CONSTRUCTIVE DECOMPOSITION OF FUZZY MEASURES IN TERMS OF POSSIBILITY OR NECESSITY MEASURES

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Abstract. Based on previous Mathematical Morphology results on the decomposition of mappings between complete lattices, constructive decompositions of fuzzy measures by intersection of possibility measures or union of necessity measures are presented.

1. Introduction

The search for uncertainty models culminates at the end of the seventies with the paper by Zadeh [1] on the *Theory of Possibility*. In Banon [2] we have a whole panorama of definitions and relationships between the main subsets of fuzzy measures used to model uncertainty.

Among the most popular models are the one based on possibility measures and its dual version based on necessity measures. These models are interesting because, as pointed out by Nguyen [3], they can be characterized by distributions (or, equivalently, by fuzzy sets).

In this paper, one more advantage is pointed out. It is shown that one can decompose (or represent) *any* fuzzy measure as a conjunctive combination of possibility measures *or* as a disjunctive combination of necessity measures. Actually, these decompositions appear to be a special case of the decomposition of mappings between complete lattices presented by Banon & Barrera [4, 5] in the framework of Mathematical Morphology applied to image processing.

As part of our main result, it is interesting to note that the possibility measures involved in the decomposition of a probability satisfy the *possibility/probability consistency principle* and that the combination of weak sources of information modeled by possibility measures may represent *exactly* a stronger source of information modeled by a probability. This observation can be interesting as part of the debate between probability and fuzzy sets [6, 7].

In Section 2, we recall the important contribution of Achache [8] to derive the different ways to characterize the possibility and necessity measures. In Section 3, we introduce two decompositions results extracted from Mathematical Morphology. Finally, in Section 4, we give an example of a probability decomposition in terms of possibility measures.

2. Axiomatic Definition and Characterization of Possibility and Necessity Measures

Let Ω be a nonempty set and I the interval $[0, 1]$ of real numbers.

A *fuzzy measure* μ on $\mathcal{P}(\Omega)$ or simply $\mathcal{P}$ (the collection of all subsets of Ω) is an increasing (isotone) mapping from $(\mathcal{P}, \subset)$ to $(I, \leq)$ (where $\subset$ is the inclusion between sets and $\leq$ is the usual order relation between real numbers) such that $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$ [9].

Since $(\mathcal{P}, \subset)$ and $(I, \leq)$ are complete lattices, any fuzzy measure satisfies both of the equivalent statements

$$\max \mu(\mathcal{H}) \leq \mu(\bigcup \mathcal{H}) \quad (\mathcal{H} \subset \mathcal{P}) \tag{1}$$

$$\mu(\bigcap \mathcal{H}) \leq \min \mu(\mathcal{H}) \quad (\mathcal{H} \subset \mathcal{P}), \tag{2}$$

where $\mu(\mathcal{H})$ denotes the image of $\mathcal{H}$ through μ.

If (1) remains true by replacing $\leq$ with $=$ then μ is called a *possibility measure* [1]. If it is (2) that remains true then μ is called a *necessity measure* [10].

Possibility (resp., necessity) measures commute with union (resp., intersection) and are called *dilation* (resp., *erosion*) in *Mathematical Morphology* [11, 4, 12].

We denote by Π the set of possibility measures and by N the set of necessity measures.

If $\pi \in \Pi$ and $\nu \in N$, then we must have $\pi(\emptyset) = 0$ and $\nu(\Omega) = 1$; but we may have $\pi(\Omega) \neq 1$ and $\nu(\emptyset) \neq 0$. When $\pi(\Omega) = 1$ and $\nu(\emptyset) = 0$, these measures are said to be *normalized*.

Let $U \in \mathcal{P}$ and $v \in I$, the mappings $\pi_{U,v}$ and $\nu_{U,v}$ from $\mathcal{P}$ to I given by

$$\pi_{U,v}(X) = \begin{cases} 0 & \text{if } X = \emptyset \\ v & \text{if } X \subset U \text{ and } X \neq \emptyset \quad (X \in \mathcal{P}) \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

and

$$\nu_{U,\nu}(X) = \begin{cases} 1 & \text{if } X = \Omega \\ \nu & \text{if } U \subset X \text{ and } X \neq \Omega \quad (X \in \mathcal{P}) \quad (4) \\ 0 & \text{otherwise,} \end{cases}$$

are, respectively, examples of possibility measure [2] and necessity measure. The latter is called *simple support function* by Shafer [13].

Let $\underset{(I)}{\leq}$ or simply $\leq$ the partial order between mappings from $\mathcal{P}$ to I, induced by $\leq$ on I, that is, given by

$$\mu_1 \leq \mu_2 \Leftrightarrow (\mu_1(X) \leq \mu_2(X) \quad (X \in \mathcal{P})).$$

We know [8] that $(\Pi, \leq)$ and $(N, \leq)$ are complete lattices.

By rewriting Proposition 2 of Achache [8], we have that the set of Galois connections (G.c.) [14] between $(\mathcal{P}, \subset)$ and $(I, \geq)$ is the graph of a dual lattice–isomorphism between Π and $E(I, \mathcal{P})$, the set of erosions from $(I, \leq)$ to $(\mathcal{P}, \subset)$. In the same way, the set of Galois connections between $(\mathcal{P}, \supset)$ and $(I, \leq)$ is the graph of a dual lattice–isomorphism between N and $\Delta(I, \mathcal{P})$, the set of dilations from $(I, \leq)$ to $(\mathcal{P}, \subset)$.

The former set of G.c. (α, β) forms a complete lattice with respect to the partial order given by

$$(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \Leftrightarrow \alpha_1 \underset{(I)}{\geq} \alpha_2 \text{ and } \beta_1 \underset{(\mathcal{P})}{\leq} \beta_2,$$

where $\underset{(\mathcal{P})}{\leq}$ is the partial order induced by $\subset$ on $\mathcal{P}$.

The latter set of G.c. (α, β) forms a complete lattice with respect to the partial order given by

$$(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \Leftrightarrow \alpha_1 \underset{(I)}{\leq} \alpha_2 \text{ and } \beta_1 \underset{(\mathcal{P})}{\geq} \beta_2.$$

The set I^Ω of mappings from Ω to I forms a complete lattice with respect to the partial order given by

$$f_1 \leq f_2 \Leftrightarrow (f_1(\omega) \leq f_2(\omega) \quad (\omega \in \Omega)).$$

Depending on the context, the mappings from Ω to I are called *possibility distributions* [1] or *necessity distributions*.

Let $\cdot^c$ be the complementation on sets. By rewriting Corollary 4 of Achache [8], we have the following propositions.

Proposition 1 (characterization of possibilities) – The mapping $f \mapsto (\pi, \beta)$ from the complete lattice I^Ω of possibility distributions on Ω to the complete lattice of G.c. between $(\mathcal{P}, \subset)$ and $(I, \geq)$, defined by

$$\pi(X) = \max f(X) \quad (X \in \mathcal{P}) \tag{5}$$

and

$$\beta(t) = \{\omega \in \Omega : f(\omega) \leq t\} \quad (t \in I),$$

is a dual lattice–isomorphism. Its inverse $(\pi, \beta) \mapsto f$ is given by

$$f(\omega) = \pi(\{\omega\}) \quad (\omega \in \Omega)$$

or equivalently

$$f(\omega) = f_\beta(\omega) \triangleq \min\{t \in I : \omega \in \beta(t)\} \quad (\omega \in \Omega).$$
□

Proposition 2 (characterization of necessities) – The mapping $f \mapsto (\nu, \alpha)$ from the complete lattice I^Ω of necessity distributions on Ω to the complete lattice of G.c. between $(\mathcal{P}, \supset)$ and $(I, \leq)$, defined by

$$\nu(X) = \min f(X) \quad (X \in \mathcal{P}) \tag{6}$$

and

$$\alpha(t) = \{\omega \in \Omega : t \leq f(\omega)\}^c \quad (t \in I),$$

is a lattice–isomorphism. Its inverse $(\nu, \alpha) \mapsto f$ is given by

$$f(\omega) = \nu(\{\omega\}^c) \quad (\omega \in \Omega)$$

or equivalently

$$f(\omega) = f_\alpha(\omega) \triangleq \max\{t \in I : \omega \notin \alpha(t)\} \quad (\omega \in \Omega).$$
□

Proposition 1 (resp., 2) leads to two different characterizations for a possibility measure π (resp., necessity measure ν) on $\mathcal{P}(\Omega)$.

The first one, given by (5) (resp., (6)) shows that any π (resp., ν) can be expressed in terms of a mapping (distribution) from Ω to I.

The second one can be derived from the fact that (π, β) (resp., (ν, α)) is a G.c.; then, for any $X \in \mathcal{P}$,

$$\pi(X) = \pi_\beta(X) \triangleq \min\{t \in I : X \subset \beta(t)\} \tag{7}$$

(resp.,

$$\nu(X) = \nu_\alpha(X) \triangleq \max\{t \in I : \alpha(t) \subset X\}). \tag{8}$$

That is, any π (resp., ν) can be expressed in terms of an erosion in $E(I, \mathcal{P})$ (resp., a dilation in $\Delta(I, \mathcal{P})$).

We observe that the erosions in $E(I, \mathcal{P})$ are increasing families of subsets of Ω indexed by I and having Ω as greatest element. The dilations in $\Delta(I, \mathcal{P})$ are increasing families having $\emptyset$ as least element.

3 Fuzzy measure decomposition

Let μ be a fuzzy measure on $\mathcal{P}(\Omega)$. By observing that for any $U \in \mathcal{P}$, there exists a $\pi \in \Pi$ (resp., $\nu \in N$) such that $\mu \leq \pi$ (resp., $\nu \leq \mu$) and $\pi(U) \leq \mu(U)$ (resp., $\mu(U) \leq \nu(U)$), namely $\pi = \pi_{U,\nu}$ (expression (3)) (resp., $\nu = \nu_{U,\nu}$ (expression (4))), with $\nu = \mu(U)$, one can prove the following lemma in the same way as for Lemma 6.1 of Banon & Barrera [4].

Lemma 3 (fuzzy measure decomposition) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \inf\{\pi \in \Pi : \mu \leq \pi\}$$
$$= \sup\{\nu \in N : \nu \leq \mu\}. \qquad \square$$

In other words, any fuzzy measure can be decomposed or represented in terms of an intersection of normalized possibility measures or an union of normalized necessity measures.

By applying Theorem 1.2 of Serra [11] we have the following constructive decompositions.

Theorem 4 (constructive decomposition of fuzzy measures) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \bigwedge (\pi_{U,\mu(U)} : U \in \mathcal{P})$$

$$= \bigvee (\nu_{U,\mu(U)} : U \in \mathcal{P}),$$

where $\pi_{U,v}$ and $\nu_{U,v}$ are given, respectively, by (3) and (4). $\square$

By applying Corollary 6.1 of Banon & Barrera [4] we can state our main result. It is an alternative and more general way to constructively decompose a fuzzy measure.

Theorem 5 (general constructive decomposition of fuzzy measures) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \bigwedge (\pi_\beta : \beta \in \mathrm{E}(I, \mathcal{P}) \text{ and } \mu\beta \leq \iota)$$

$$= \bigvee (\nu_\alpha : \alpha \in \Delta(I, \mathcal{P}) \text{ and } \iota \leq \mu\alpha),$$

where ι is the identity mapping from I to I, and where π_β and ν_α are given, respectively, by (7) and (8). $\square$

In the decomposition by an intersection, the condition on the erosions β is that the compositions of μ by β are *anti–extensive*. In the decomposition by an union the condition on the dilations α is that the compositions of μ by α are *extensive*.

In terms of possibility and necessity distributions, we have, for any $X \in \mathcal{P}$,

$$\mu(X) = \bigwedge (\max f_\beta(X) : \beta \in \mathrm{E}(I, \mathcal{P}) \text{ and } \mu\beta \leq \iota)$$

$$= \bigvee (\min f_\alpha(X) : \alpha \in \Delta(I, \mathcal{P}) \text{ and } \iota \leq \mu\alpha),$$

where f_β and f_α are given, respectively, in Propositions 1 and 2.

The decomposition given in Theorem 5 can be simplified by dropping out *redundant* terms.

A subset E' of E (resp., Δ' of Δ), such that, for any $\beta' \in \mathrm{E}'$ (resp., $\alpha' \in \Delta'$) $\mu\beta' \leq \iota$ (resp., $\iota \leq \mu\alpha'$), is said to *satisfy the inf–decomposition* (resp., *sup–decomposition*) *condition for* μ iff., for any $\beta \in \mathrm{E}$ (resp., $\alpha \in \Delta$) such that $\mu\beta \leq \iota$ (resp., $\iota \leq \mu\alpha$), there exists $\beta' \in \mathrm{E}'$ (resp., $\alpha' \in \Delta'$) such that $\beta \leq \beta'$ (resp., $\alpha' \leq \alpha$).

Under theses conditions, the decomposition formulae of Theorem 5 become [4, Section 7]

$$\mu = \bigwedge (\pi_\beta : \beta \in \mathrm{E}')$$

$$= \bigvee (\nu_\alpha : \alpha \in \Delta').$$

When the elements of E' (resp., Δ') are all *maximal* (resp., *minimal*) then the above decompositions are said to be *minimal*. Nevertheless, simpler decompositions may still exist as it is shown in Section 4.

By dropping out unredundant terms we can *approximate* the fuzzy measure μ by an interval whose upper and lower limits are, respectively, an intersection of possibility measures and an union of necessity measures:

$$\mu \in [\bigvee (\nu_\alpha : \alpha \in \Delta''), \bigwedge (\pi_\beta : \beta \in \mathrm{E}'')],$$

where $\Delta'' \subset \Delta'$ and $\mathrm{E}'' \subset \mathrm{E}'$.

4 Example of probability measure decompositions in terms of possibility measures

We now assume that Ω is finite. Let n be number of elements of Ω, $n = \#\Omega$. A probability measure on $\mathcal{P}(\Omega)$, as a fuzzy measure [2] can be decomposed, as shown in Section 3, in terms of possibility measures.

As an example, we consider the case of a probability μ on $\mathcal{P}(\Omega)$ with a uniform (i.e., constant) distribution, that is,

$$\mu(X) = (\#X)/n \quad (X \in \mathcal{P}(\Omega)).$$

Since Ω is finite, the decomposition of μ can be simplified by choosing $I = \{0, 1/n, 2/n, ..., 1\}$

The decomposition derived from Theorem 4 has 2^n terms. Two typical possibility distributions are given in Figure 1 for $n = 4$. The distribution value is 1 outside a given subset of Ω and is k/n inside it (where k is the number of elements in the subset).
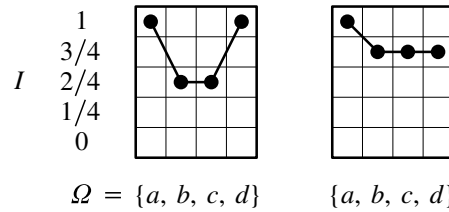

Fig. 1 – Typical possibility distributions in the decomposition by Theorem 4.

The minimal decomposition derived from Theorem 5 has $n!$ terms. Two typical possibility distributions are given in Figure 2 for $n = 4$.
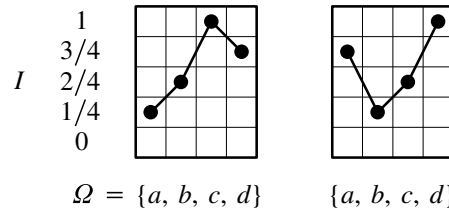

Fig. 2 – Typical possibility distributions in the minimal decomposition by Theorem 5.

The $n!$ terms in the minimal decomposition correspond to the $n!$ maximal chain in $(\mathcal{P}(\Omega), \subset)$ between $\emptyset$ and Ω.

Actually, we can find *simpler* decompositions by observing that we can cover $\mathcal{P}(\Omega)$ with $\binom{n}{n/2}$ maximal chains, if n is even, or $\binom{n}{(n-1)/2}$ chains, if n is odd. These maximal chains, that we call *chains of interest* can be found by using the search technique given below.

For any $i = 0, 1, ..., n$ and any $A \in \mathcal{P}$, let

$$\mathcal{X}_i = \{X \in \mathcal{P} : (\#X) = i\}$$

$$\overrightarrow{\mathcal{X}}_{i,A} = \{X \in \mathcal{P} : (\#X) = i \text{ and } A \subset X\}$$

$$\overleftarrow{\mathcal{X}}_{i,A} = \{X \in \mathcal{P} : (\#X) = i \text{ and } X \subset A\}.$$

Let n be even (resp., odd), to each element X of $\mathcal{X}_{n/2}$ (resp., $\mathcal{X}_{(n-1)/2}$) we associate the chain of interest that contain X.

Let $0 \leq i < n/2$ (resp., $0 \leq i < (n-1)/2$), to find the successors (w.r.t. the chains of interest) in $\mathcal{X}_{i+1}$ of the elements in $\mathcal{X}_i$, we go through the following steps:

1 – let $\mathcal{B}$ be the empty subcollection of $\mathcal{P}$;
2 – let $\mathcal{A}$ be $\mathcal{X}_i$;
3 – let A be an element of $\mathcal{A}$;
    let B be an element of $\overrightarrow{\mathcal{X}}_{i+1,A} - \mathcal{B}$ (B is a *successor* of A);
    let $\mathcal{B}$ be $\mathcal{B} + \{B\}$;
    if $\mathcal{B}$ is $\mathcal{X}_{i+1}$ then end;
      else let $\mathcal{A}$ be $\mathcal{A} - \{A\}$;
    if $\mathcal{A}$ is empty then go to 2
      else go to 3.

Let $n/2 < i \leq n$ (resp., $(n-1)/2 < i \leq n$), to find the antecedents (w.r.t. the chains of interest) in $\mathcal{X}_{i-1}$ of the elements in $\mathcal{X}_i$, we go through the above steps, just by replacing $\overrightarrow{\mathcal{X}}_{i+1,A}$ with $\overleftarrow{\mathcal{X}}_{i-1,A}$ and successor with antecedent.

For example, for $n = 4$ we have 6 chains of interest. By applying the above search technique we can get the 6 possibility distributions of Figure 3.
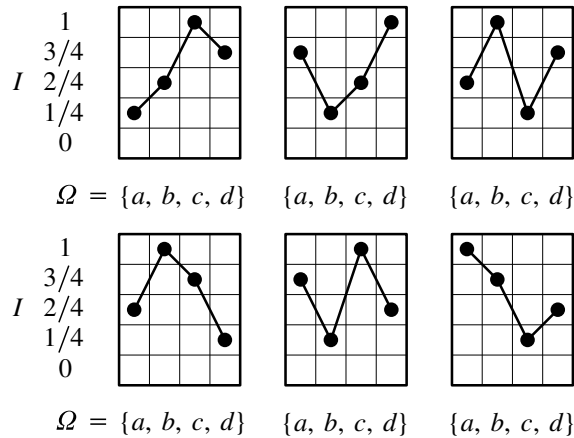

$$\Omega = \{a, b, c, d\} \quad \{a, b, c, d\} \quad \{a, b, c, d\}$$

$$\Omega = \{a, b, c, d\} \quad \{a, b, c, d\} \quad \{a, b, c, d\}$$

Fig. 3 – Possibility distributions in the minimal decomposition of a probality by Theorem 5.

We verify, for example, that

$$\mu(\{b, c\}) = \#\{b, c\}/4 = 2/4$$
$$= \wedge (1, 2/4, 1, 1, 1, 3/4) = 2/4$$

and

$$\mu(\{a, c, d\}) = \#\{a, c, d\}/4 = 3/4$$
$$= \wedge (1, 1, 3/4, 3/4, 1, 1) = 3/4.$$

6. References

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