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Nei Yoshihiro Soma
Horacio Hideki Yanasse

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SUMMARY

A PLANAR SOLUTION PROCEDURE FOR LINEAR DIOPHANTINE EQUATIONS

Nei Yoshihiro Soma
ITA/CTA - SJCampos - SP - Brazil

Horacio Hideki Yanasse
INPE/MCT - SJCampos - SP - Brazil

ABSTRACT

In this paper we present a planar procedure for solving linear diophantine equations based on the calculation of the rank of the Mignosi's matrix. We also suggest a procedure for improving the bound on a linear diophantine problem of Frobenius.

RESUMO

Neste trabalho apresenta-se um procedimento planar de resolução de equações diofantinas lineares baseado no cálculo do posto da matriz de Mignosi. Também sugere-se um procedimento para melhorar o limitante do problema de Frobenius em equações diofantinas lineares.

1. INTRODUCTION

The diophantine equations appeared with Diophantus 2000 years ago and deal with the integer solution of the equation

$$\sum_{J=1}^{N} A(J).\ X(J)^{C(J)} = B$$

where A(J), C(J) and B, are integers.

Several interesting problems are derived from this general equation:

a) the famous Fermat's last theorem - "Are there three natural numbers such that the equation $X(1)^N + X(2)^N = B^N$ is satisfied, $N \geq 3$, $N \in \mathbb{N}$?";

b) the Goodbach's conjecture - "Are there even numbers greater or equal to 4 that cannot be expressed as the sum of two prime numbers?".

We analyse here the linear diophantine equation (LDE), that is, "is there a natural N-tuple (X(1), X(2),..., X(N)) such that

$$\sum_{J=1}^{N} A(J)\ X(J) = B,\ A(J),\ B \in \mathbb{N},\ J=1,2,\ldots,N?". \tag{1}$$

This particular problem appears in a letter by Leibnitz to Bernoulli in 1669 and has been the focus of study of several famous mathematicians like Gauss, Cauchy, Silvester, Hardy, Ramanujan and others.

In practical settings, LDEs appear in several models in a great number of situations (see, for instance, Kluyver and Salkin, 1975), therefore, there is a great interest in solving this problem in an efficient manner.

To solve this problem Gilmore and Gomory (1966) use a dynamic programming recursion that requires an O(B) of memory requirements and an O(NB) of computational time. Recently, Yanasse and Soma (1985) presented an algorithm for the unidimensional knapsack problem that has an improved performance as compared with the dynamic programming methods.

In the present work we are interested solely whether equation (1) has or does not have a solution. The LDE is NP-complete (see Garey and Johnson, 1979), so a polynomial algorithm probably does not exist unless P=NP.

To find a solution for (1) we present a pseudopolynomial algorithm that, in the worst case, the computational time is limited to $O(N(B-A(1)) - \sum_{J=1}^{N} A(J))$ and the memory requirements is $O(B-A(1))$ where it is assumed $A(1) \leq A(J)$ for all J. This algorithm is based on observations made from the Mignosi's matrix (Mignosi, 1980).

2. THE ALGORITHM

We assume, without loss of generality, that our data is already sorted, that is, $0<A(1)<A(2)<\ldots<A(N)$. We also assume that there is no A(J), J=1,2,...,N such that $B/A(J) \in \mathbf{N}$, otherwise the solution to (1) is trivial. Also, we can assume that $B \geqq A(1)+A(N)$ for, otherwise, we can reduce our problem to one with N-1 variables since $X(N)=0$.

Mignosi (1908) stated that the number of solutions of a LDE is given by

$$\eta_B = \frac{1}{B!}\left|\det\begin{bmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(B-1) & \sigma(B) \\ -B+1 & \sigma(1) & \dots & \sigma(B-2) & \sigma(B-1) \\ 0 & -B+2 & \dots & \sigma(B-3) & \sigma(B-2) \\ 0 & 0 & \ddots & \dots & \dots \\ \dots\dots & & -2 & \sigma(1) & \sigma(2) \\ 0 & \dots & 0 & -1 & \sigma(1) \end{bmatrix}\right|$$

where $\sigma(t) = \sum_{j \in J} A(j)$ if $A(j)$ divides t and

$J = \{1,2,\dots, N\}$; and zero, otherwise.

In fact, η_B is integer and will be different than zero if and only if the rank of M is B, where

$$M \triangleq \begin{bmatrix} \sigma(1) & \sigma(2) & \dots\dots\dots\dots & \sigma(B-1) & \sigma(B) \\ -B+1 & \dots\dots\dots\dots & & & \sigma(B-1) \\ & \ddots & & & \vdots \\ & 0 & & \ddots & \sigma(2) \\ & & & -1 & \sigma(1) \end{bmatrix}$$

Our problem reduces to the determination of the rank of M. Here we propose to perform elementary column operations and try to make column B, the last column of matrix M, all zero.

Notice that M has a very special structure. All elements above the main diagonal is nonnegative and each column has at most one negative element. By adding a nonnegative linear combination of the columns of M to column B, we can make it all zero, with the exception perhaps of its first element, m'_{1B}, obtained after all these operations. Only if $m'_{1B}>0$, the rank of M is B, otherwise rank(M) = B-1.

It is important to observe that only the signs of m_{ij}, the elements of M, are sufficient to determine the rank of M. Consider matrix R with elements r_{ij} such that

$$r_{ij} = \begin{cases} 1 & \text{if} \quad m_{ij}>0 \\ 0 & \text{if} \quad m_{ij}=0 \\ -1 & \text{if} \quad m_{ij}<0 \end{cases} \qquad \begin{array}{l} \\ i = 1,\ldots,B \\ j = 1,\ldots,B \end{array}$$

Then,

<u>Theorem 1</u>: rank(M) = rank(R).

<u>Proof</u>: To zero the $j^{\underline{th}}$ element of column B of matrix M or R, we can add to it a positive multiple of column j-1 since matrix's M or matrix's R j-th element in column j-1 is negative. Since each column of M or R has at most one negative element, after this elementary column operation, what was positive in column B of matrix M or R remains positive (except for the $j^{\underline{th}}$ element), what becomes positive in column B of matrix M becomes positive in column B of matrix R, what remains zero in M, remains zero in R. So, rank(M)=B if and only if rank(R)=B. Since rank(M)≥B-1 and rank(R)≥B-1 then rank(M) = rank(R).

The proof of theorem 1 provide us with the following observation:

<u>Corollary 1</u>. If the first element of column B becomes positive after anyone of the elementary column operations in the process of zeroing a j-th element of column B of M (or R), then rank(M) = rank(R) = B.

Theorem 1 also shows that when calculating the rank of M, any operation that preserves the sign can be performed.

If we had to build matrix M with all the $\sigma(t)$'s greater than zero, this would require computing for each t, t=1,...,B whether A(j) divides t, j=1,...,N. For matrix R, although simpler, still we will have to check if A(j) divides t for at least one j, j=1,...N. In the proof of theorem 1 it became apparent that a simpler matrix can be considered for the rank's calculation. Consider matrix S having elements s_{ij} such that

$$s_{ij} = \begin{cases} -1 & \text{for} \quad i=j+1 \\ 1 & \text{for} \quad i=j-A(k)+1 \qquad k=1,\dots,N \\ 0 & \text{otherwise} \end{cases}$$

$$i=1,\dots,B.$$
$$j=1,\dots,B.$$

This matrix is, in general, sparse and it is related to R in the sense that only a few subset of the diagonals that are different than zero in R are present in S.

We will prove that.

<u>Theorem 2</u>: rank(S) = rank(R).

Proof. We know that rank(R) is B if and only if

$$\sum_{J=1}^{N} A(J)\ X(J) = B \text{ for some } X(J) \geq 0 \text{ integer. Otherwise, } \operatorname{rank}(R) =$$

B-1. Also, rank(S)≥B-1 and rank(R)≥rank(S). This latter inequality can be proved if we observe that in the process of zeroing the last column of S we add to it, linear positive combination of columns of S. The same combination can be used in R. Since the columns of R are lexicographically greater or equal to the columns of S, those elementary column operations preserve this ordering.

Assume rank(R)=B. Then $B = \sum_{J=1}^{N} m_J\ A(J)$ for some $m_J \varepsilon\ \mathbb{Z}^+$, J=1,...,N.

In S, we can add to column B, the following columns: if $m_1 \geq 1$, take column B-A(1)-1, column B-2A(1)-1,..., column B-m_1 A(1)-1; if $m_2 \geq 1$, take column B-m_1A(1)- A(2)-1, column B-m_1A(1)-2A(2)-1,..., column B-m_1A(1)-m_2A(2)-1;...; if $m_N \geq 1$, take column B-m_1A(1)-m_2A(2)- ... - m_{N-1}A(N-1) - A(N)-1, column B-m_1A(1)-m_2A(2)- ... -m_{N-1} A(N-1)-2A(N) - 1; ..., column B-m_1A(1)-m_2A(2)-...-m_{N-1}A(N-1)-(m_N-1) A(N)-1.

Since $B = \sum_{J=1}^{N} m_J\ A(J)$, the column B-$m_1$A(1)-$m_2$A(2)-...-($M_N$-1) A(N)-1 = A(N)-1. But this column has its first element $s_{1,\ A(N)-1} = 1$. Since the last column B remains non-negative when we add all these columns and the first element of column B becomes positive, the rank(S)=B and we are done.

The planar procedure we suggest follows the steps that are performed to determine the rank of S. Consider the following example:

$$5x_1 + 7x_2 + 9x_3 = 32, \quad x_1,x_2,x_3 \varepsilon \mathbf{N}. \tag{2}$$

Is there a solution to equation 2?

Our procedure starts building a square grid of size B where diagonals in the positions corresponding to A(1), A(2),..., A(N) are drawn. Also, a guideline which is a secondary diagonal is drawn as shown is Figure 1.

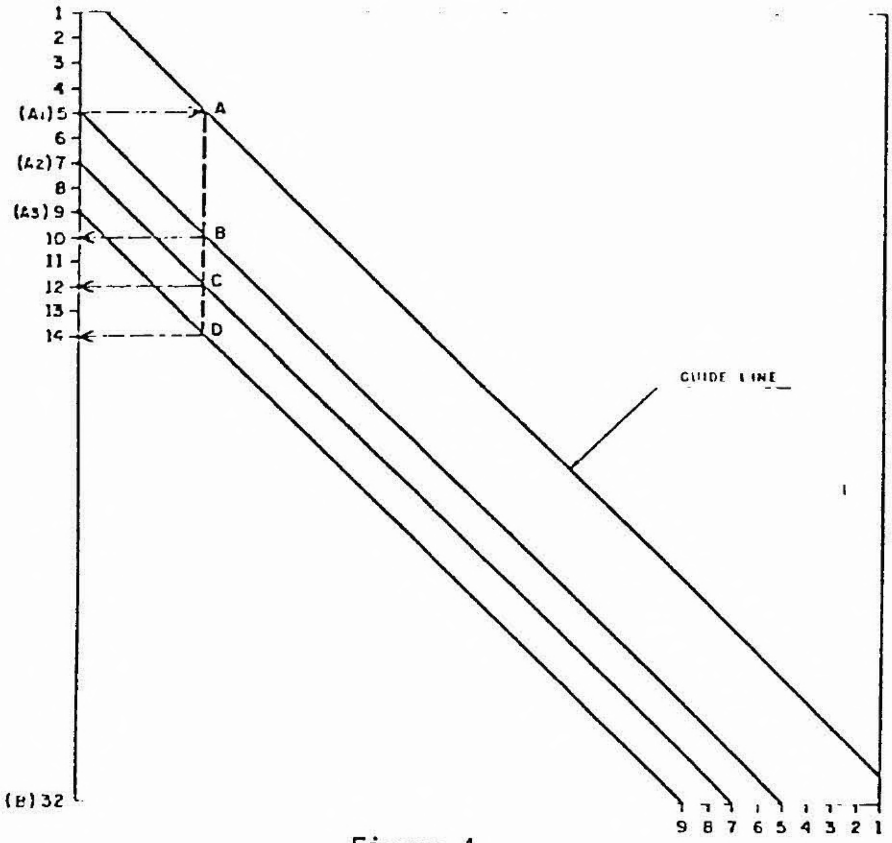

Figure 1.

We mark with a black dot the initial positions A(1),..., A(N).

Starting from the first black dot from the top left, we draw a horizontal line that crosses the guideline at A (see Figure 1). From A, we draw a vertical line that crosses the diagonal lines at

B, C and D, respectively. From B, C, D we draw horizontal lines that cross the vertical scale at 10, 12 and 14 for this particular example. So, these positions are marked with black dots too and they indicate values for which equation 1 has a solution when the right hand-side equals that value. Notice that what we do is analog to an elementary column operation in S.

We proceed to the immediately next black dot and perform these same operations. This is schematized in Figure 2.

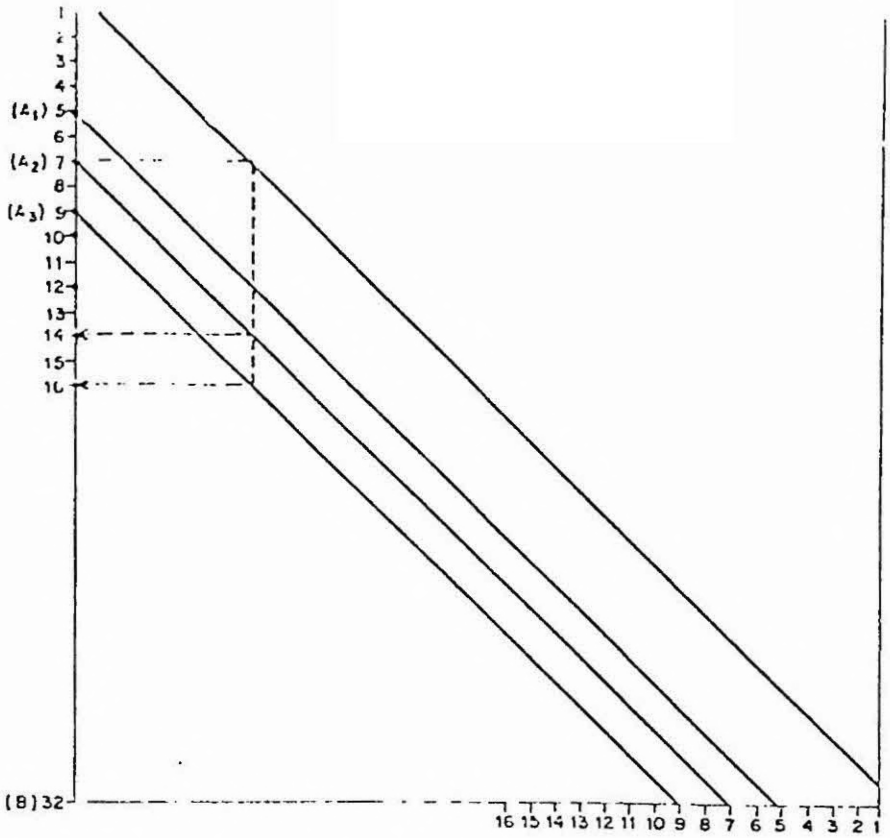

Figure 2.

After a few iterations, we arrive at the position shown in Figure 3 indicating in this example that the equation 2 has a feasible solution.

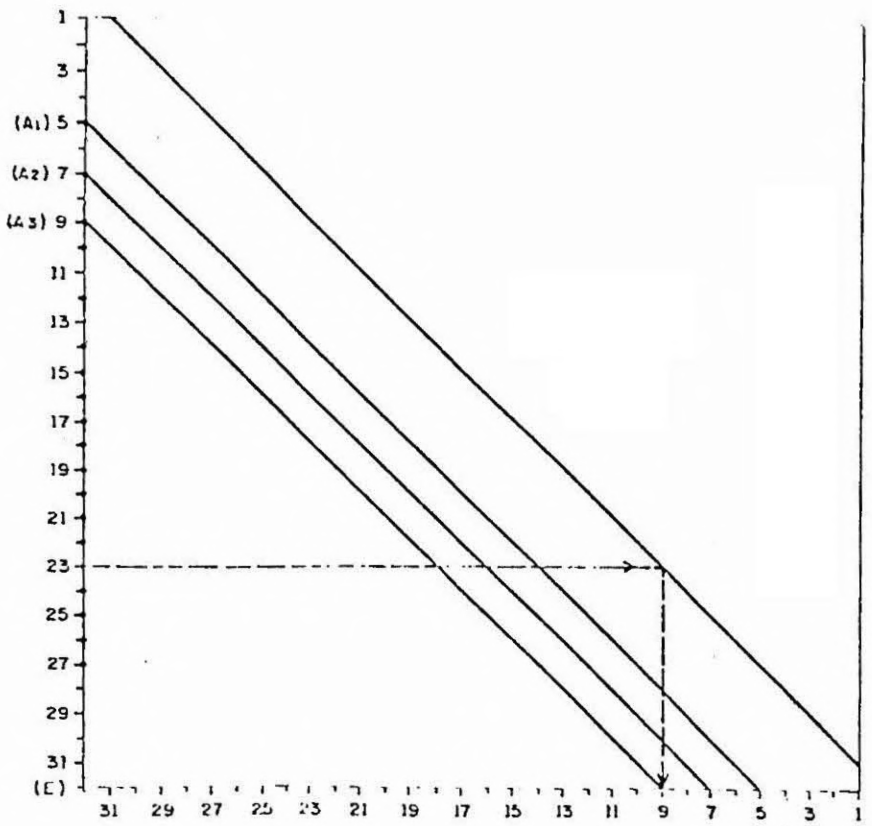

Figure 3.

As can be seen from the previous example, the problem was reduced to one in the plane. We only need to work with a grid of size B, draw diagonals corresponding to A(1), A(2),...,A(N) and mark dots conveniently, according to some specified rules. Observe that these operations can be done for any linear diophantine equation of the form (1), which is quite interesting.

3. OBSERVATIONS

The procedure just presented is of $O(N(B-A(1) - \sum_{J=1}^{N} A(J))$ as indicated in Yanasse and Soma (1985). The fact that in this case we are not carrying an objective function does not modify the performance of the algorithm. There are some variations that can be suggested at this point, for instance, making black dots in this horizontal scale in correspondence with the black dots in the vertical scale. A black dot in the vertical scale means that there is a solution for the linear diophantine equation with that right-hand-side. A black dot in the horizontal scale means that there is a solution to the linear diophantine equation from the black dot point to B. Hence, when one draws the vertical line from a point in the guideline and one hits a black dot in the horizontal scale, one can stop immediately. In Figure 4 we illustrate what we would achieve with this variation.

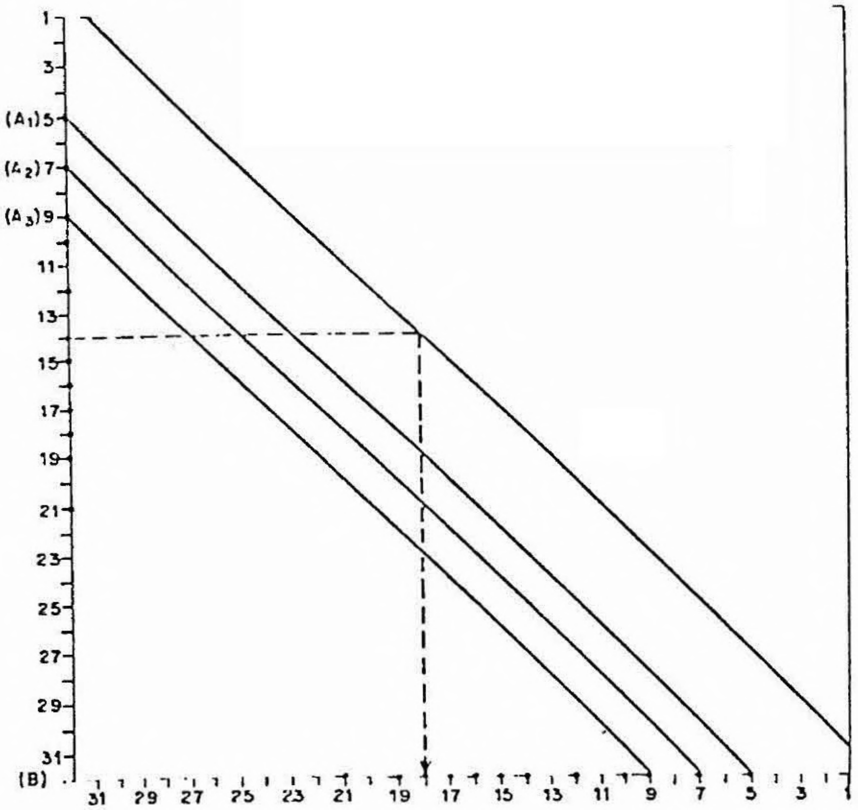

Figure 4.

Notice that if by any chance the number of black dots in the vertical scale reaches $\lfloor B/2 - A(1)/2 \rfloor + 1$ then we can stop. The linear diophantine equation 1 has solution. This can be seen by symmetry. Observe that in the horizontal scale we will have also $\lfloor B/2 - A(1)/2 \rfloor + 1$ black dots which implies that at least one vertical line drawn from a point P on the guideline obtained by a horizontal line from a black dot in the vertical scale will hit one of the horizontal black dots.

It may be worth using a sufficient test based on Mendelsohn, 1970, to check whether (1) has a solution. Consider the equation

$$Ax + Dy = B. \tag{3}$$

If the greatest common divisor of A and D, GCD (A,D), is equal to 1 and $(A-1)(D-1) \leq B+1$, then (3) has at least one non-zero natural solution.

Let $M_{IJ} \triangleq GCD(A(I), A(J))$. If there exists at least one pair of indices I and J, $I, J \in \{1,...,N\}$, $I \neq J$ such that M_{IJ} divides B and $\dfrac{(A(I)-M_{IJ})\,(A(J)-M_{IJ})}{M_{IJ}} \leq B + M_{IJ}$ then $A(I)X(I)+A(J)X(J)= B$ has at least one nonzero natural solution (by Mendelsohn, 1970), hence, $\sum_{J=1}^{N} A(J)\,X(J) = B$ has a solution. One strategy in applying this test would be to choose the pairs (I, J) in such a way that A(I).A(J) is non-decreasing.

It is convenient to observe that if M_{IJ} does not divide B then $A(I)X(I)+A(J)X(J)=B$ has no solution. In the case the previous sufficient test is not satisfied, either one of the following cases must occur:

a) for all pair (I,J), $I,J \in \{1,...,N\}$, $I \neq J$, M_{IJ} does not divide B; or

b) there is a pair (I,J), $I, J \in \{1,...,N\}$, $I \neq J$ where M_{IJ} divides B and $(A(I)-M_{IJ})\,(A(J)-M_{IJ})/_{MIJ} > B+M_{IJ}$.

In case (a), the LDE has no solution with only two variables different than zero. This suggests that we should consider at least three variables at a time in trying to determine whether (1) has a solution. This approach is not explored further in this present work.

In case (b), it appears that a pseudopolynomial algorithm which is polynomial in B might perform well since B is not relatively large compared with some coefficients of equation (1).

Concerning the Mendelsohn bound, in the next section we explore the planar procedure in an attempt to find a stronger bound on a linear diophantine problem of Frobenius.

4. IMPROVING BOUNDS ON A LINEAR DIOPHANTINE PROBLEM OF FROBENIUS

It is known that if the GCD (A(1),...,A(N))=1, then (1) has at a least one natural solution for suficiently large values of B(see, for instance, Grosswald, 1962).

The problem of determining the smallest integer above which equation 1 has always a solution, or at least, getting non trivial estimates, appeared for the first time with Frobenius and has been the focus of attention of several researchers (e.g., Bateman, 1958; Brauer, 1942; Heap and Lynn, 1964; Johnson, 1966, Erdös and Graham, 1972).

Let G(A(1),...,A(N)) be the smallest bound above which equation (1) has always at least one solution. Assume, without loss of generality, that GCD (A(1),...,A(N))=1. It is known that for A(1)X(1)+A(2)X(2)=B, $G(A(1), A(2) \leq G_M(A(1), A(2)) = (A(1)-1)(A(2)-1)-1$, as given by Mendelsohn, 1970; for A(1)X(1)+A(2)X(2)+ ... +A(N)X(N)=B, $G(A(1), (A2),...,A(N)) \leq G_B(A(1),..., A(N)) = (A(1)-1)\ (A(N)-1)-1$, as given by Brauer, 1942 or $G(A(1), A(2),...,A(N)) \leq G_{EG}(A(1),...,A(N)) = 2A(N-1). \left\lfloor \frac{A(N)}{N} \right\rfloor - A(N)$, as given by Erdös and Graham, 1972, where $\lfloor x \rfloor$ means the greatest integer smaller or equal to x.

We present next a procedure that might improve these bounds, in some cases. Let equation (1) with GCD(A(I), A(J)) = 1

for all I and J, I, Jε{1,...,N}, I$\neq$J. Let LG(A(1),...,A(N),B) $\triangleq$
$$\sum_{J=1}^{N} \left\lfloor \frac{B-A(1)}{A(J)} \right\rfloor - \left\lfloor \frac{B-2A(1)}{2} \right\rfloor - 1.$$

Theorem 3. If LG(A(1),...,A(N),B)$\geq$0 and B<A(1)A(2), then the LDE(1) has at least one solution.

Proof. Follows immediately from theorem 2 and the symmetry argument given in Section 3.

Consider the following illustrative example:

$$5X(1) + 7X(2) + 11X(3) + 13X(4) + 17X(5) = B \quad (4)$$

where the GCD (A(I), A(J)) = 1, I$\neq$J, I, Jε{1,2,3,4,5}.

The best bound for this problem is

G_M = (5-1) (7-1)-1=23, given by the Mendelsohn bound. But LG(5,7,11,13,17,22)=1>0. Therefore, equation 4 has at least one solution for all B$\geq$22. In fact, using theorem 3, we would get that LG(5,7,11,13,17,L)$\geq$0 for L=15,16,..., 23, hence, equation 4 has at least one solution for all B$\geq$15. Notice that, in this example for B=14 we have also a solution but this could not be obtained by the result on theorem 3.

In searching for an improved bound, we have to determine the lowest value L for which LG(A(1),...,A(N),B)$\geq$0 for B=L, L+1,..., G_{MIN} where G_{MIN} equals the minimum of the known bounds. This bound L may be smaller than the ones already known, as in the example shown. Unfortunately, it is not true that if

LG(A(1),...,A(N),M)$\geq$0 for some M, then LG(A(1),...,A(N),K)$\geq$0 for all K$\geq$M. Hence, a clever strategy has to be derived in order to determine the value of L. One possible way for doing this would be to keep the remaining of the division in each one of the terms in the expression of LG to compute the changes as we decrease B, from $G_{MIN}-1$, by the minimum of these remainings.

The requirement that the GCD(A(1), A(J)) = 1, for all I, J$\varepsilon\{1,\ldots,N\}$, $I \neq J$ can be relaxed and the procedure adjusted accordingly with an increased calculation effort.

5. FINAL COMMENTS

We present here a planar solution procedure for solving linear diophantine equations. The procedure is an enumeration scheme and has special features that may be explored further. For instance, at each elementary column operation in matrix S, we have to perform N additions (and comparisons if we carry also a criterion). Since these operations are independent, this might suggest that they can be done in parallel in an adequate multiprocessor computer. We will have, hence, an improved factor of N in performance. In the case where a simple YES or NO answer is required, an O(A(N)) of memory requirements implementation can be easily made. Also, recall that if the A(J)'s are prime two by two, then $B<(A(1)-1)\ (A(2)-1)-1$ and the pseudopolynomial algorithm presented should perform quite well.

Finally, the case of explicit bounds on the integer variables (e.g., 0-1 variables) can be handled by adequate modifications on the algorithm.

In this case, the enumeration is made accordingly with the tree schematized in Figure 5.

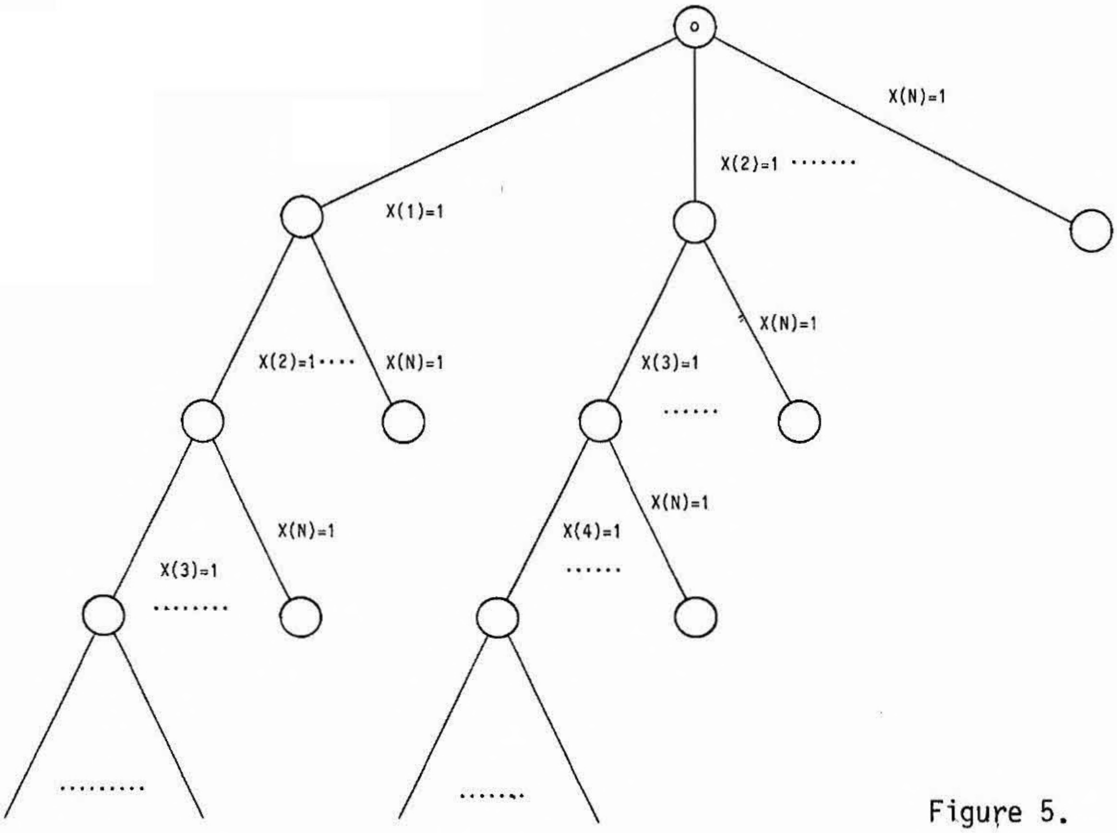

Figure 5.

What differs here from other traditional enumeration methods is the order that the nodes are visited on this tree.

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