Geometric and Topological Multi-Resolution of *n*-Dimensional Solids

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Acknowledgements

- PhD. Thesis Work
  - Esdras Medeiros Filho (IMPA)
- Co-Supervision
  - Helio Lopes (PUC-Rio)
- Collaboration
  - Thomas Lewiner (PUC-Rio)

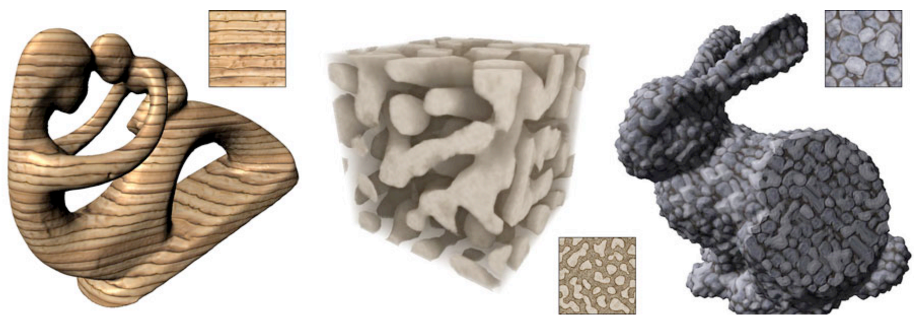
Outline

- Context and Motivation
- Mathematical Background
- *Sampling Solid Objects*
- *Geometric / Topological Multiresolution*
- *Adaptive Multi-Triangulation Structure*
- Examples and Applications
- Future Work

Motivation

Why Solids?

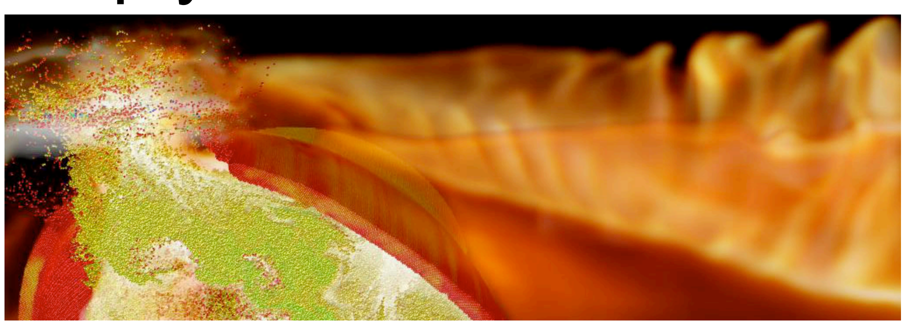
- Most Objects are NOT Hollow…



Matter Inside !

[Kopf et al, 2007]

- An Empty Glass is Full of Air…

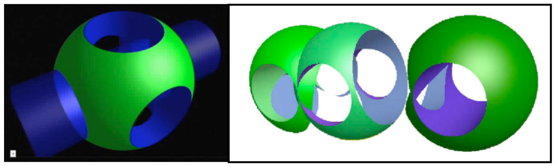


Inside Matters ;-)

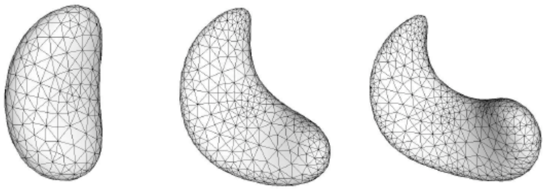
[C-SAFE, 2005]

3D Applications

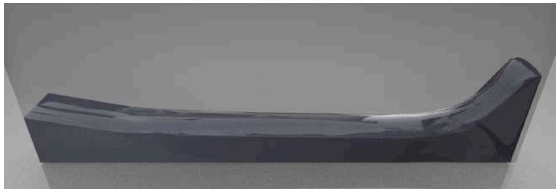
- Modeling
  - CSG
- Animation
  - Deformation
- Simulation
  - Fluids
- Scientific Visualization



[Romeiro et al, 2005]



[de Goes et al, 2007]



[de Goes et al, 2007]

2D Applications

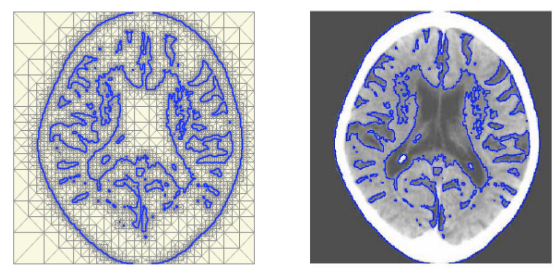
- Image Analysis
  - Segmentation

[Felzenszwalb et al, 2004]

- Image Compression
  - Adaptive Triangulation

[Demaret et al, 2000]

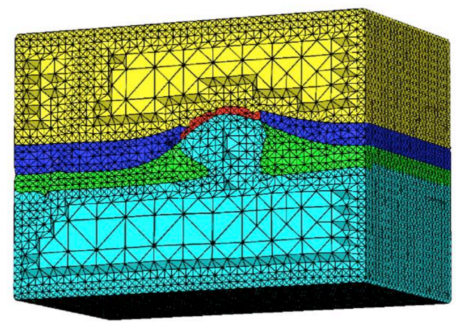
- Medical Imaging
  - Contouring



[Lewiner et al, 2006]

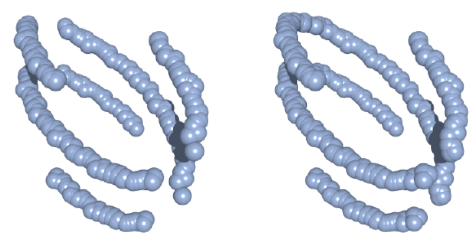
Related Work

- Geometric Multiresolution
  - Multi-Triangulations
  - Adaptive Meshes
  - Etc..



[Marroquim et al, 2005]

- Topology Control
  - Simplification
  - Filtration / Persistence
  - Etc..

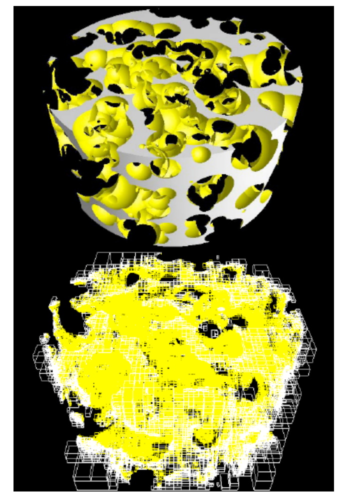


[Zomorodian et al, 2005]

What is Missing?

Unified Geometric –Topological Framework

- For Example:
  - Simplest Topology / Highest Resolution
  - Only Holes with Size Greater than *X*
  - Fine Boundary and Coarse Interior



[Romeiro et al, 2005]

** Why it is difficult ?*

- Continuous Geometry x Discrete Topology
- Representation / Data Structure
- Adaptation, Computation, etc …

Game Plan

- Characteristic Function

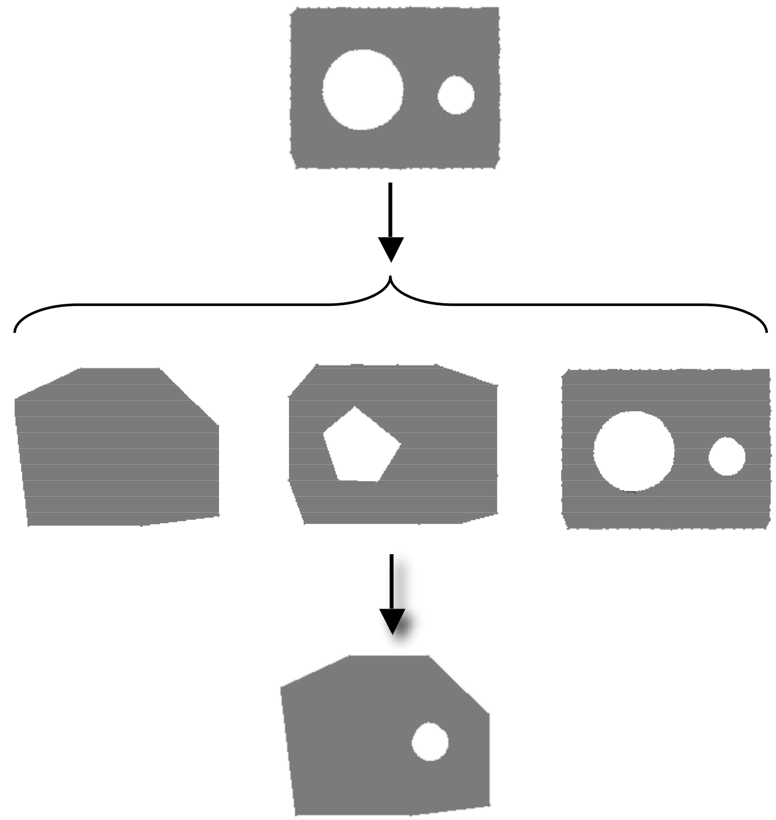
$$\chi(S)$$

- Multiresolution Hierachy

$$M_{gt}(S)$$

- Adaptation

$$A(f, M_{gt}(S))$$

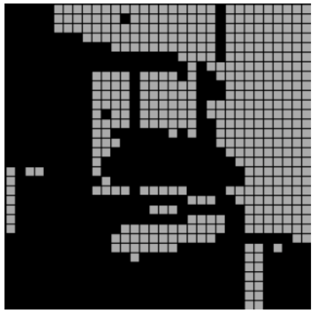


* *Obs: Attributes!*

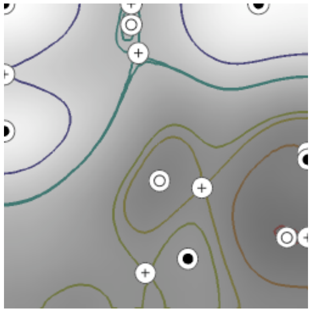
Relations with M.M.

Morphological Scale-Spaces

- Mathematical Morphology
  – Dilation / Erosion
  – Binary Image (Characteristic Function)
  * Discrete Representation



- Level Sets
  – Diffusion
  – Interface Curve (Boundary)
  * Continuous Front Evolution



Our Framework

- Key Ideas
  - Solid Topology + Stochastic Point Sampling
  - Geometric / Topological Operators
  - Variable-Resolution Structure

- Advantages
  - Integrate Geometry and Attributes
  - Natural Notion of Scale
  - General Adaptation

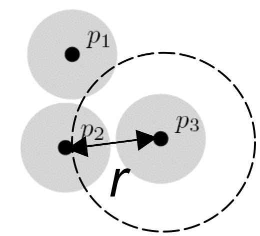
Contributions

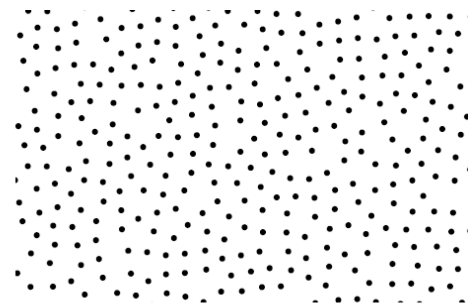
- Solid Poisson-Disk Sampling
  - Scale Space for Characteristic Function
- α-Filtration
  - Multiresolution of α-*Solids*
- G/T Multi-Triangulation
  - Adaptation with *Stellar* and *Handle* Operators

Stochastic Sampling/Reconstruction

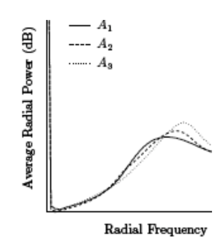
Poisson-Disk Distribution

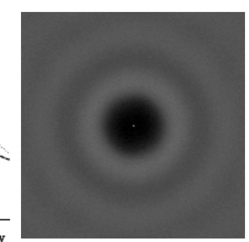
- Random Uniform Sampling
  - Irregular Pattern
  - Points *r*-Distant, at least





- Reconstruction without Aliasing
  - Combined Low-Pass Filter



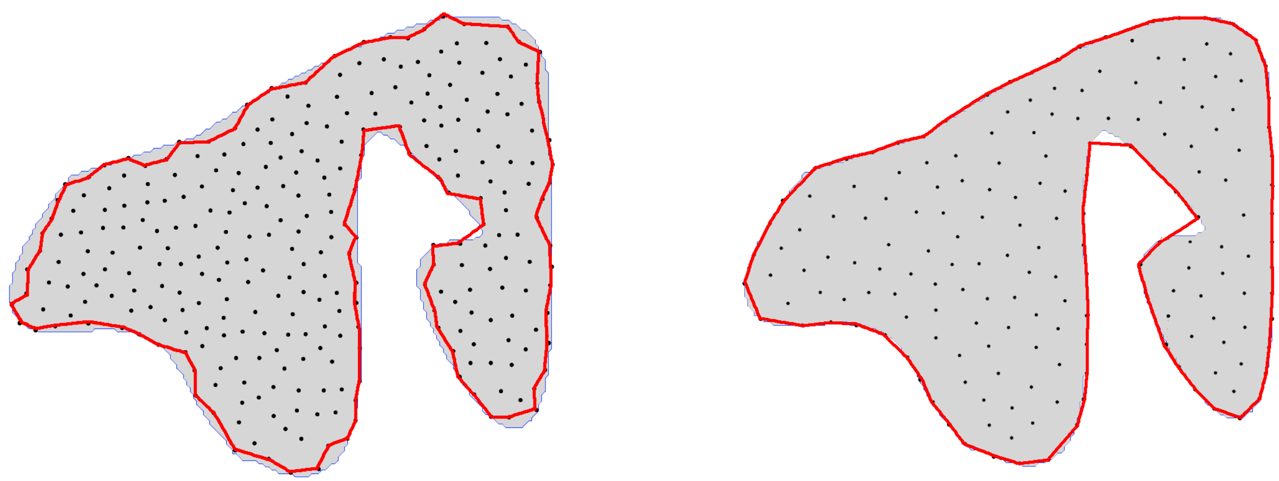


* *Natural Scale / Resolution*

Sampling Solid Regions

** Poisson-Disk Sampling is only Defined for* $\mathbb{R}^n$

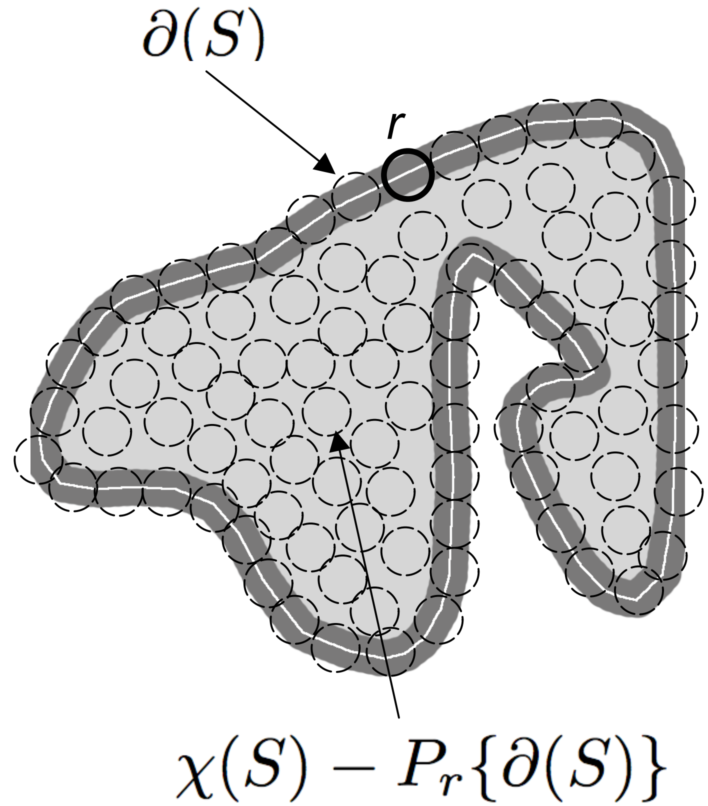
- Restriction to $\chi(S)$ does not follow geometry
- Must take into account $\partial(S)$



Algorithm (1)

- Two-Steps

1. Sample the Boundary
2. Sample the Interior minus *r*-Tubular Neighborhood



* *Stratified Sampling*

- *Feature Sensitive*

➢ Implementation

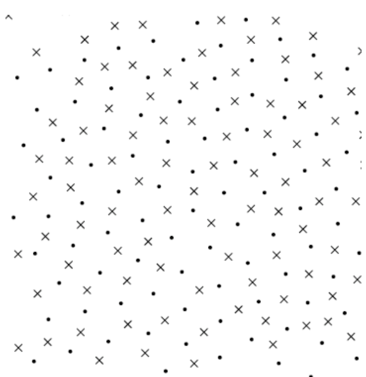
- Dart Throwing
- Quad-Tree Acceleration

Multiresolution Sampling Spaces

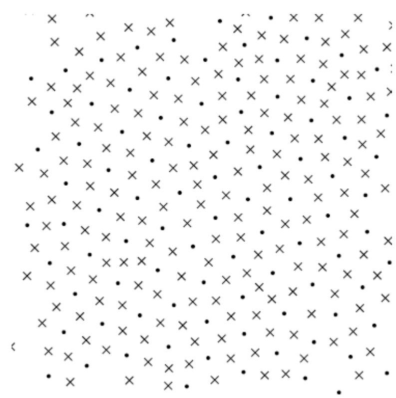
- Nested Poisson-Disk Scale Spaces
  - Disk Radius $r = 2^j$

- Multiresolution Hierarchy $\{P_j\}, \quad j \in \mathbb{Z}$

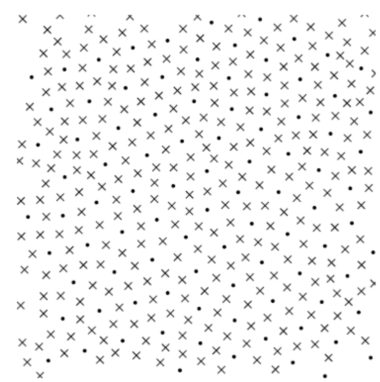
$$\{0\} \cdots \subset P_1 \subset P_0 \subset P_{-1} \cdots L^2(\mathbb{R})$$



P_1



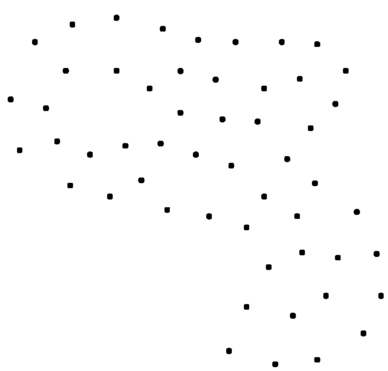
P_0



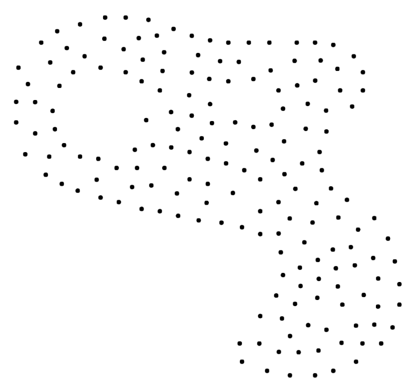
P_{-1}

Multiresolution Solid Sampling

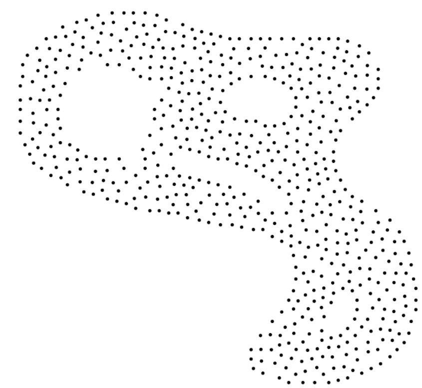
- Apply Algorithm (1) for $r = 2^j$
- Tagged Samples:
  - Boundary / Interior
  - Resolution Level



S_4



S_2



S_1

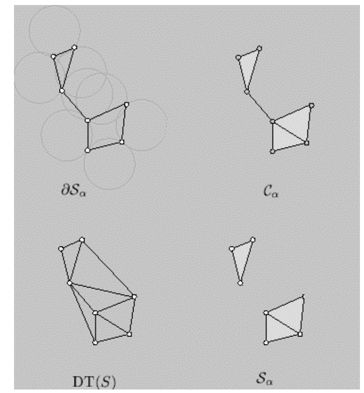
Structuring and Reconstruction

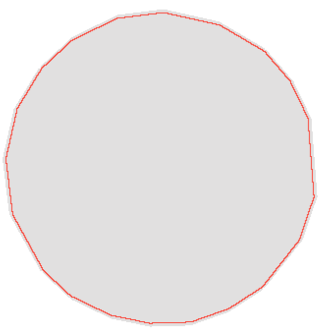
Piecewise-Linear Approximation

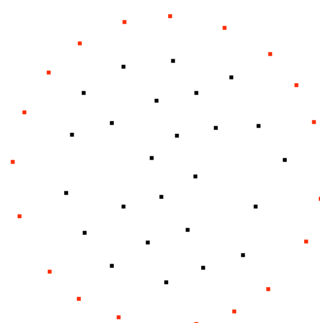
- α-Solids

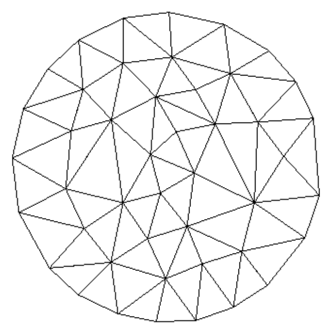
  (α-shape + Regularization)

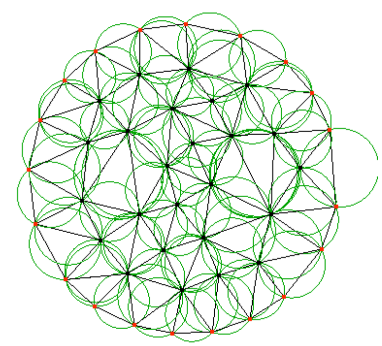
* Subset of Delaunay Triangulation











circumradius < α

Looking for the Right Operators…

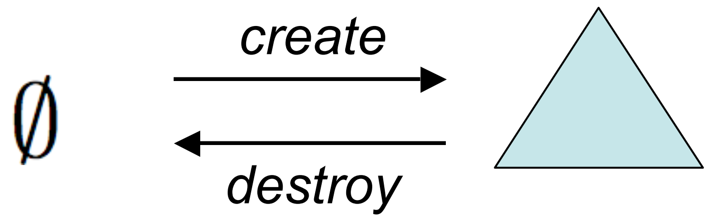
- Operations on Simplicial Complexes
  - Building
  - Change Resolution
  - Change Topology

- Types of Operators
  - Topological
    - Handlebody Theory
  - Geometric
    - Stellar Theory

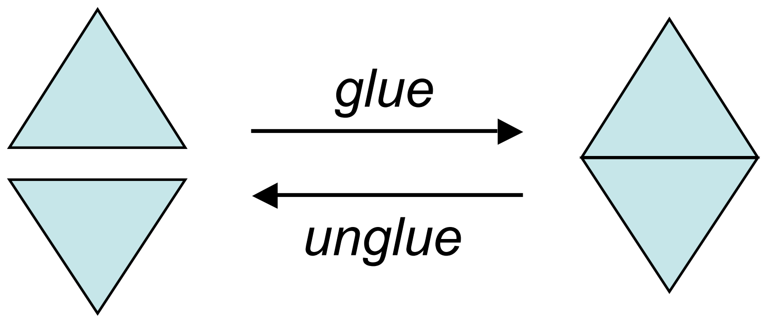
Handle Operators

Change Topology

- Connected Components



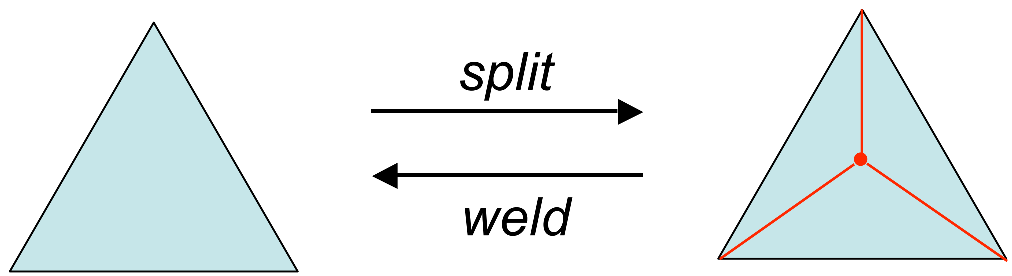
- Boundary



Stellar Operators

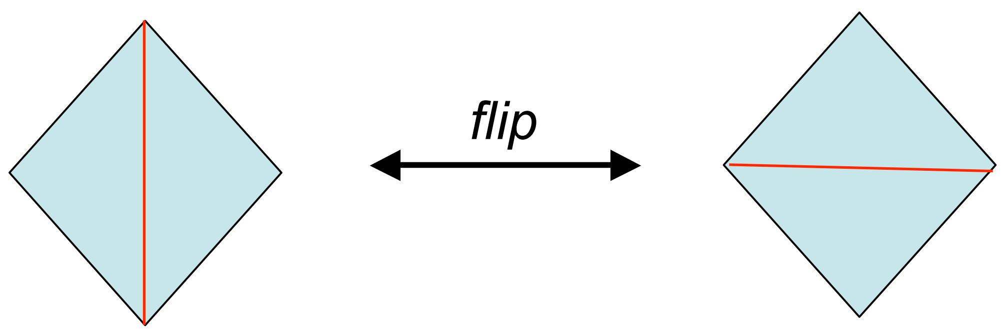
Change Combinatorial Structure

- Resolution



$\sqrt{3}$

- Structure



Integrated Framework

Combinatorial Manifold Operators

- Handle
- Stellar

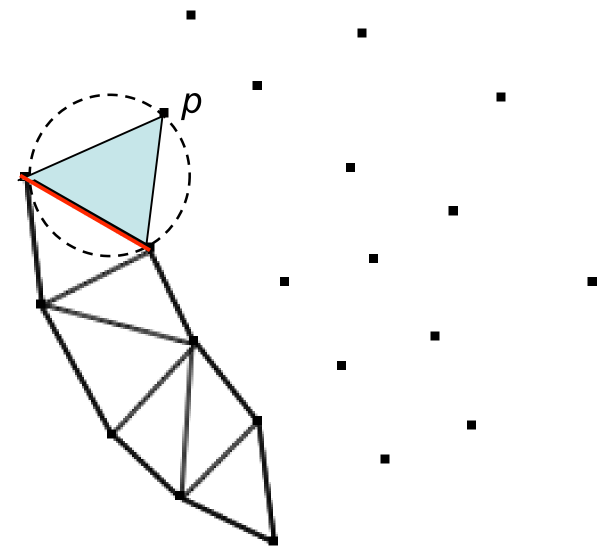
- Properties
  - Atomic
  - Minimal Set
  - Consistent (*by construction*)

* Effective Abstractions

Reconstruction from α-Sampling

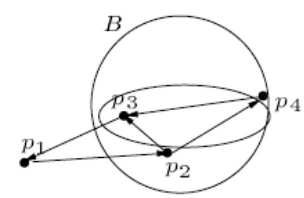
Advancing Front Triangulation

- Ball-Pivoting Algorithm (2)



while (points to process)
    while (e=candidate edge)
        p = *ball_pivot* e
        create σ_p
        glue σ_p
    if (t=find seed)
        create σ_t

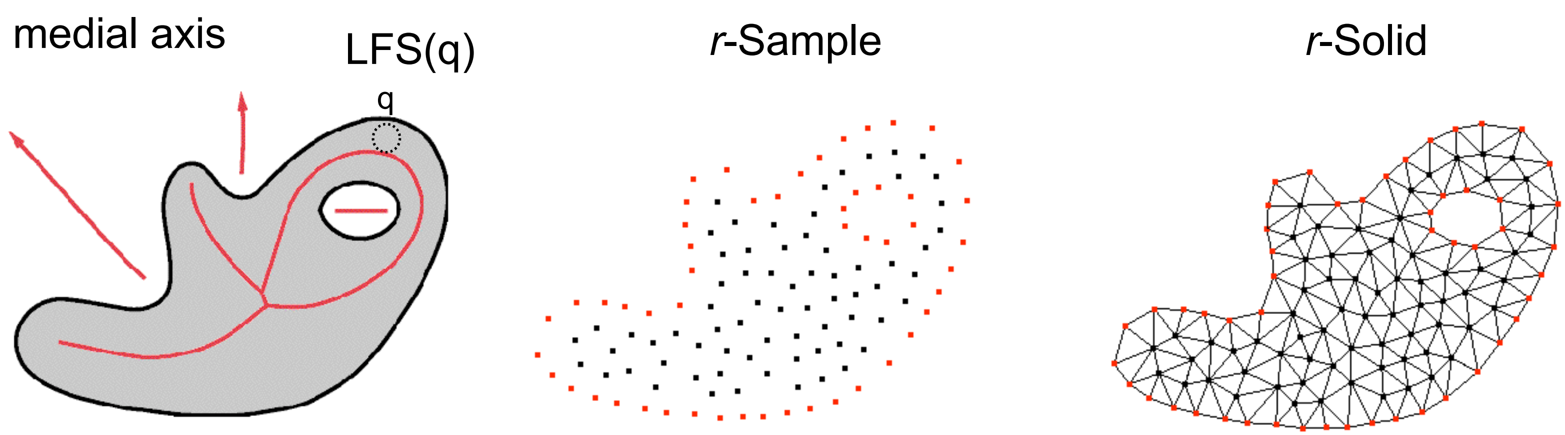
* Obs: Roll α-Ball in $\mathbb{R}^{n+1}$



Topological Guarantees

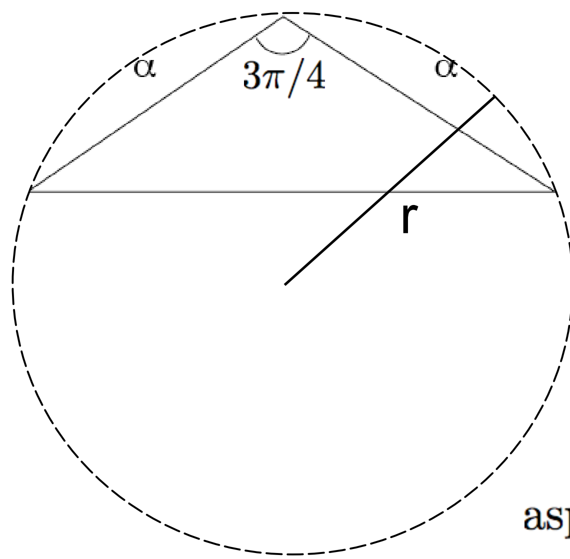
- Theorem (Amenta, 1999):
  *“If **P_r** is an* r-Sample *for* r < k.LFS(**S**)*,*
  then the crust is homeomorphic to **S**.*”*

* Applies to α-Solids, (α = r)



Quality Guarantees

- Theorem (Medeiros et al, 2007):
  "*Let P_α be an α-Sampling of S, then the aspect ratio of $C_\alpha(P_\alpha)$ is bounded by* $4\sqrt{3}$ *".*



Worst case:

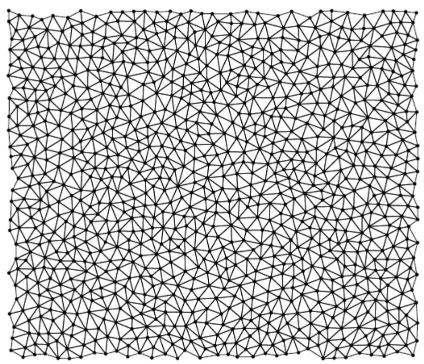
$a = b = r = \alpha$

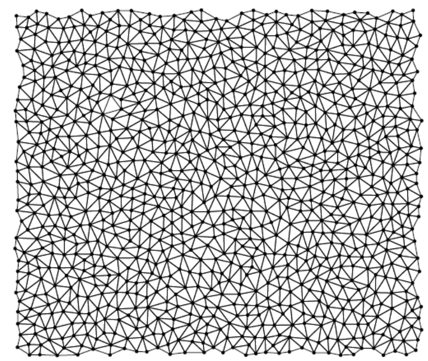
$c = \alpha\sqrt{3}$

$$\text{aspect ratio}(\tau) = \frac{c^2}{A(\tau)} = 4\sqrt{3}$$

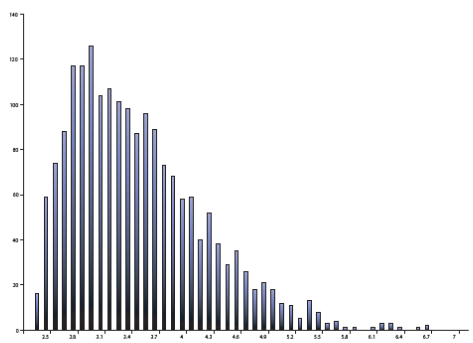
Sampling Independence

- Different α-Samplings

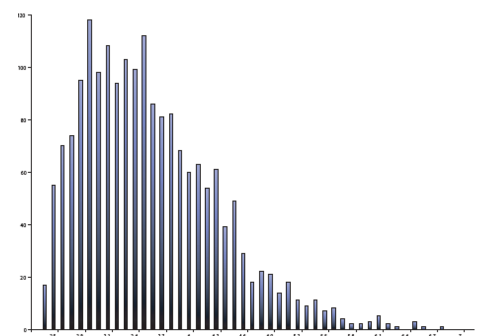




- Aspect Ratio Distribution



2.31 3.46 6.68



2.32 3.52 6.70

α-Filtration

- Defining the Multiresolution of Solids
  - Ball-Pivoting at Each Level…

- *How to Move* <u>*Gradually*</u> *Between Levels*?
  - Change radius: $2^j < \alpha < 2^{j+1}$
  - Apply Stellar and Handle Operators
  - * *g(α) defines an Order for Samples*

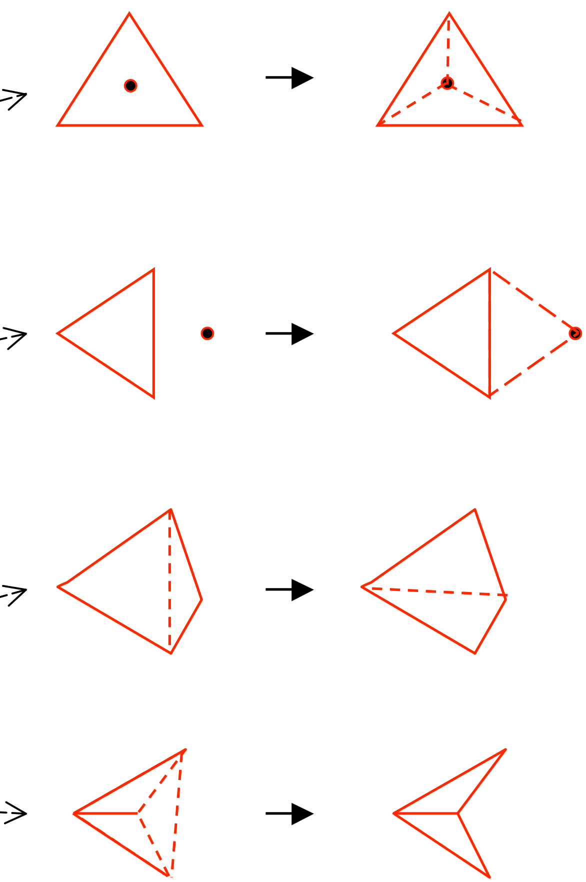
- Construction Strategies:
  - Refinement
  - Simplification

Top-Down Construction

Point Insertion

- Algorithm (3)

while (samples to insert)
    if (inside)
        split σ_p
    else
        create σ_p
        glue σ_p
    while (not Delaunay)
        switch (condition)
            case 1: *flip e*
            case 2: *destroy* σ_t

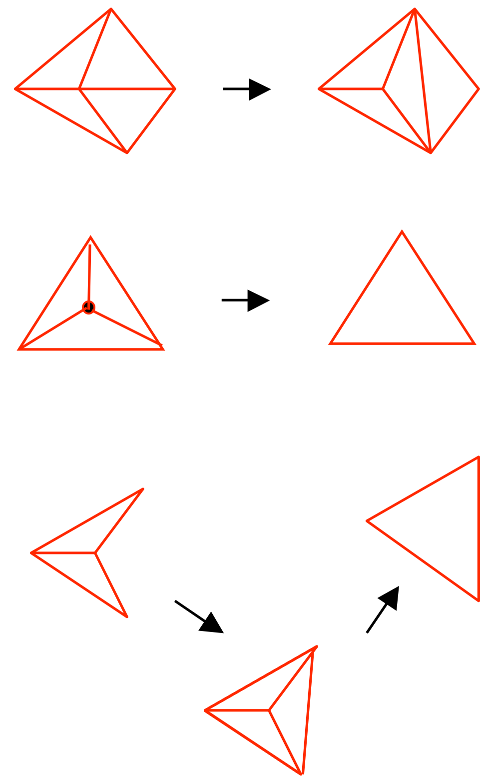


Bottom-Up Construction

Point Removal

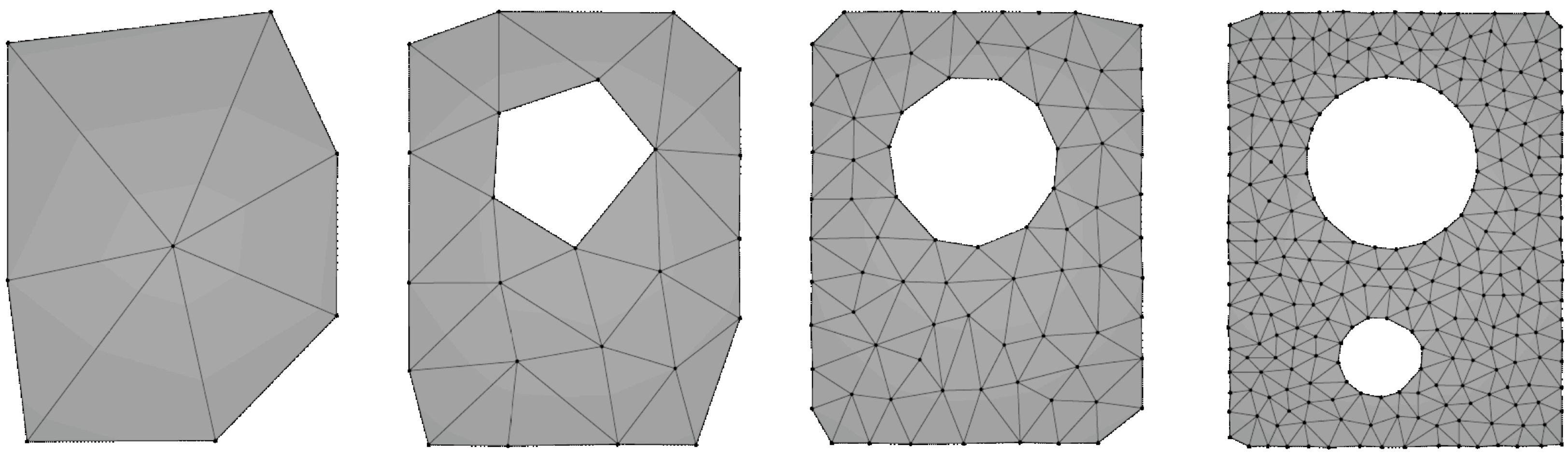
- Algorithm (4)

while (samples to remove)
    while (valence $\neq$ 3)
        flip e
    if (inside)
        weld σ_t
    if (boundary)
        create σ_p
        weld σ_p



Resolution Levels

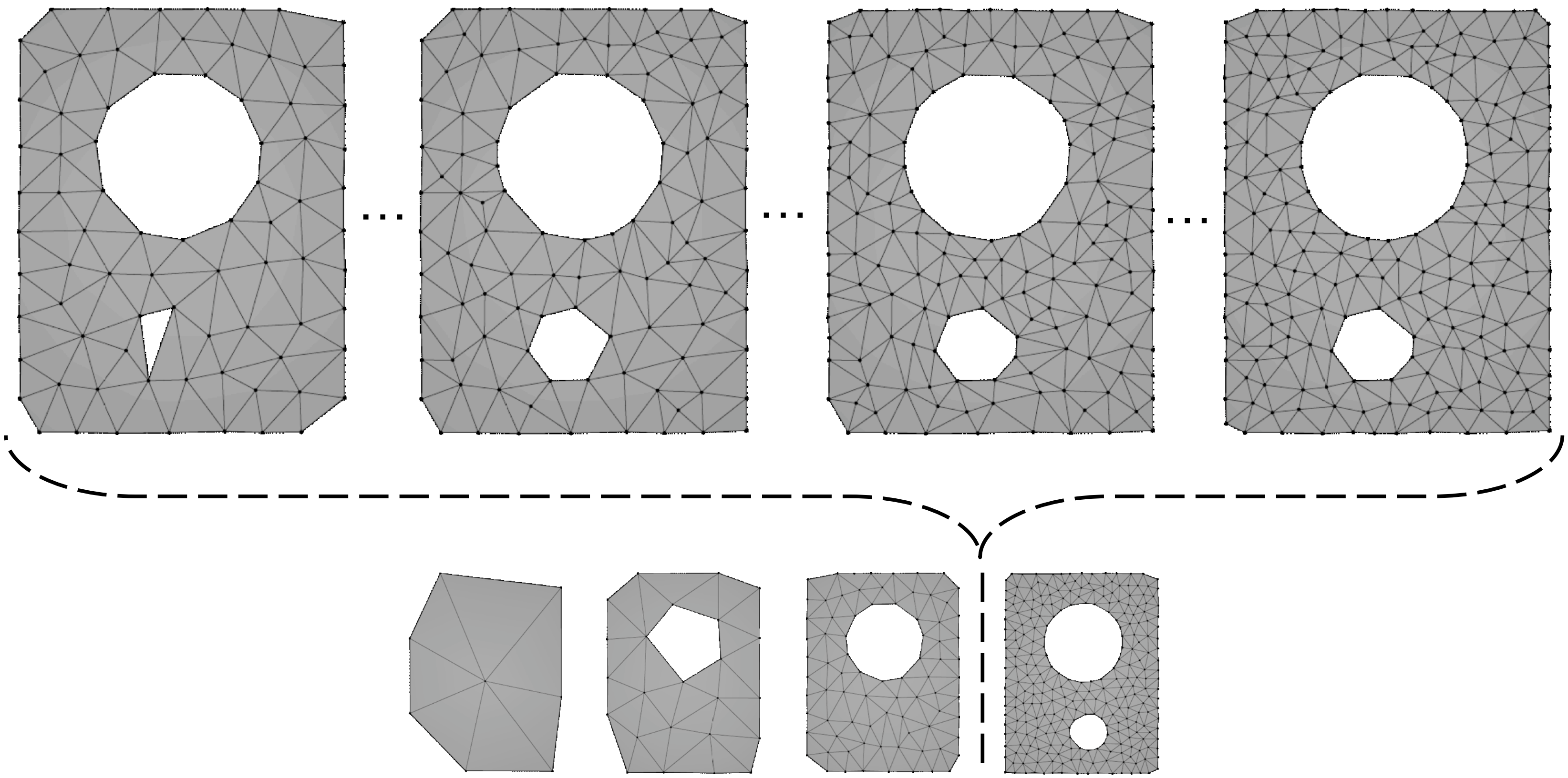
- Coarse Grain
  - Global



2^3 2^2 2^1 2^0

Intra Levels

- Fine Grain
  - Local



Properties

- Theorem (*Uniqueness*) :
  "Given $\mathbf{S}_\alpha$ *and* **g**, *the* α*-Filtration generates a* <u>*unique family*</u> *of triangulations* $\mathbf{T}_\alpha$*"*

- Theorem (*Symmetry*) :
  "In α*-Filtration, the refinement sequence is the* <u>*inverse*</u> *of the simplification sequence"*

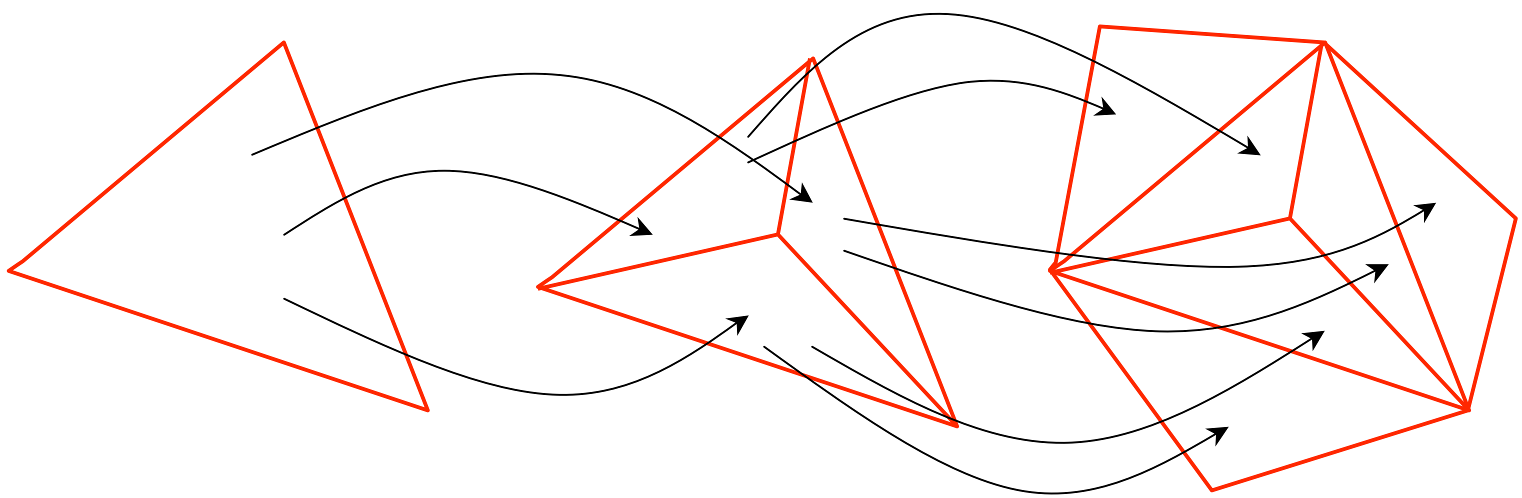
insert p_1 *split* t_1 *flip* e_1 insert p_2 *create* t_2 *glue* t_2 . . . insert p_n *split* t_n	*weld* t_n remov p_n . . . *unglue* t_2 *destroy* t_2 remov p_2 *flip* e_1 *weld* t_1 remov p_1

G/T Multi-Triangulation

- Operators (*stellar / handle*) make only local changes (*geometry / topology*)
- Changes are weakly interdependent!
- Multi-Triangulation Structure
  - Encodes
    - Hierarchy
    - Dependencies
- *Representation for All Possible Meshes!*

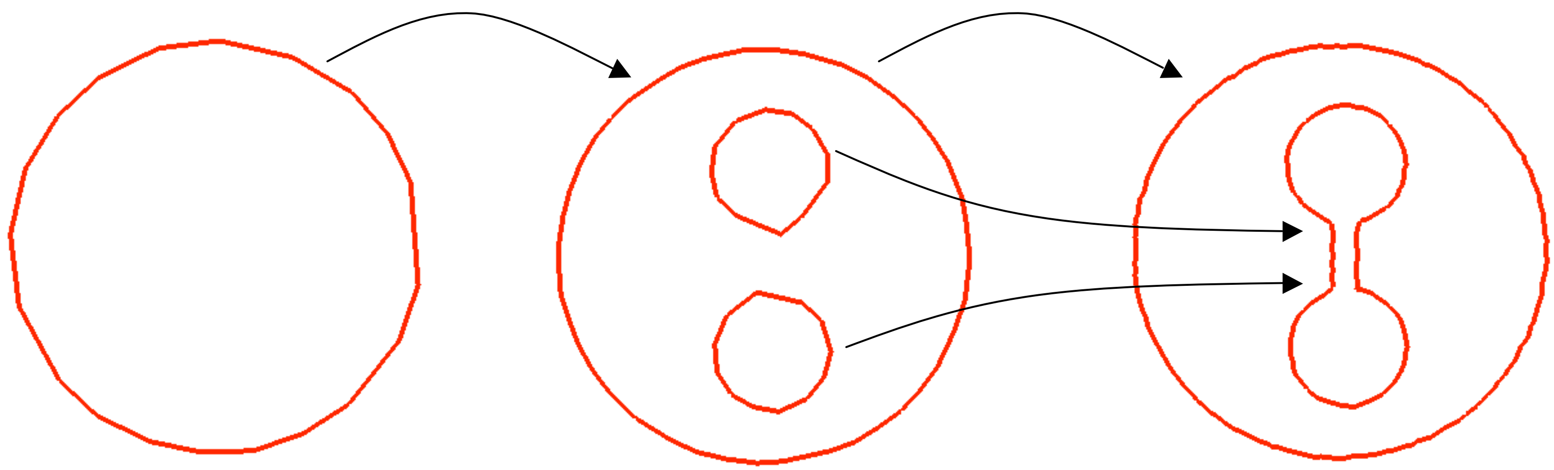
Geometric Hierarchy

- Lattice
  - Interior Decomposition



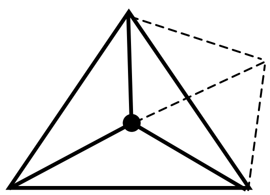
Topological Hierarchy

- Graph
  - Boundaries

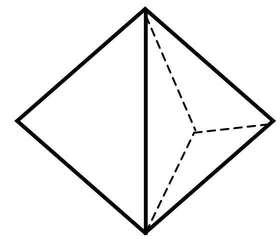


Dependencies

- Geometric
  - Incident Faces at Same Level

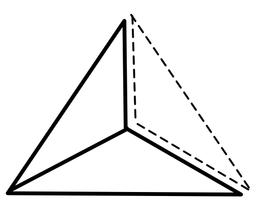


weld

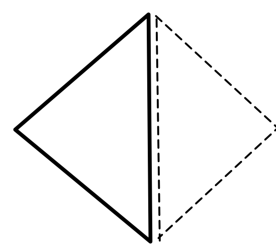


flip

- Topological
  - All Edges must be Interior



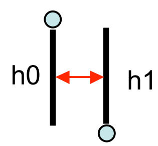
weld



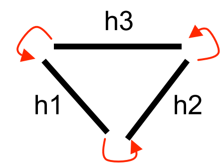
flip

Data Structure

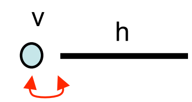
- Mesh (*half-edge*)



edge

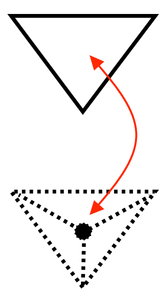


face

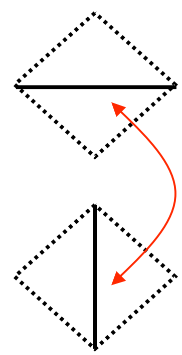


vertex

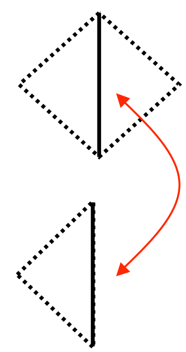
- Multi-triangulation



face-vertex



edge-edge



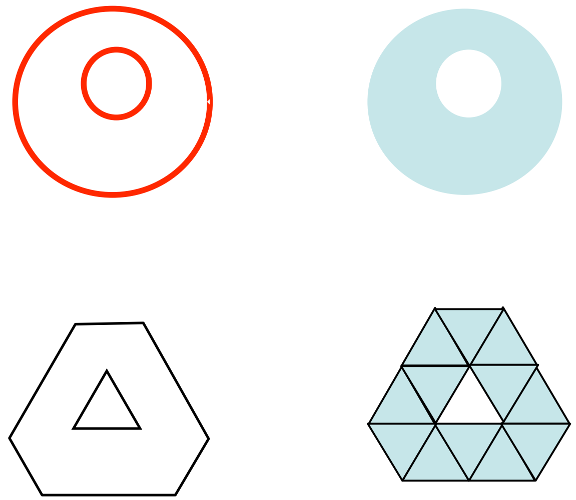
int-bd edge

Adaptation

- Spatially Variant Function

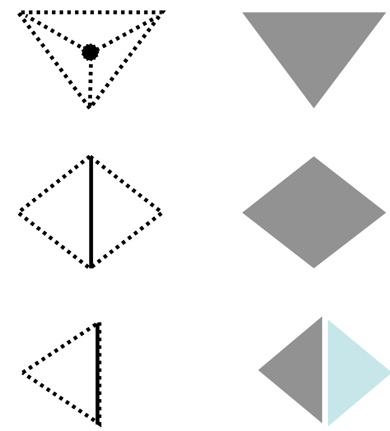
$$f : D \to \mathbb{R}^+$$

- Domains
  - Geometric
    - Boundary (Shape)
    - Interior (Attributes)
  - Structure
    - Components (Topology)
    - Combinatory (Quality)



Covering

- Elements Partition the Mesh in Regions
  - simp.Face ↔ ref.Vertex
  - int.Edge ↔ int.Edge
  - int.Edge ↔ bd.Edge



- Priority Queues
  - Sort Regions based on Adaptation Function

refinement

simp-faces	*int-edges*	*int-edges*
ref-vertices	*int-edges*	*bd-edges*

simplification

Maintaining the Queues

- Algorithm (5)

```
while (top_val > target)
    pop queue
    apply transition
    update elements
```

- *Same Algorithm for All Tasks!*
  - Refinement / Simplification / Improvement
- *Transition includes Dependencies*
- *Update re-evaluates Changes*

Dynamic Adaptation

- Algorithm (6)

T_0 = base_triangulation

initialize priority_queues

while (adapting)

    read parameters
    evaluate f on T_k
    simplify T_k
    refine T_k
    improve T_k

Algorithm (5)

** Obs: Conservative!*

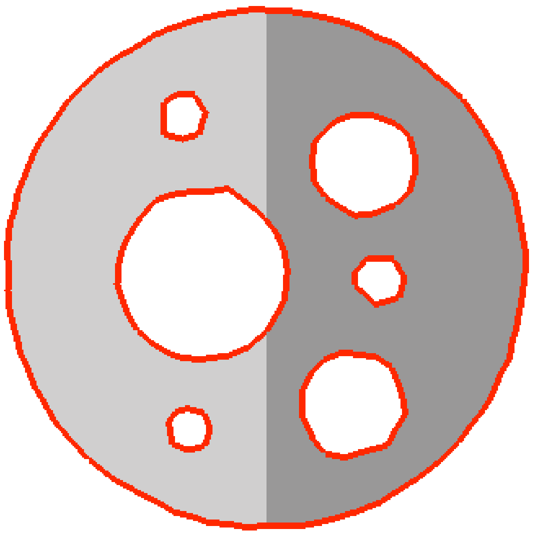
Applications

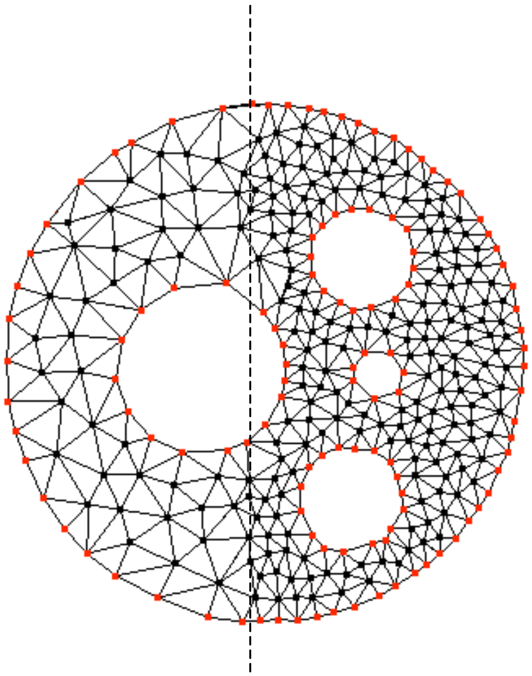
- Adaptation Criteria
  - Single Criteria
    (*defaults for others*)
  - Multiple Criteria
    (*must solve conflicts*)

- Examples
  - Attribute Resolution
  - Topology Granularity
  - Geometry Detail

Attribute Example

- Area of Triangles

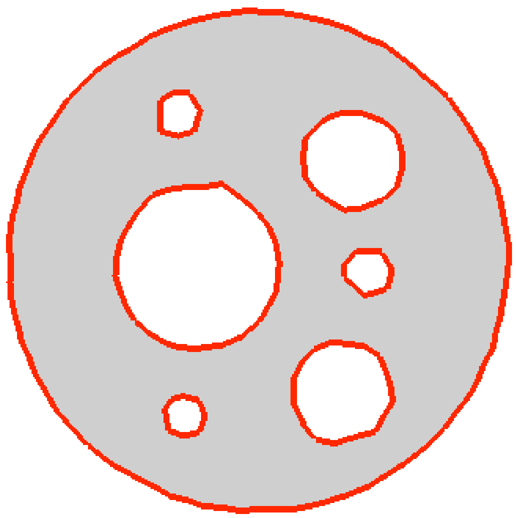


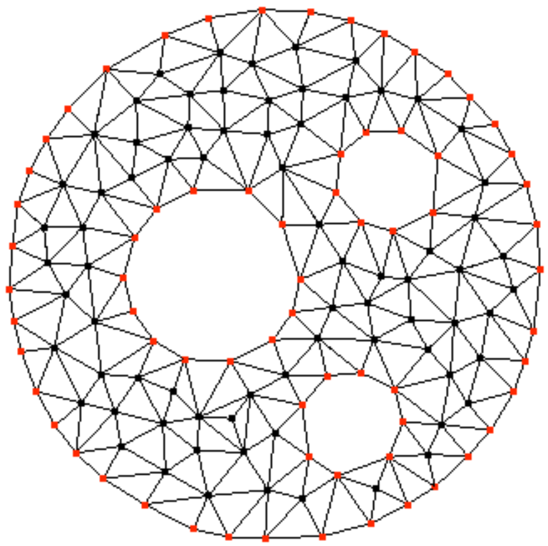


$a \approx n$ $\qquad$ $a \approx \frac{n}{3}$

Topological Example

- Size of Holes





$s > m$

Final Remarks

- Future Work
  - Applications
  - Extend to 3D
- Conclusions
  - Theoretical Results
  - Computationally Efficient
  - Effective in Practice

Thanks !