General Approach for Fuzzy Mathematical Morphology

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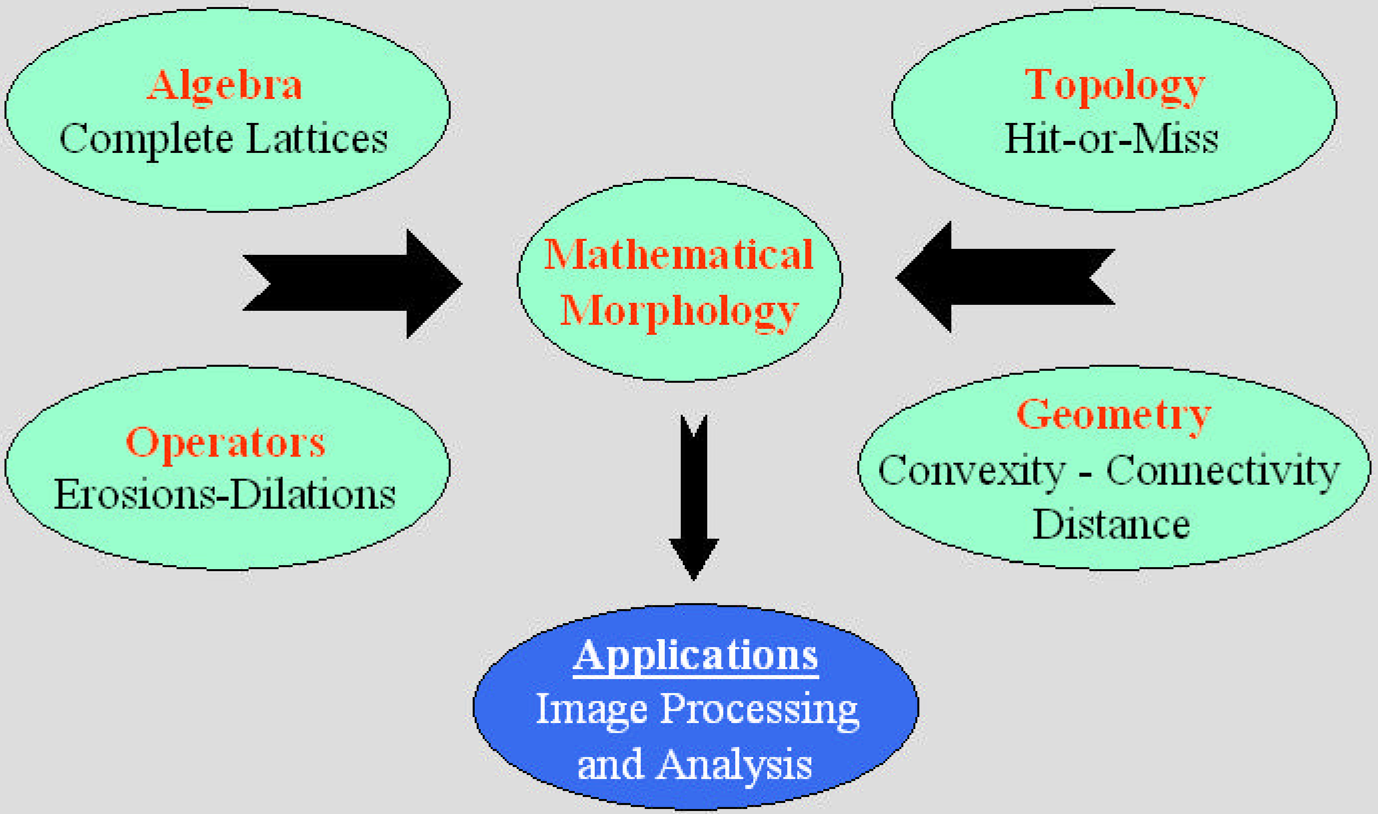


A mathematical tool that studies operators on complete lattices

ALGEBRAIC DILATION AND EROSION

Algebraic dilation: commutes with the supremum

$$\delta(\vee_i X_i) = \vee_i \delta(X_i)$$

Algebraic erosion: commutes with the infimum

$$\varepsilon(\wedge_i X_i) = \wedge_i \varepsilon(X_i)$$

Adjunction: (ε, δ) such that

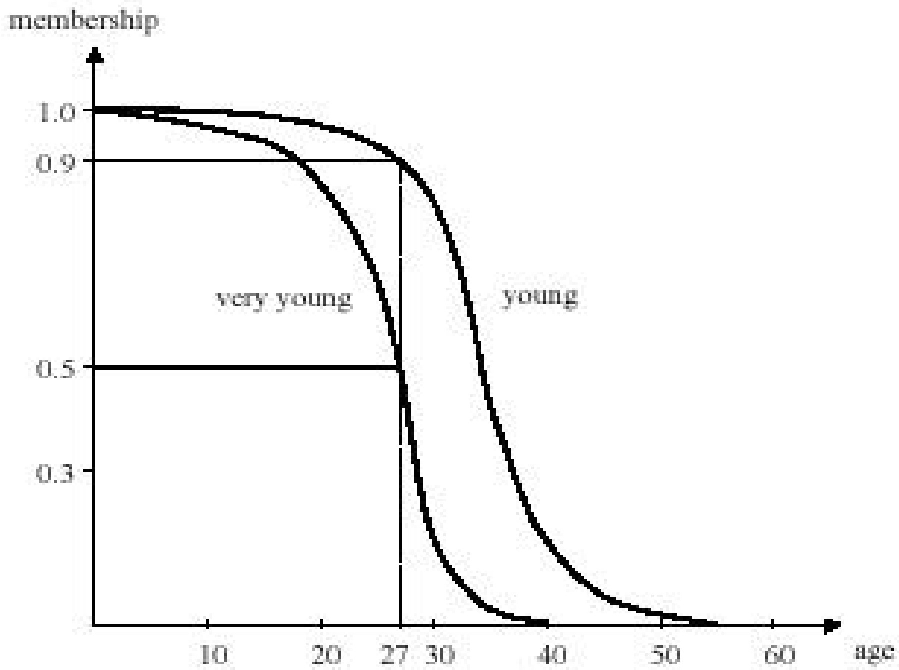
$$\forall (X, Y), \delta(X) \leq Y \text{ iff } X \leq \varepsilon(Y)$$

(ε, δ) adjunction $\Rightarrow$

- ε = algebraic erosion
- and δ = algebraic dilation.

FUZZY SETS = membership functions

$\mu_A(27) = 0.9$, $\mu_B(27) = 0.5$



A = “young”
B= “very young”

Instead of $\mu_A(x)$ we could write $A(x)$

$$m_{U-X}(x) = 1 - m_X(x)$$

$$m_{X \cup Y}(x) = max(m_X(x), m_Y(x))$$

$$m_{X \cap Y}(x) = min(m_X(x), m_Y(x))$$

An operation c: [0,1]x[0,1]? [0,1] is a conjunctor (a fuzzy generalization of the logical AND operation), or t-norm, if it is commutative, increasing in both arguments,
c(x,1) = x for all x, c(x,c(y,z)) = c(x(x,y),z).

An operation I: [0,1]x[0,1]? [0,1] is an implicator if it decreases by the first and increases by the second argument,
I(0,1) = I(1,1)=1 , I(1,0) = 0.

Lukasiewicz: c(x,y) = max (0,x+y-1) ; I(x,y) = min(1,y-x+1),
“classical” : c(x,y) = min(x,y) ; I(x,y) = y if y<x, and 1 otherwise.

Grey –scale images can be represented as fuzzy sets!

Say that a conjunctor and implicator form an ADJUNCTION when

C(b,y) = x if and only if y = I(b,x)

Having an adjunction between implicator and conjunctor, we

define

adjoint pair of fuzzy erosion and dilation:

$$\delta_B(A)(x) = \sup_y c(B(x-y), A(y)),$$

$$\varepsilon_B(A)(x) = \inf_y i(B(y-x), A(y)).$$

GENERAL DEFINITION OF FUZZY MORPHOLOGICAL OPERATIONS

First, let us consider the universal set E and let us define a class of fuzzy sets $\{A_y, | y \in E\}$. Then for any fuzzy subset X of the universal set E we define fuzzy dilation and erosion as follows:

$$\delta(X)(x) = \bigvee_{y \in E} c(A_x(y), X(y)),$$

$$\varepsilon(X)(x) = \bigwedge_{y \in E} i(A_y(x), X(y)).$$

Proposition $(\varepsilon(X), \delta(X))$ *is adjunction.*

A, M- fuzzy sets in an universal set E

$$\inf_{z\in\Gamma} A(z) \geq \min(A(x), A(y))$$

x and y are connected in the fuzzy set A

$$d_M(x,y) = \frac{\mathrm{len}\,(x,y)}{C_M(x,y)},$$

len(x,y) is the length of the shortest continuous path between x and y in M

$$C_M(x,y) = \sup_{\Gamma} \inf \{M(z) | \, z \in \Gamma\}$$

? - an arbitrary path between x and y in the universe E

$$[B_M(y,r)](x) = \begin{cases} 1 \text{ if } d_M(x,y) \leq r \\ 0 \ \text{ otherwise.} \end{cases}$$

A fuzzy geodesic ball in M

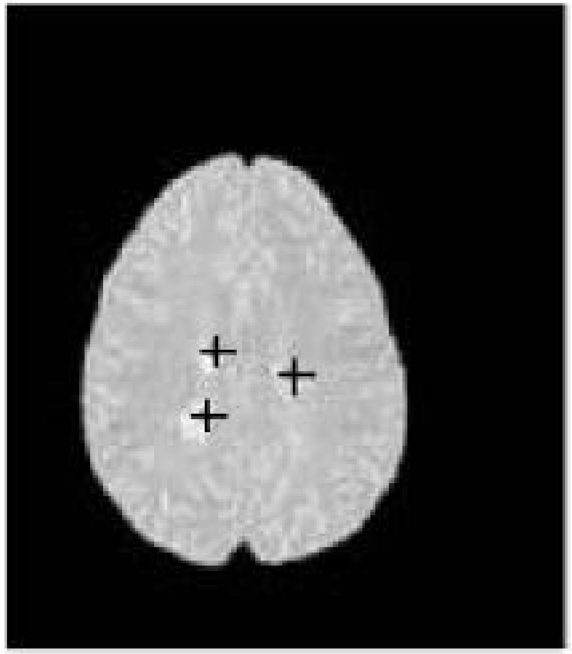
Definition of fuzzy geodesic operations

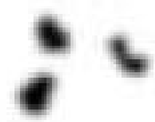
$$\Delta_M^r(X)(x) = \bigvee_{y\in E} c[(B_M(x,r))(y), X(y)]$$

$$\mathcal{E}_M^r(X)(x) = \bigwedge_{y\in E} i[(B_M(y,r))(x), X(y)]$$

This is an sdjunction !

DETECTION OF SCLEROTIC LESIONS DUE TO THE MARKERS ON LEFT (THE SET ***X)*** BY GEODESIC RECONSTRUCTION BY DILATION





INTERVAL OPERATIONS (S. Markov)

Outer operations: A+B, A – B, A X B

Inner operations:

$$A + B = [a^- + b^-, a^+ + b^+], \quad A +^- B = [a^{-s} + b^s, a^s + b^{-s}],$$
$$A - B = [a^- - b^+, a^+ - b^-], \quad A -^- B = [a^{-s} - b^{-s}, a^s - b^s],$$

where

$$s = \begin{cases} +, & \omega(A) \geq \omega(B), \\ -, & \omega(A) < \omega(B). \end{cases}$$

Here and further on, a^s with $s \in \{+, -\}$ denotes certain endpoint of the interval A — the left one if $s = -$ and the right one if $s = +$. We define the 'product' st for $s, t \in \{+, -\}$ by $++ = -- = +$, $+- = -+ = -$, i.e. $a^{++} = a^{--} = a^+$ and $a^{+-} = a^{-+} = a^-$.

$$A + B = A \oplus B$$

$$A +^- B = A \ominus (-B) \cup B \ominus (-A)$$

By analog we can define outer multiplication by appropriate algebraic T-invariant erosion, where T is multiplicative group

OPERATIONS BY FUZZY NUMBERS

$$(A+B)(x) = \bigvee_{z+y=x} \min(A(y), B(z));$$

$$(A \times B)(x) = \bigvee_{z.y=x} \min(A(y), B(z));$$

$$(A-B)(x) = \bigvee_{y-z=x} \min(A(y), B(z)) = (A+(-B))(x);$$

$$\frac{A}{B}(x) = \bigvee_{zx=y} \min(A(y), B(z)) = \left(A \times \frac{1}{B}\right)(x).$$

Let us consider a universal set E. Let also there exists an Abelian group of automorphisms T in $\mathcal{P}(E)$ such that T acts transitevely on the supremum-generating family $l = \{\{e\}|e \in E\}$ as defined previously. In this case, for shortness we shall say that T *acts transitively on* E. Then having an arbitrary fuzzy subset B from E, we can define a family of fuzzy sets in $\{A_y^B \,|\, y \in E\}$ such as $A_y^B(x) = B(\tau_y^{-1}(x))$.

$$\delta_B(X)(x) = \bigvee_{y \in E} c(A_x^B(y), X(y))$$

$$\varepsilon_B(X)(x) = \bigwedge_{y \in E} i(A_y^B(x), X(y))$$

$$\text{Let}(\delta_B(A))(x) \;=\; \bigvee_{y*z=x} \min(A(y), B(z)),$$

$$(\varepsilon_B(A))(x) \;=\; \inf_{y\in\mathbb{R}} \left(h\left(A(y) - B(\tau_x^{-1}(y)\right)(1 - A(y)) + A(y)\right),$$

where $h(x) = 1$ when $x \geq 0$ and is zero otherwise.

Now it is clear that if $\tau_b(x) = x + b$ and $* = +$ then

$$(\delta_B(A)) \;=\; A + B.$$

We can also define an *inner addition* operation by

$$A +^- B \;=\; \varepsilon_{-B}(A) \cup \varepsilon_{-A}(B).$$

If $\tau_b(x) = xb$ for $b \neq 0$ and $y * z = yz$ then

$$(\delta_B(A)) \;=\; A \times B.$$

In this case an inner multiplication exists as well:

$$A \times^- B \;=\; \varepsilon_{\frac{1}{B}}(A) \cup \varepsilon_{\frac{1}{A}}(B).$$

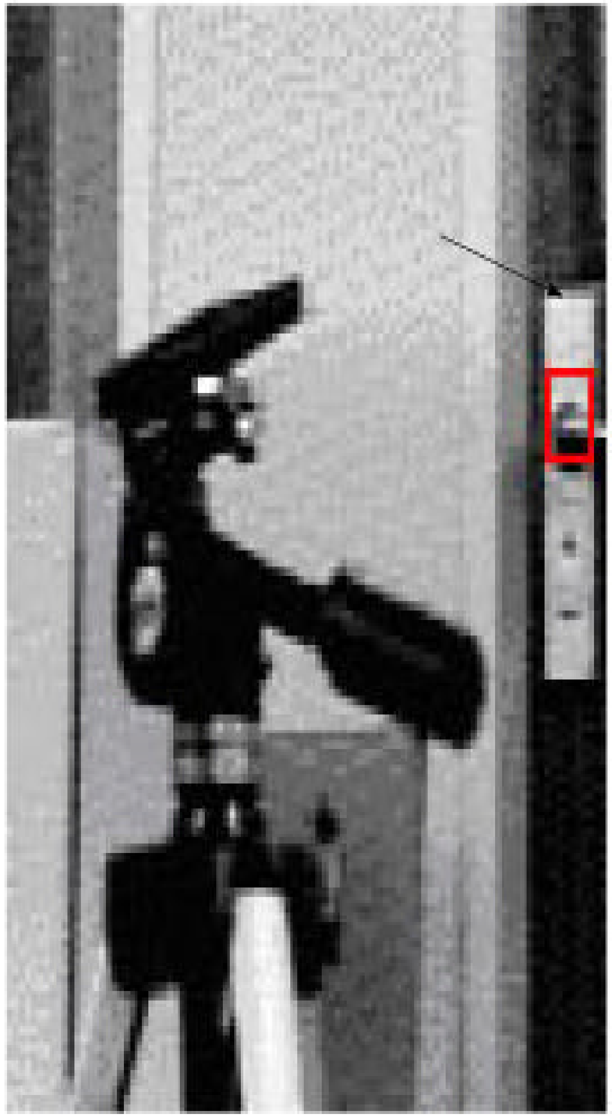
FUZZY HIT-OR-MISS TRANSFORM BY INTUITIONISTIC FUZZY SE's

Remind that an intuitionistic fuzzy subset A from the universal set E is characterised by two functions: the degree of membership $\mu_A(x)$ and the degree of nonmembership $\nu_A(x)$. for every point $x \in E$ we have that $\mu_A(x) + \nu_A(x) \leq 1$. Then one can define intersection ot two intuitionistic sets by taking a t-norm Δ of their membership functions for the resulting membership function, and taking the associated s-norm ∇ of their nonmembership functions for the resulting nonmembership functions. Remind that the associated s-norm is defined by $x \nabla y = 1 - ((1-x)\Delta(1-y))$.

Fuzzy hit-or-miss transform, by arbitrary T-invariant erosion:

$$\tilde{\pi}_{A,B}(X) = \varepsilon_A(X)\,\Delta\,\varepsilon_B(X^c)$$

A and *B* can be considered as the MEMBERSHIP and NONMEMBERRSHIP part of a **Intuitionistic Fuzzy Set**



0	0	0	0.6/0.2	0.6/0.2	0.6/0.2	0	0	0
0	0	0	0.6/0	1/0	0.6/0	0	0	0
0	0	0	0.6/0	0.6/0	0.6/0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0.6/0.2	0.6/0	0.6/0	0	0	0/0.8	0/0.8	0/0.8	0/0.8
0.6/0.2	1/0	0.6/0	0	0/1	0/1	0/1	0/1	0/1
0.6/0.2	0.6/0	0.6/0	0	0	0/0.8	0/0.8	0/0.8	0/0.8
0	0	0	0	0	0	1/0	0	0
0	0	0	0.8/0	0.8/0	0.8/0	0	0	0
0	0	0	0.8/0	1/0	0.8/0	0	0	0
0	0	0	0.6/0.2	0.6/0.2	0.6/0.2	0	0	0

Intuitionistic structuring element

Finding a c-like shape with degree of truth 0.54

Questions

? ? ?