

The problem of short times for brownian motion in the context of the diffusion equation

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As we know, the problem underlying the parabolic diffusion equation is that this equation is an approximation only valid on time scale which are large compared to the time scale at which the diffusion-causing collisions take place. One of the nonphysical properties of the diffusion equation is that an infinite velocity of information propagation is inherent to it since it is a parabolic equation. Other nonphysical property underlying this equation is that this behavior is caused by the model assumption that the collision frequency is infinite.

Such as investigated by Einstein (3), the parabolic diffusion equation

$$\frac{\partial \eta}{\partial t} = D \frac{\partial^2 \eta}{\partial x^2} \quad (1)$$

with the initial condition $\eta(x, 0) = \delta(x)$ have solution given by

$$\eta(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}, \quad (2)$$

where D is the so-called diffusion coefficient. As one may check, the second moment obtained of this distribution function is the mean square displacement $\langle x^2 \rangle = \int x^2 \eta(x, t) dx = 2Dt$, or equivalently, the Einstein's relation (ER). In order to understand some difficulties present in the Einstein approach let us to begin by calculating the velocity which the particles are diffusing. It is straightfoward to show that, by using the (ER) we gives $v \propto t^{-1/2}$. Note that, when the times scale goes to zero, we have $v \rightarrow \infty$ and, of course, this is a nonphysical result being the limit for the propagation velocity in a fluid of the order of the sound speed. As one can see, another way to observe this difficulty is given in terms of the distribution function. In this way, the particles behaves as diffusive gaussian process: initially the function $\eta(x, t)$ represents a delta centered around the origin $x = 0$, but in the course of time it evolves as a gaussian of variable width.

In order to obtain the hyperbolic diffusion equation describing the brownian motion of the free particle it will be supposed that each particle executes motions independent of the other. Let us suppose that there are N particles supended in a liquid. Considering that in the intervals of time τ , the coordinates of each particle undergone a displacement $\Delta x = l$, since l assume negative and positive values for each particle. Then, the fraction of particles that suffers a displacement between x and $x + l$ in the time τ is given by a probabilistic law (3) following the form:

$$\frac{dN}{N} = \phi(l) dl, \quad (3)$$

with $\phi(l)$ obeying the normalization condition $\int_{-\infty}^{+\infty} \phi(l) dl = 1$ and $\phi(l)$ is an odd function, that is, $\phi(l) = -\phi(-l)$. In this way, following the Einstein approach (3), the particle's number, which in the τ is in the position x and $x + l$, is given by

$$\eta(x, t + \tau) dx = dx \int_{l=-\infty}^{l=+\infty} \eta(x + l, t) \phi(l) dl. \quad (4)$$

As τ and length l are very small in comparison with t and x variables it is possible to expand the function $\eta(x, t)$ until the second order term:

$$\eta(x, t + \tau) \cong \eta(x, t) + \tau \frac{\partial \eta(x, t)}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 \eta(x, t)}{\partial t^2} + \dots \quad (5)$$

$$\eta(x+l, t) \cong \eta(x, t) + l \frac{\partial \eta(x, t)}{\partial x} + \frac{l^2}{2!} \frac{\partial^2 \eta(x, t)}{\partial x^2} + \dots \quad (6)$$

$$\eta + \frac{\partial \eta}{\partial t} \tau + \frac{\tau^2}{2} \frac{\partial^2 \eta}{\partial t^2} = \eta \int_{-\infty}^{+\infty} \phi(l) dl + \frac{\partial \eta}{\partial x} \int_{-\infty}^{+\infty} l \phi(l) dl + \frac{\partial^2 \eta}{\partial x^2} \int_{-\infty}^{+\infty} \frac{l^2}{2} \phi(l) dl.$$

On the right side of this expression the second term is identically null since that $\phi(l) = \phi(-l)$. Then, considering the normalization condition we have:

$$\frac{\tau}{2} \frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \eta}{\partial t} = D \frac{\partial^2 \eta}{\partial x^2}, \quad (7)$$

where $D = \frac{1}{2\tau} \int_{-\infty}^{+\infty} l^2 \phi(l) dl$ is the diffusion coefficient, which is defined by the second moment of the distribution function.

The solution of this equation have been given by Straton (18). Thus, if one consider the boundary conditions $\eta(x, 0) = 0$ to be consistent with the Einstein approach, we obtain the solution

$$\eta(x, t) = e^{-t/\tau} \left\{ \frac{1}{2} \delta(x+vt) + \frac{1}{2} \delta(x-vt) \right\} + \frac{1}{2v\tau} J_0 \left\{ \frac{(x^2 - v^2 t^2)^{1/2}}{v\tau} \right\} + \frac{t}{2\tau} \frac{J_1 \left\{ \frac{(x^2 - v^2 t^2)^{1/2}}{v\tau} \right\}}{(x^2 - v^2 t^2)^{1/2}},$$

that is valid for $|x| \leq vt$ and

$$\eta(x, t) = e^{-t/\tau} \left\{ \frac{1}{2} \delta(x+vt) + \frac{1}{2} \delta(x-vt) \right\} \quad (8)$$

for $|x| > vt$. If one consider the asymptotic expansion for de Bessel functions the expression (8) can be rewright as

$$\eta(x, t) = e^{-t/\tau} \left\{ \frac{1}{2} \delta(x+ct) + \frac{1}{2} \delta(x-ct) \right\} + \frac{e^{-(t/\tau)y^2}}{4c\tau} \left\{ \frac{\pi t}{2\tau} (1-y^2)^{1/2} \right\} [1 + (1-y^2)^{-1/2}],$$

where $y = x/vt < 1$. The physical meaning from this equation is very clear. The first therm describe the D'Alambert solution for the case of plane wave which rapidly becomes negligible while the second therm represents the diffusive regime. By considering the regime $y \ll 1$, or equivalently, $x \ll vt$, we recovered the solution for the case of the parabolic diffusion equation $\eta(x, t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt)$.

At this point, in order to compare the PDE and HDE approaches, let us to plot the solutions for parabolic and hyperbolic diffusion equation. The result from our numerical experiment, as shown in Figure 1, shows that the

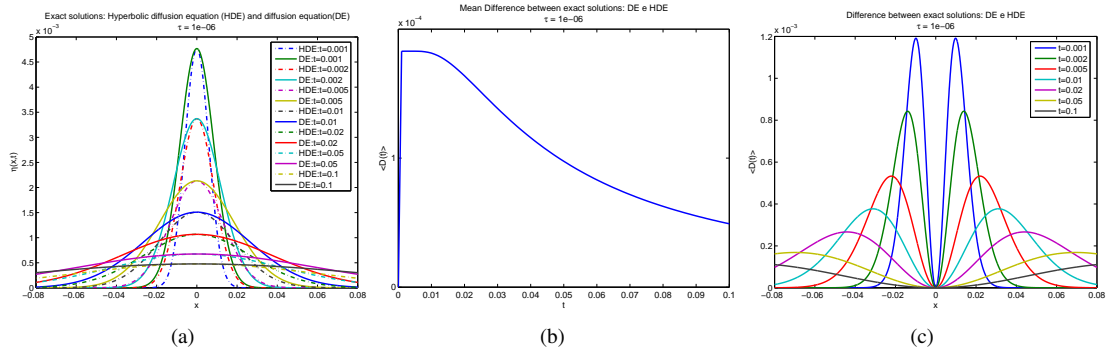


Figure 1: Comparison between exact solutions of PDE and HDE for $\tau = 10^{-6}$: a) Exact Solutions, b) Mean difference between exact solutions, c) Absolut difference between exact solutions.

intrinsic inconsistency that appears in parabolic approach, due to very short time intervals, can be removed by adding the second derivative term in the hyperbolic equation. The inconsistency pointed out here was removed using the diffusion coefficient $D = 0.035$ for both PDE and HDE. In this limit it is important to emphasize the transition found for the diffusion coefficient values. It happens from the time scale of order 10^{-5} to 10^{-6} when the influence of the value of the time interval should be taken into account to understand the diffusion process.

Despite many investigations on the hyperbolic approach of the diffusive process, all of them are restricted to address the heat diffusion as the fundamental scenario. In this paper, we focus on the Brownian motion showing

for the first time a numerical result where the values for the diffusion coefficient and its consistency in time are reported ($D = 0.035$). It means that using the hyperbolic approach the Brownian motion can be considered even for very short time intervals as 10^{-6} s. As a consequence of this result, at least two more fundamental investigations can be performed. The first is to provide solutions into the context of the Ito diffusion addressing the k-problem discussed in the introduction of this paper. The second one is related to the sub-diffusive interpretation when time goes to zero. In that case, a possible breaking of the extensive property of the system's entropy can be addressed including a scenario where non-gaussian fluctuation can be the key to construct a Brownian scenario for very short time intervals. Work along these lines is in progress and will be communicated later.

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