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The Geomorphologic Structure of Hydrologic Response

AUTOR

IGNACIO RODRÍGUEZ-ITURBE AND JUAN B. VALDÉS

Graduate Program in Hydrology and Water Resources, Universidad Simón Bolívar, Caracas, Venezuela

A unifying synthesis of the hydrologic response of a catchment to surface runoff is attempted by linking the instantaneous unit hydrograph (IUH) with the geomorphologic parameters of a basin. Equations of general character are derived which express the IUH as a function of Horton's numbers R_A, R_B, and R_L; an internal scale parameter L_Ω; and a mean velocity of streamflow v. The IUH is time varying in character both throughout the storm and for different storms. This variability is accounted for by the variability in the mean streamflow velocity. The underlying unity in the nature of the geomorphologic structure is thus carried over to the great variety of hydrologic responses that occur in nature. An approach is initiated to the problem of hydrologic similarity.

INTRODUCTION

The quantitative analysis of drainage networks has gone through dramatic advances since the 1960's, mainly after *Shreve*'s [1966] classical paper which led the way for a theoretical foundation of Horton's well-known empirical laws and provided a new perspective for many other problems in fluvial geomorphology. Although these developments are of great importance for hydrologists, there has been a void in the coupling of quantitative geomorphological analysis with the most important hydrologic variable, namely, the streamflow response to surface runoff of the geomorphological unity, the watershed. This paper is a first step in that direction with the conviction that the search for a theoretical coupling of quantitative geomorphology and hydrology is an area which will provide some of the most exciting and basic developments of hydrology in the future.

Figure 1 shows a hypothetical watershed with the Strahler ordering procedure: (1) Channels that originate at a source are defined to be first-order streams. (2) When two streams of order ω join, a stream of order $\omega + 1$ is created. (3) When two streams of different order join, the channel segment immediately downstream has the higher of the orders of the two combining streams. The quantitative expressions of Horton's laws are

Law of stream numbers

$$N_\omega/N_{\omega+1} = R_B$$

Law of stream lengths

$$\bar{L}_\omega/\bar{L}_{\omega-1} = R_L$$

Law of stream areas

$$\bar{A}_\omega/\bar{A}_{\omega-1} = R_A$$

[*Schumm*, 1956] where N_ω is the number of streams of order ω, $\bar{L}_\omega$ is the mean length of streams of order ω, and $\bar{A}_\omega$ is the mean area of the basins of order ω. R_B, R_L, and R_A represent the bifurcation ratio, the length ratio, and the area ratio whose values in nature are normally between 3 and 5 for R_B, between 1.5 and 3.5 for R_L, and between 3 and 6 for R_A.

A detailed description of channel networks (which also contains an outstanding synthesis of the geomorphologic aspects of interest in hydrologic response analysis) is that of *Smart* [1972]. We refer to Smart's work for an in-depth understanding of many of the implications of the above laws.

A basic question at this moment is, Given an ordered system of the geomorphologic elements of a basin and given that this system, in all its many possible forms and natural appearances, is well described by laws which respond to well-defined theories [*Shreve*, 1966, 1967], is there a manner to relate this order to hydrologic response characteristics? The implications of such a question are many. Basically, an understanding would be provided of the role of the geomorphologic properties in watershed hydrology instead of the so many and not very illuminating regressions we keep using in the field. The above question also holds the key for flood analysis in areas of insufficient or inexistent data as well as for the transposition of rainfall-runoff event data from one basin to another.

Hydrologists are familiar with the fantastic variety of forms and shapes that drainage networks may possess, and they are familiar with the variety of ways that nature may respond to precipitation inputs into a watershed. We know now that those shapes and forms of the drainage basin arise in their infinite variety from some basic themes, the geomorphological laws, that nature plays to interpret the structures we encounter in natural watersheds. It seems to us that there also should exist some basic themes in the structure of the hydrologic response of a basin. These themes should be related to the nature of the geomorphological structure and should contain the key to the grand synthesis which hydrologists always dream of. Many researchers long ago declared that this synthesis could never be quite attained. We do not share this view.

Even more important is the point that just the quest for the key or for pieces of it at least will lead to exciting new perspectives in hydrology and will get not only into the questions, What will happen . . . ?, but even more importantly into the questions, Why will it happen . . . ?, from which we seem to have been drifting during the last years because of pressing operational problems.

The search for a link between geomorphologic laws and hydrologic response needs some measure of description of the hydrologic response structure of a basin. The description used here is the instantaneous unit hydrograph (IUH) that is equivalent to the unit impulse response function of the basin.

THE TIME HISTORY OF ONE DROP OF EFFECTIVE RAINFALL

Consider a watershed such as the one in Figure 1 with a bucket at the outlet of the basin. We are interested in how fast the bucket is filled when a volume of rainfall excess of certain temporal and spatial characteristics is imposed on the water-

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shed. To make things simpler and to generalize the results, assume the input is a unit volume of effective precipitation uniformly distributed over the basin and instantaneously imposed upon it. The bucket at the outlet will start empty and will reach a final volume equal to the total volume of rainfall excess over the basin. A plot of this volume throughout time is the cumulative response of the basin, or what is the same, the total volume yielded as output up to a certain time t,

$$\text{volume } (t) = \int_0^t q(t)\, dt = V(t) \tag{1}$$

The derivative of the observed $V(t)$ gives the hydrograph of discharges $q(t)$ resulting from the rainfall input. This hydrograph $q(t)$ is the IUH. A different manner in which to look at the previous situation will be to search for the probability that a rainfall drop chosen at random from the input has reached the bucket at time t. A function describing this probability will appear as shown in Figure 2, starting at zero at the origin and reaching unity as time goes to infinity. The ordinate axis of Figure 2 can be interpreted as the percentage of drops reaching the outlet of the watershed at time t and thus is equivalent to $V(t)$ in (1). The derivative of $V(t)$, as in Figure 2, is the IUH of the basin.

A Statistical Framework for the IUH

Lienhard [1964] provided an approach to the study of the IUH from a purely statistical mechanical point of view. Our approach is necessarily different, since we want the geomorphologic structure to play an explicit role. The derivation of the probability that a rainfall drop chosen at random has reached the outlet at time t will be tackled by defining first some terms.

1. State is the order of the stream in which the drop is located at time t. When the drop is still in the overland phase, the state is the order of the stream to which the land drains directly. A drop may begin in any state, but all drops eventually terminate in the highest numbered state, $\Omega + 1$.
2. Transition is the change of state.
3. N is the number of states, i.e., $\Omega + 1$, where Ω is the order of the basin and the extra state is the bucket or trapping state.

The probabilistic description of the drainage network is made through its transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\ p_{21} & p_{22} & p_{23} & & p_{2N} \\ \vdots & & & & \\ p_{N1} & p_{N2} & p_{N3} & \cdots & p_{NN} \end{bmatrix} = \begin{bmatrix} 0 & p_{12} & p_{13} & \cdots & p_{1\Omega} & 0 \\ 0 & 0 & p_{23} & \cdots & p_{2\Omega} & 0 \\ 0 & 0 & 0 & \cdots & p_{3\Omega} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2}$$

where p_{ij} is the probability that the drop makes a transition from state i to state j. This is the same as the proportion of drops that, having entered state i, move next to state j. The Nth state is the bucket which is a trapping state.

The $\mathbf{P}$ matrix is not enough to describe the basin for our purposes because it does not take into account the dynamic characteristics which influence the time a drop spends in a state on its way to the outlet.

If the process of a drop going through the basin were one

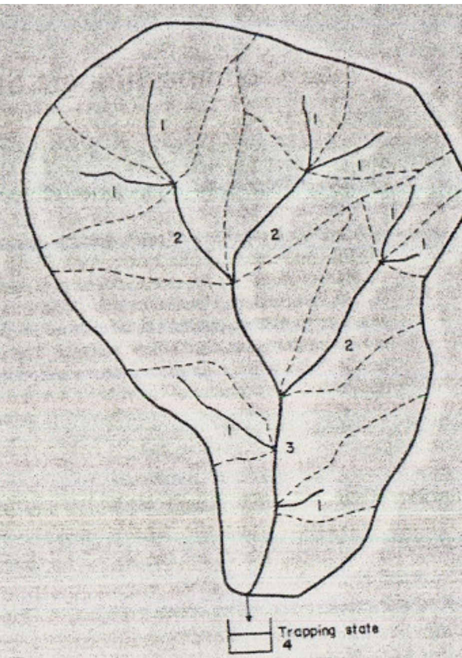


Fig. 1. Third-order basin with Strahler's ordering system and its trapping state.

where in each time step the drop made a transition (or, in other words, we were worried about the number of transitions and not interested in the time dimension as such), then $\mathbf{P}$ would be enough to describe the situation. But transitions occur at various times, not at the same time. Indeed, because there are an infinite number of drops and because time is treated as being continous, the simple concepts of Markov chains do not apply without modification to this problem. Suppose, nevertheless and for the moment, that in each time step the drop makes a transition, and suppose that the transition from one state to the next state only depends on the state where the drop is at this moment (Markovian hypothesis, which is reasonable), then our problem would be reduced to finding the state probability matrix $\Theta(n)$:

$$\Theta(n) = \Theta(0) \cdot \Phi(n) = \Theta(0) \cdot \mathbf{P}^n \tag{3}$$

where $\Theta(n)$ is a row vector whose elements $\theta_i(n)$ give the probability that the process (drop) is found in state i at step n. The matrix $\Phi(n)$ is the multistep transition probability matrix whose elements $\phi_{ij}(n)$ give the probability that the process goes from state i to state j after n transitions. Vector $\Theta(0)$ is the initial state probability vector (a row vector) whose elements $\theta_i(0)$ give the probability that the process starts at state i

or, in other words, that the drops starts its travel in a stream of order i.

Unfortunately, the simple scheme described above is not applicable to our problem because the state at a given time depends on the time between transitions as well as the number of steps, or transitions, to reach a certain state. In a watershed the time between transitions depends on the location of the drop because different streams in the same catchment have different dynamic characteristics. We think of this as a semi-Markovian process whose successive state occupancies are governed by the transition probabilities of a Markov process but whose time of stay in any state is described by a random variable that depends on the state presently occupied and on the state to which the next transition will be made. Thus at transition instants the semi-Markovian process behaves just like a Markov process. We call this process the imbedded Markov process. Nevertheless, the times at which transitions occur are governed by a different probabilistic mechanism.

The Formal Model

The order of the streams occupied by the drop on successive transitions are governed by the transition probabilities p_{ij} of the imbedded Markov process. But the time τ_{ij} that the drop will spend in state i before making a transition to state j is a random variable that can take on any positive value with probability density function $h_{ij}(\tau)$. We define now an unconditional waiting time in state i, τ_i, as the time spent by the drop in state i when one does not know its successor state. The τ_i is a random variable described by the waiting time density function

$$w_i(\tau) = \sum_{j=1}^{N} p_{ij} h_{ij}(\tau) \tag{4}$$

Following *Howard* [1971], we can define

- $\mathbf{H}(\ )$ matrix of holding time density functions, $N \times N$;
- $\mathbf{W}(\ )$ $N \times N$ diagonal matrix whose ith diagonal element is the unconditional waiting time density function $w_i(\ )$;
- ${}^{>}\mathbf{W}(\ )$ $N \times N$ diagonal matrix whose ith diagonal element is the complementary cumulative distribution ${}^{>}w_i(t) = \sum_{j=1}^{N} p_{ij} \cdot \text{Prob}\,[\tau_{ij} > t]$.

In this general model of the continuous time semi-Markov process the interval transition probabilities are given by *Howard* [1971]:

$$\phi_{ij}(t) = \delta_{ij}\,{}^{>}w_i(t) + \sum_{\alpha=1}^{N} p_{i\alpha} \int_0^t d\tau \cdot h_{i\alpha}(\tau)\phi_{\alpha j}(t-\tau) \tag{5}$$

$$i = 1, 2, \cdots, N \qquad j = 1, 2, \cdots, N$$

where $\phi_{ij}(t)$ represents the probability for the drop to go from state i to state j in the time interval t and δ_{ij} is

$$\delta_{ij} = 1 \qquad i = j$$
$$\delta_{ij} = 0 \qquad i \neq j$$

In matrix notation,

$$\mathbf{\Phi}(t) = {}^{>}\mathbf{W}(t) + \int_0^t d\tau[\mathbf{P} \,\square\, \mathbf{H}(\tau)]\mathbf{\Phi}(t-\tau) \tag{6}$$

where the operation $\mathbf{P} \,\square\, \mathbf{H}(\tau)$ stands for multiplication of corresponding elements.

Equation (6) will not take us very far because it is quite difficult to solve and it is impossible to generalize the results. Nevertheless, we can make two assumptions which make things considerably simpler:

1. Holding times τ_{ij} are independent of destination state. Then

$$h_{ij}(\tau) = w_i(\tau)$$

$$\mathbf{P} \,\square\, \mathbf{H}(\tau) = \mathbf{W}(\tau) \cdot \mathbf{P}(\tau)$$

2. Times between events are well described by the functionally appealing exponential density function. Thus the waiting time of the drop on a stream of order i is given by

$$w_i(\tau) = \lambda_i e^{-\lambda_i \tau} \qquad {}^{>}w_i(\tau) = e^{-\lambda_i \tau}$$

where λ_i is a different mean waiting time for each stream order.

Assumption 1 is quite realistic for the traveling of a drop; assumption 2 will be shown to be a reasonable hypothesis later on in this paper.

The mean waiting time matrix is Λ^{-1}, where

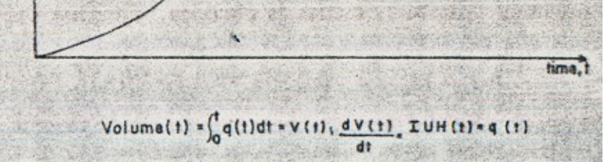
$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

the λ_i being the inverse of the mean waiting time in streams of order i.

The two previous hypotheses allow a drastic simplification of (6). Defining a transition rate matrix as

$$\mathbf{A} = \Lambda(\mathbf{P} - \mathbf{I})$$

$$\mathbf{A} = \begin{bmatrix} -\lambda_1 \sum_{j>1} p_{1j} & \lambda_1 p_{12} & \lambda_1 p_{13} & \cdots & 0 \\ 0 & -\lambda_2 \sum_{j>2} p_{2j} & \lambda_2 p_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$



$$\text{Volume}(t) = \int_0^t q(t)dt = V(t), \quad \frac{dV(t)}{dt} = IUH(t) = q(t)$$

Fig. 2. Effective rainfall volume collected at the trapping state as function of time, resulting from a unit input of precipitation.

the interval transition probability matrix becomes [*Howard,* 1971]

$$\Phi(t) = e^{\mathbf{A}t} \tag{7}$$

where $e^{\mathbf{A}t}$ is defined as $\mathbf{I} + \mathbf{A}t + (\mathbf{A}^2t^2/2!) + \cdots$.

Our final goal is the state probability matrix $\theta(t)$ whose elements $\theta_i(t)$ give the probability that the drop occupies state i at time t,

$$\theta(t) = \theta(0) \cdot \Phi(t) \tag{8}$$

where $\theta(0)$ represents the initial state probability row vector with the same interpretation we gave it in (3). The $\theta(0)$ depends on the spatial character of the rainfall, but under our assumption of uniform precipitation it will be extremely easy to compute.

Really, we are interested only in the last term of the row vector $\theta(t)$, which gives us the probability that our drop is at the bucket or outlet of the basin at time t and which we have pictured in Figure 2. *Howard* [1971] shows that the exponential transform of (7) is given by

$$\Phi^e(t) = [s\mathbf{I} - \mathbf{A}]^{-1} \tag{9}$$

Thus in order to find $\Phi(t)$ we need only to carry out (9) and then make a straightforward inversion of the transform. We will show this in detail for a third-order basin.

Third-Order Basin IUH

In this case, $N = 4$, and we have

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+\lambda_1 & -\lambda_1 p_{12} & -\lambda_1 p_{13} & 0 \\ 0 & s+\lambda_2 & -\lambda_2 & 0 \\ 0 & 0 & s+\lambda_3 & -\lambda_3 \\ 0 & 0 & 0 & s \end{bmatrix} \tag{10}$$

where use has been made of the fact that

$$p_{14} = p_{24} = 0 \qquad p_{23} = p_{34} = 1$$

Next we evaluate the inverse matrix $[s\mathbf{I} - \mathbf{A}]^{-1}$:

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{s(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}$$

$$\begin{bmatrix} s(s+\lambda_2)(s+\lambda_3) & s\lambda_1 p_{12}(s+\lambda_3) & s\lambda_1\lambda_2 p_{12} + s\lambda_1 p_{13}(s+\lambda_2) & \lambda_1\lambda_2\lambda_3 p_{12} + \lambda_1\lambda_3 p_{13}(s+\lambda_2) \\ 0 & s(s+\lambda_1)(s+\lambda_3) & s\lambda_2(s+\lambda_1) & \lambda_2\lambda_3(s+\lambda_1) \\ 0 & 0 & s(s+\lambda_1)(s+\lambda_2) & \lambda_3(s+\lambda_1)(s+\lambda_2) \\ 0 & 0 & 0 & (s+\lambda_1)(s+\lambda_2)(s+\lambda_3) \end{bmatrix}$$

and proceed to write it in a partial fraction expansion form,

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{s}[a_{ij}] + \frac{1}{s+\lambda_1}[b_{ij}] + \frac{1}{s+\lambda_2}[c_{ij}] + \frac{1}{s+\lambda_3}[d_{ij}] \tag{11}$$

Equation (11) is the expression of $\Phi^e(t)$, and the interval transition probability matrix is obtained by inverse exponential transformation,

$$\Phi(t) = [a_{ij}] + e^{-\lambda_1 t}[b_{ij}] + e^{-\lambda_2 t}[c_{ij}] + e^{-\lambda_3 t}[d_{ij}] \tag{12}$$

As we discussed for (8), we are only interested in the terms of the last column of $\Phi(t)$, namely, $\phi_{i4}(t)$, where $i = 1, 2, 3, 4$. This column, when multiplied by the row vector $\theta(0)$ of (8), yields $\theta_4(t)$, or state probability for state 4.

It is straightforward to obtain the above terms,

$$a_{14} = 1 \qquad a_{24} = 1 \qquad a_{34} = 1 \qquad a_{44} = 1$$

$$b_{14} = \frac{\lambda_3(\lambda_2 - \lambda_1 p_{13})}{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)} \qquad b_{24} = 0 \qquad b_{34} = 0 \qquad b_{44} = 0$$

$$c_{14} = \frac{\lambda_1\lambda_3 p_{12}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \qquad c_{24} = \frac{\lambda_3}{\lambda_2 - \lambda_3}$$

$$c_{34} = 0 \qquad c_{44} = 0$$

$$d_{12} = \frac{\lambda_1\lambda_2 - \lambda_1\lambda_3 p_{13}}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)} \qquad d_{24} = \frac{\lambda_2}{\lambda_3 - \lambda_2}$$

$$d_{34} = -1 \qquad d_{44} = 0$$

The probability that a drop chosen at random in state i (i = 1, 2, 3, 4) has reached the outlet at time t is given by

$$\phi_{14}(t) = 1 + \frac{\lambda_3[\lambda_2 - \lambda_1 p_{13}]}{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)} e^{-\lambda_1 t}$$

$$+ \frac{\lambda_1\lambda_3 p_{12}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} e^{-\lambda_2 t}$$

$$+ \frac{\lambda_1\lambda_2 - \lambda_1\lambda_3 p_{13}}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t} \tag{13}$$

$$\phi_{24}(t) = 1 + \frac{\lambda_3}{\lambda_2 - \lambda_3} e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_3 - \lambda_2} e^{-\lambda_3 t} \tag{14}$$

$$\phi_{34}(t) = 1 - e^{-\lambda_3 t} \tag{15}$$

Trapping state

$$\phi_{44}(t) = 1 \tag{16}$$

It is easy to check that in all cases, when $t \to \infty$, $\phi_{i4}(t) \to 1$, and when $t \to 0$, $\phi_{i4}(t) \to 0$, which have to hold from a physical point of view.

The probability that a drop chosen at random has reached the outlet at (or before) time t is given by

$$\theta_4(t) = \theta_1(0)\cdot\phi_{14}(t) + \theta_2(0)\cdot\phi_{24}(t) + \theta_3(0)\cdot\phi_{34}(t) \tag{17}$$

where use has been made of the fact that $\theta_4(0) = 0$. At this point we should add the irrelevance of the random entry, entering a continuous Markov or semi-Markov process at a random time rather than when a transition is completed does not affect the statistics of the process in any way [*Howard,* 1971].

We have defined $\theta_i(0)$ as the probability that the process starts at state i, or in other words that the drop starts its travel in a stream of order i. Thus we can write

$$\theta_1(0) = \frac{A_1^*}{A_T} \qquad \theta_2(0) = \frac{A_2^*}{A_T} \qquad \theta_3(0) = \frac{A_3^*}{A_T} \tag{18}$$

where A_i^* (i = 1, 2, 3) represents the total area of order i draining directly into a stream of order i and A_T is the total area of the basin.

The IUH for the third-order basin is now

$$\mathrm{IUH}(t) = \frac{d\theta_4(t)}{dt} = \theta_1(0)\cdot\frac{d\phi_{14}(t)}{dt} + \theta_2(0)\,\frac{d\phi_{24}(t)}{dt} + \theta_3(0)\,\frac{d\phi_{34}(t)}{dt} \qquad (19)$$

Our goal is to relate the $\theta_i(0)$ terms and the $d\phi_{i4}(t)/dt$ terms to the geomorphologic Horton numbers.

For a basin of any order there are two types of terms making up $d\phi_{iN}(t)/dt$, namely, the λ_i terms and the p_{ij} terms. The p_{ij} terms can be related directly to the geomorphological parameters. The p_{ij} stands for the probability that a drop goes from a stream of order i to a stream of order j; for the third-order basin we only have p_{12} and p_{13}. The question may arise, given that we can compute directly in each case from topographical maps the terms $\theta_i(0)$ in (18) as well as p_{12} and p_{13} (the proportion of first-order streams draining into second-order and third-order streams), Why express the p_{ij} and the $\theta_i(0)$ as functions of geomorphologic parameters? The reason for this is that one of the main goals of this research is to find out if the geomorphologic order of things is related to the hydrologic response. Thus instead of using (19) (or similar equations for basins of other orders) as just a tool for synthetic IUH derivation in each particular case it is important to write it as a function of those parameters which express the geomorphologic order as a result of the structural dictates of space. This we see as a way of bringing harmony and explanation to the infinite patterns of hydrologic response that nature creates and that arise, maybe, from the working of a few formal themes. Thus

$$p_{1i} = \frac{\text{number of streams of order 1 draining into order } i}{\text{total number of streams of order 1}} \qquad i = 2, 3 \qquad (20)$$

There are N_1 streams of order 1 of which $2N_2$ make up for the streams of order 2. The remaining $(N_1 - 2N_2)$ streams of order 1 drain into streams of orders 2 and 3. Following *Smart* [1968], we will assume that the lengths of interior links in a given network are independent random variables drawn from a common population. This assumption implies that the distribution of interior link lengths is independent of order, magnitude, or any other topologic characteristic, and then we may write that the $(N_1 - 2N_2)$ streams of order 1 join streams of orders 2 and 3 according to

$$(N_1 - 2N_2)\cdot\frac{\text{number of links of order } i}{\text{total number of links of orders 2 and 3}} \qquad i = 2, 3 \qquad (21)$$

The mean number of links of order ω in a finite network of order Ω is [*Smart*, 1971]

$$E(\nu, \omega, \Omega) = N_\omega \prod_{\alpha=2}^{\omega} \frac{N_{\alpha-1} - 1}{2N_\alpha - 1} \qquad \omega = 2, 3, \cdots, \Omega$$

Thus (1) the number of links of order 2 equals $N_2[(N_1 - 1)/(2N_2 - 1)]$, say, x, and (2) the number of links of order 3 equals $N_3[(N_1 - 1)/(2N_2 - 1)]\cdot[(N_2 - 1)/(2N_3 - 1)]$, say, y. Since $N_3 = 1$, the ratio $x/(x + y)$ gives $N_2/(2N_2 - 1)$.

Thus on the average the number of streams of order 1 that drain into second-order streams is

$$2N_2 + \frac{N_2}{2N_2 - 1}\cdot(N_1 - 2N_2)$$

We may then write

$$p_{12} = \frac{R_B^2 + 2R_B - 2}{2R_B^2 - R_B} \qquad (22)$$

Similarly, we can estimate the value of p_{13} for basins of order 3 as

$$p_{13} = \frac{R_B^2 - 3R_B + 2}{2R_B^2 - R_B} \qquad (23)$$

Having written the p_{ij} as a function of Horton's geomorphologic parameters, we will proceed now to do the same with the initial probabilities $\theta_i(0)$ of (19). Equation (18) shows

$$\theta_1(0) = \frac{A_1^*}{A_T} = \frac{N_1\bar{A}_1}{A_3} = R_B^2 R_A^{-2}$$

$$\theta_2(0) = \frac{A_2^*}{A_T} \qquad \theta_3(0) = \frac{A_3^*}{A_T} \qquad (24)$$

some analysis being necessary to rewrite A_2^* and A_3^*.

The number of streams of order 1 available to be tributaries of orders 2 and 3 is $N_1 - 2N_2$; of those the number going into second-order streams may be written as

$$(N_1 - 2N_2)\cdot\frac{\text{number of links of order 2}}{\text{total number of links of orders 2 and 3}} = (N_1 - 2N_2)\cdot\frac{N_2}{2N_2 - 1}$$

Thus on the average a stream of order 2 has

$$(N_1 - 2N_2)\cdot\frac{1}{2N_2 - 1} + 2$$

streams of order 1 that drain into it.

The average area draining directly into a second-order stream is

$$\bar{A}_2 - \bar{A}_1\left[\frac{N_1 - 2N_2}{2N_2 - 1} + 2\right]$$

and

$$\theta_2(0) = \frac{N_2}{A_T}\left[\bar{A}_2 - \bar{A}_1\left(\frac{N_1 - 2N_2}{2N_2 - 1} + 2\right)\right]$$

$$= \frac{R_B}{R_A} - \frac{R_B^3 + 2R_B^2 - 2R_B}{R_A^2(2R_B - 1)} \qquad (25)$$

We proceed along similar lines to write $\theta_3(0)$ as a function of Horton's ratios. There is one stream of order 3 into which drain all the N_2 streams of order 2. In addition, there are

$$N_1 - \frac{N_2\cdot(N_1 - 2N_2)}{2N_2 - 1} - 2N_2$$

first-order streams draining directly into the third-order stream.

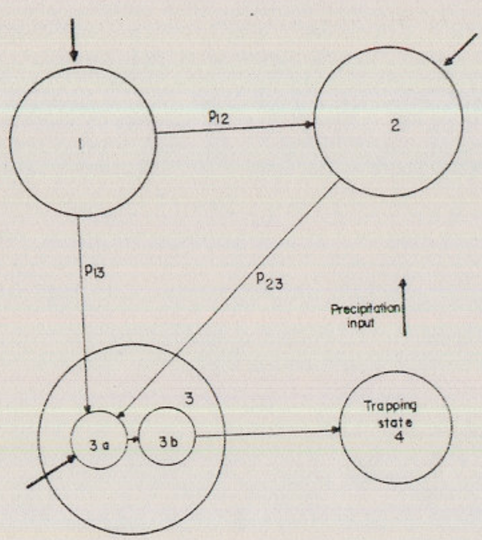


Fig. 3. Representation of a third-order basin as a continuous Markov process.

The area A_3^* draining directly into the third-order stream can then be written

$$A_T - N_2\bar{A}_2 - \bar{A}_1\left[N_1 - \frac{N_2(N_1 - 2N_2)}{2N_2 - 1} - 2N_2\right]$$

Finally,

$$\theta_3(0) = 1 - \frac{R_B}{R_A} - \frac{1}{R_A^2}\left[\frac{R_B(R_B^2 - 3R_B + 2)}{2R_B - 1}\right] \qquad (26)$$

There are some mathematical restrictions imposed on the values which R_B and R_A can take. Obviously, all the $\theta_i(0)$ have to be between zero and 1 and also $\sum_i \theta_i(0) = 1$. Thus from the expression of $\theta_i(0)$ for $\Omega = 3$ or for any other Ω one immedi-ately needs $R_A > R_B$. Similarly, other restrictions appear in the ratio R_A/R_B for higher-order $\theta_i(0)$. From a simple evaluation of the equations we concluded that the generalization of the $\theta_i(0)$ as a function of R_A and R_B can be carried out wherever the ratio $R_A/R_B \geq 1.2$. Even for much smaller values than 1.2, in most cases the generalization is still valid. The above mathematical problem, that for highly unusual values of R_A and R_B one may get negative $\theta_i(0)$ (which of course have no meaning), does not seem to impose major limitations for the study of drainage basins. In any case these restrictions follow from the basic assumption of random topologic development of drainage networks. Going back to (19), the only remaining terms to be expressed in a general manner are the parameters $\lambda_i (i = 1, 2, 3)$. The waiting time of a drop in a state of order i is assumed to be a random variable exponentially distributed with parameter λ_i. Therefore

$$E\,[\text{waiting time in state } i] = \lambda_i^{-1} \qquad (27)$$

In this manner, λ_i^{-1} is the mean time spent by a drop in state i when consideration has been made of both the time spent as overland flow and the time spent as streamflow. The importance of the overland waiting time appears to be rather smaller than that of the stream waiting time under the framework of analysis taken in this paper. When one considers drops traveling through a stream of order i, most of them will come from the two streams of order $i - 1$, which make up for the stream in question, or from tributary streams which drain along the route of our stream of order i. The only drops affected by overland waiting time will be those draining directly by overland flow into the stream of order i. These drops are in number considerably fewer, in general, than the above ones, and thus we feel that in average terms the mean waiting time in state i will be the streamflow waiting time. Only for streams of order 1 would one expect that most of the drops, except for channel precipitation, are affected by overland waiting time; because of the smaller size of the order 1 areas, this time is nevertheless considered to be of minor importance in the overall IUH.

It would also be possible to extend Horton's stream order concept to make the first order be overland flow under the frame work presented in this paper. Nevertheless, this was not done in our analyses in order to maintain the simplicity of the results.

The Waiting Time Mechanism

As mentioned earlier in this paper, the impacting advances of quantitative geomorphology show dramatically how the attribures of a watershed are decided by the constraints of space. This is even more explicit when one considers the small range of variation that Horton ratios have in natural river basins. Most of the basic principles governing the hydrologic response are believed to be known, but the apparent complexity of the phenomena prevents us from understanding and explaining them. As *Weiskopf* [1977] beautifully put it in his definition of the external frontier of science, 'the external frontier delimits the exploration of those realms of nature that lie beyond currently understood principles . . . an understanding of the principles by no means implies an understanding of the world of phenomena.' To explain the world of hydrologic phenomena, it will be necessary to develop scientific theories of general character. In respect to the structure of the hydrologic response these theories by necessity will have to be linked to the geomorphologic structure.

The IUH by (19) has been expressed as a function of R_A, R_B, the watershed order Ω, and the λ_i. We know that river basins can change the shape of their IUH in response to a change in scale, and yet, at the same time, and in seeming contradiction, have the same shape at different scales. Since the scale does not depend on R_A, R_B, or Ω, the reason for the above observation should lie in the λ_i, which should contain both a size effect and the dynamic component of the response.

How realistic is the assumption of an exponential distribution for the random variable describing the waiting time for streams of order i? We feel it is quite a workable one; consider a basin of, say, order 3 with $\bar{L}_1 = 250$ m and assume a velocity of 2 m/s. For the first-order streams the time of residence is 1.25 min for those drops traveling the whole course of the average first-order stream. With an $R_L = 3$ ($L_3 = 2250$ m) the time of residence for the drop traveling the whole third-order stream is about 18 min. Thus except for long streams of the highest order in larger-order basins the average waiting time of a drop seems to be localized in the first two intervals in which we will be estimating the IUH (for example, intervals of 10 min). The true distribution for the waiting time will be something like a gamma type starting at zero and positive skewness; if the mean of this distribution is as we have seen above, close (for our purposes) to the origin, then the mode will be even closer to the origin, making the exponential assumption a realistic one.

For the stream of the highest order we prefer to modify the exponential distribution for two reasons:

1. As discussed before, the mode starts to shift to the right.
2. More important is the fact that the assumption of an exponential distribution is equivalent to that of a linear reservoir. This for the highest-order stream implies that the basin excited by an instantaneous input responds with an exponential type of outflow. This exponential type of response coming from the highest-order stream will produce a hydrograph for the whole basin which does not start at zero but at an ordinate equal to the ordinate at the origin of the partial unit impulse response function corresponding to the highest-order subbasin.

Since the mathematical theory becomes extremely cumbersome for nonexponential distributions of the waiting times, the subbasin of the highest order is artificially represented as two linear reservoirs. This is shown in Figure 3, which pictures the connections between the different parts which make up the structure of the basis for the case $\Omega = 3$. Notice that drops in the second-order streams can only go to the third-order stream, but now the third-order stream is represented by two states $3a$ and $3b$. State $3a$ receives the drops from all second-order streams, part of first-order streams, and those drops draining directly into the third-order stream. All these drops are passed to the state $3b$, which is the one that feeds the bucket. We wish that the combination of $3a$ and $3b$, that is to say, the third-order state, had a mean waiting time of λ_3^{-1}, corresponding to the dynamic characters of the third-order stream. We assign to $3a$ and $3b$ identical exponential distributions with mean waiting times of $0.5\lambda_3^{-1}$; the sum of these two exponentials is pictured in Figure 4. The distribution of the waiting time for the third-order stream is now

$$w_3(\tau) = \mu^2 \tau e^{-\mu\tau} \tag{28}$$

with mean value of $\lambda_3^{-1} = 2\mu^{-1}$.

The adoption of the extra state $3b$ changes the expressions for $\phi_{14}(t)$, $\phi_{24}(t)$, and $\phi_{34}(t)$ given by (13), (14), and (15), but the methodology to obtain them remains exactly the same.

For a basin of order 3 the transition probability matrix is now given by

$$\mathbf{P} = \begin{bmatrix} 0 & p_{12} & p_{13} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

and the transition rate matrix becomes

$$\mathbf{A} = \begin{bmatrix} -\lambda_1 & \lambda_1 p_{12} & \lambda_1 p_{13} & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & 0 \\ 0 & 0 & -2\lambda_3 & 2\lambda_3 & 0 \\ 0 & 0 & 0 & -2\lambda_3 & 2\lambda_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

Calling the bucket state 5 and calling former states $3a$ and $3b$ states 3 and 4, respectively, one finds

$$\mathrm{IUH} = \frac{d\theta_5(t)}{dt} = \theta_1(0)\,\frac{d\phi_{15}(t)}{dt} + \theta_2(0)\,\frac{d\phi_{25}(t)}{dt} + \theta_3(0)\,\frac{d\phi_{35}(t)}{dt} \tag{31}$$

where use has been made of the fact that $\theta_4(0) = 0$. The derivative terms in (31) are given by

$$\phi_{15}(t) = 1 + A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t} + A_3 \cdot t \cdot e^{-\lambda_3^* t} + A_4 e^{-\lambda_3^* t}$$

where $\lambda_3^* = 2\lambda_3$ and

$$A_1 = \frac{(\lambda_3^*)^2 \{\lambda_1 p_{13} - \lambda_2\}}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_1)^2}$$

$$A_2 = \frac{(\lambda_3^*)^2 \lambda_1 p_{12}}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_2)^2}$$

$$A_3 = \frac{\lambda_3^* \{\lambda_1 \lambda_2 - \lambda_1 \lambda_3^* p_{13}\}}{(\lambda_1 - \lambda_3^*)(\lambda_3^* - \lambda_2)}$$

$$A_4 = \{(\lambda_3^*)^3 \lambda_1 p_{13} (\lambda_3^* - \lambda_1)(\lambda_2 - \lambda_3^*) -) - [3(\lambda_3^*)^2 - 2\lambda_2\lambda_3^* - 2\lambda_1\lambda_3^* + \lambda_1\lambda_2][\lambda_1\lambda_2(\lambda_3^*)^2 - \lambda_1(\lambda_3^*)^3 p_{13}]\} \div [(\lambda_3^*)^2 (\lambda_1 - \lambda_3^*)^2 (\lambda_2 - \lambda_3^*)^2]$$

$$\phi_{25}(t) = 1 - \frac{(\lambda_3^*)^2}{(\lambda_3^* - \lambda_2)^2} e^{-\lambda_2 t} + \frac{\lambda_3^* \lambda_2}{\lambda_3^* - \lambda_2} e^{-\lambda_3^* t} \cdot t + \frac{\lambda_2 (2\lambda_3^* - \lambda_2)}{(\lambda_2 - \lambda_3^*)^2} e^{-\lambda_3^* t}$$

$$\phi_{35}(t) = 1 - \lambda_3^* e^{-\lambda_3^* t} \cdot t - e^{-\lambda_3^* t}$$

The partial IUH corresponding to the highest-order subbasin (3 in this case) is given by $d\phi_{35}(t)/dt$ and will now start from zero at the origin.

The equations for the initial probabilities $\theta_i(0)$ and for the transition probabilities p_{ij} as function of the Horton numbers remain as given before, since they are unaffected by the extra state $3b$.

As discussed at the beginning of this section, the λ_i should contain both a size or scale effect and the dynamic component of the response. We need a number of λ_i equal to the order Ω of the basin; nevertheless, this can be tackled in a simple manner. Let v be the average streamflow velocity in the catchment. Then

$$\lambda_i = v/\bar{L}_i \tag{32}$$

which implies

$$\lambda_1 = v/\bar{L}_1 \qquad \lambda_2 = \lambda_1 \cdot R_L^{-1} \qquad \lambda_3 = \lambda_1 \cdot R_L^{-2} \tag{33}$$

assuming that for a given rainfall-runoff event the velocity at any moment is approximately the same throughout the whole drainage network.

The above assumption is based on the pioneer work of *Leopold and Maddock* [1953] and has been experimentally validated by many studies, the most recent one by *Pilgrim* [1977]. Leopold and Maddock show that the change in velocity in the downstream direction when considering a discharge of given frequency throughout the basin is very small. Changes in width, depth, and possibly roughness more than compensate for the effects of slope, producing in theory a very small increment in the velocity when advancing downstream during a discharge of fixed frequency. Pilgrim found from

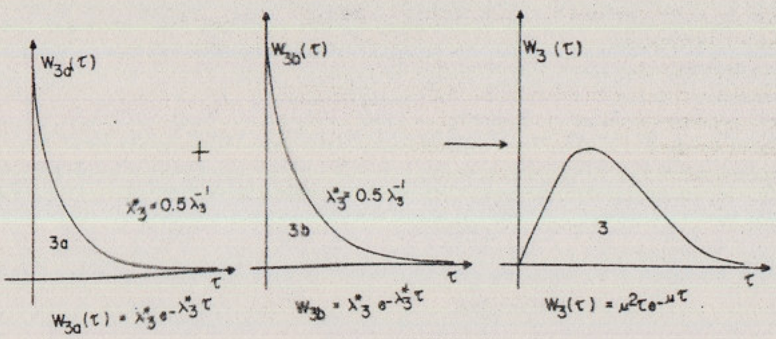


Fig. 4. Representation of the waiting time probability function at the highest-order stream.

tracer experiments that the average velocity tends to be nearly constant in a downstream direction.

Equation (33) now gives all the λ_i as function only of a dynamic parameter v, the Horton length ratio R_L, and a scale or size factor $\bar{L}_1$ (or any other $\bar{L}_i$). Since L_Ω is easier to measure with higher precision than $\bar{L}_1$, it is better to use L_Ω as scale factor and write the λ_i as function of λ_Ω.

The analytical derivation of the IUH was carried out for $\Omega = 3$, 4, and 5. The resulting equations show some similitudes which immediately suggest the possibility of a general synthesis, which will be attempted later in this paper.

Figure 5 shows examples of IUH's computed for different values of the geomorphologic parameters but with a fixed velocity. Figure 6 shows examples of IUH's computed for the same set of geomorphologic parameters when v is varying.

The Geomorphologic IUH

We have expressed the IUH as a function of R_A, R_B, R_L, the velocity v, and the scale parameter L_Ω. What is the meaning of the velocity v? It tells us that the IUH varies both from storm to storm and also throughout the same storm. It gives us the key to the time-varying IUH analysis. The dependence of the

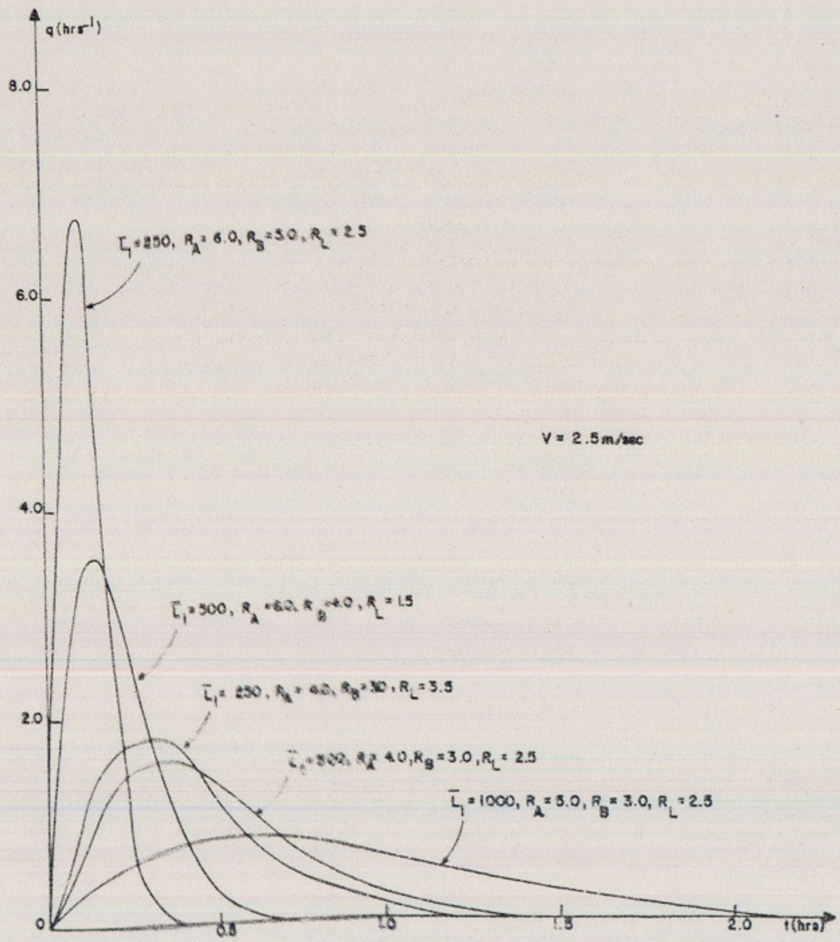


Fig. 5. Examples of the changes in the IUH when velocity is fixed but the geomorphologic characteristics are changing, ($\bar{L}_1$ in meters).

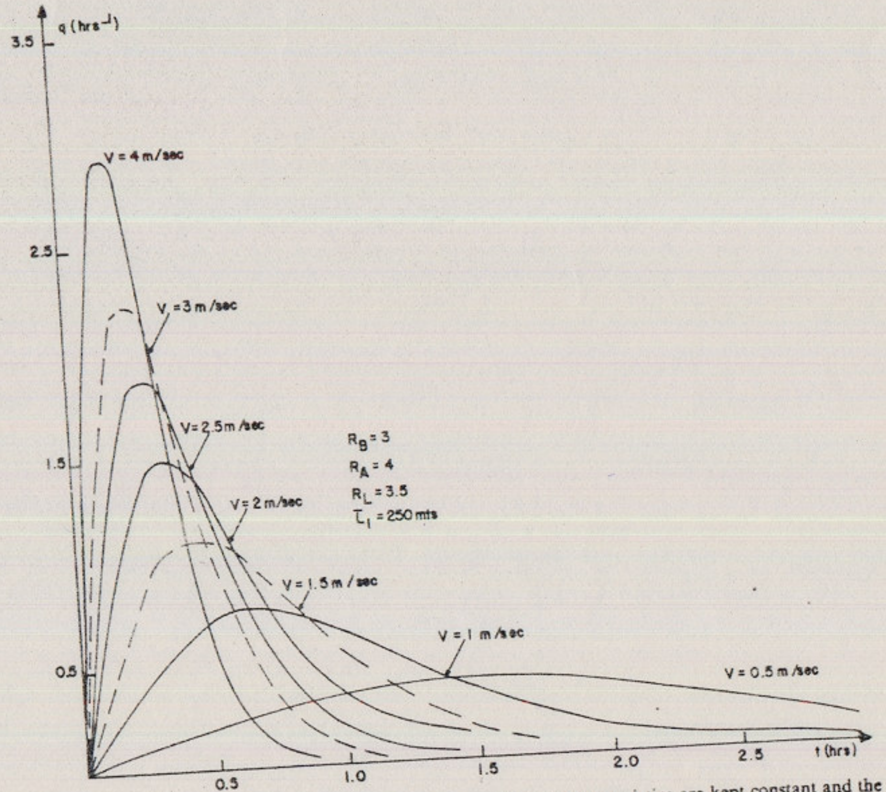


Fig. 6. Examples of the changes in the IUH when the geomorphologic characteristics are kept constant and the velocity varies.

IUH on velocity has serious implications in the ways to approach a design problem or, in general, in the estimation of the peak flow and time to peak flow of real storms when using unit hydrograph methods. This topic is discussed by *Rodriguez-Iturbe et al.* [1979]. We believe that many of the criticisms of the IUH analysis based on the fact that different IUH's are obtained for different storms and which are commonly attributed to the nonlinearities of the system, which of course exist, may be addressed in terms of a time-varying IUH. This is substantiated in the results of the companion papers by *Valdés et al.* [1979] and *Rodriguez-Iturbe et al.* [1979]. The effect of v on the IUH will be shown in the experiments of the next section of this paper. What we are saying is that the results indicate that the nonlinear effects imbedded in the response of a basin manifest themselves in the velocity of the discharge; thus a time-varying linear framework evolving with the velocity is a valid one to the problem.

To test the framework described in the previous section, four natural basins and three synthetically built ones were analyzed in great detail, and a very disaggregated representation of each of them was carried out by means of a rainfall-runoff model. Through very controlled experiments a set of IUH's for each basin was derived from the rainfall-runoff model such that each IUH corresponded to a different flow velocity kept constant during the event. These IUH's were compared to the IUH's derived from the geomorphologic approach. The experiments, the results, and their implications are described in the papers by *Valdés et al.* [1979] and *Rodriguez-Iturbe et al.* [1978]. In all cases the agreement was excellent, suggesting the proposed framework is a valid one. Nevertheless, let us point out again that the goal of this research is not to implement a design tool useful by itself specially in ungaged basins. We need the experiments in order to feel confident we are on stable grounds, but the aim of the effort is to understand the nature and development of existing hydrologic hierarchies. This requires a much deeper insight into the interactions and their manifestations than we have gained today in hydrology and geomorphology.

The proposed mathematical framework also makes it possible to study on a systematic basis

(1) some effects of nonuniform rainfall in the derived response function (This study may be carried out by varying the initial state probabilities $\theta_i(0)$ and will shed light on the relative importance of the different structures which make up the basin in the hydrologic response of the watershed. A forthcoming paper by the authors analyzes this problem.) and (2) the effect of infiltration and other losses in different geomorphologic subunits of the basin in the hydrologic response of the watershed (This study is carried out by adding another state to the representation of the basin. This state accounts for the transformation from precipitation to effective rainfall, and there is a transition probability from each stream order to this new state.). An important point is that the above analyses can be carried out on a general basis without being subscribed to particular basins.

As mentioned before, this paper gives the equations for the geomorphologic IUH of a third-order basin. But equations for higher-order basins can be derived with exactly the same framework. Although the derivation is simple, the procedures are quite lengthy. This is irrelevant, nevertheless, because the

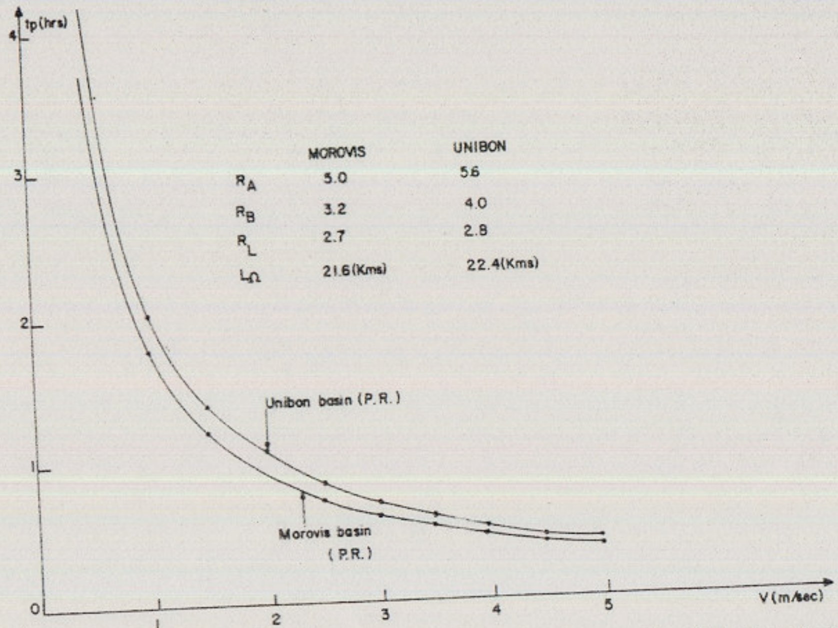


Fig. 7. Examples of the variation of the time to peak of the IUH as function of the flow velocity.

equations for all orders are related as will be shown in the next section of this paper. It is important to notice at this moment that different hydrologists may assign different Ω to the same basin, depending on judgment and the scale of the map. R_A, R_B, and R_L, on the other hand, do not depend on the scale of the map. Clearly, the IUH should be the same for both hydrologists, but the equations are different in their functional structure because they represent two differents Ω. It turns out, as the experiments of the next section show, that both IUH's agree almost perfectly as long as one compares, say, an IUH of fourth order with a certain $\bar{L}_1$ with an IUH of third order with $\bar{L}_1{}^* = L_1 \cdot R_L$ maintaining in both cases the same R_A, R_B, and R_L.

The Peak and Time to Peak of the IUH: A Geomorphologic Synthesis

The most important characteristics of an IUH are the peak q_p and the time to peak t_p. As long as these two factors are correct, the exact form of the IUH is not very important, and a triangular approximation is quite satisfactory [*Henderson*, 1963]. Unfortunately, the sum of exponential functions in the IUH expression does not lend itself to mathematical manipu-

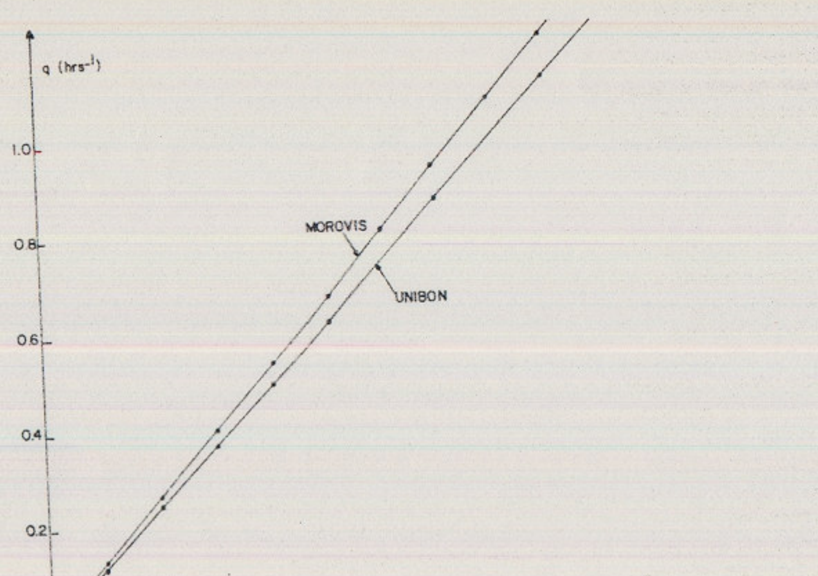


Fig. 8. Examples of the variation of the peak of the IUH as a function of the flow velocity.

lation in order to obtain the maximum of the function. Thus we resorted to an accurate approximation involving values of q_p and t_p obtained in the computer from the expressions of the IUH for different velocities in the range 0.5–6 m/s and for $\Omega = 3, 4, 5$ with $\bar{L}_1$ (the scale factor) varying from 125 to 2000 m. These calculations were carried out for 126 combinations of values of R_B, R_A, and R_L in the ranges 2.5–5.0, 3.0–6.0, and 1.5–4.1. For fixed R_A, R_B, R_L, $\bar{L}_1$, and Ω one notices that q_p and t_p are very simply related to the velocity v.

Figures 7 and 8 show the points obtained for q_p and t_p from the IUH equations for a typical computation and illustrate how these points can be fitted extremely well by some simple functional dependence with v. The chosen relationships are

$$q_p = \theta \cdot v \qquad (34)$$

$$t_p = k/v \qquad (35)$$

where θ and k depend on R_A, R_B, R_L, $\bar{L}_1$, and Ω. Equations (34) and (35) adjust extremely well the dependence of q_p and t_p on v, the R^2 are indistinguishable of 1, and, more important, each value of the geomorphologically derived q_p and t_p was compared with the ones yielded by (34) and (35). This was carried out for all the 126 combinations of R_B, R_A, and R_L which are calculated for each $\bar{L}_1$ and for each Ω. In all cases, differences between the exact values of the IUH equations and those of (34) and (35) were under 10%.

The functional dependence of q_p and t_p on v contained in (34) and (35) is somewhat expected; if one approximates the IUH with a triangle, then

$$(q_p \cdot t_b)/2 = 1$$

where t_b stands for the base time or total duration of the IUH. The t_b is the time that it takes the last drop of the unit impulse rainfall to reach the outlet of the basin. Thus t_b is some length over a certain velocity, and q_p then will be a velocity over a length. Therefore θ and k have dimensions of L^{-1} and L, respectively.

The task is now to find the geomorphologic dependence of θ and k. With fixed $\bar{L}_1$ and Ω a regression analysis was performed between the 126 combinations of R_B, R_A, and R_L versus θ and k. The regressions giving a better fit are of multiplicative form, for example,

$$k = \alpha R_B^{\beta_1} R_A^{\beta_2} R_L^{\beta_3}$$

With all the R^2 above 0.97 and most of them above 0.99 they are shown in detail in the report by *Rodriguez-Iturbe et al.* [1979].

It is crucial to understand that the regression analyses performed here are not empirical; we knew the functional relationship of the geomorphologic IUH, and thus the regressions have to yield excellent fits. Their only purpose is of an operational character in order to present general results which are very difficult to obtain with straight mathematics from IUH equations because their form, sum of exponentials, does not lend itself to clean mathematics.

The generalization of the results may be better understood in terms of an example taken from the computations. For a third-order basin ($\Omega = 3$) and a size parameter $\bar{L}_1 = 500$ m the following regression equations are obtained for θ and k:

$$\theta = 2.61\ R_L^{-1.57} \qquad R^2 = 0.997 \qquad (36)$$

$$k = 0.22 R_B^{0.56} R_A^{-0.55} R_L^{1.62} \qquad R^2 = 0.993 \qquad (37)$$

For $\Omega = 3$ and $\bar{L}_1 = 1000$ m the equations are

$$\theta = 1.31 R_L^{-1.57} \qquad R^2 = 0.997 \qquad (38)$$

$$k = 0.44 R_B^{0.56} R_A^{-0.55} R_L^{1.62} \qquad R^2 = 0.992 \qquad (39)$$

The important point is that for fixed Ω the exponents of R_B, R_A, and R_L variables remain practically the same for all values of $\bar{L}_1$. The coefficient in front of the equation for both θ and k is in almost exact proportion to the size of $\bar{L}_1$ in all the analyzed cases. In this manner, for $\Omega = 3$ we can write the general equations

$$\theta = 1.31/\bar{L}_1 \cdot R_L^{-1.57} \qquad (40)$$

$$k = 0.44 \bar{L}_1 R_B^{0.55} R_A^{-0.55} R_L^{1.62} \qquad (41)$$

where $\bar{L}_1$ is expressed in kilometers, since we have used the coefficients obtained for $\bar{L}_1 = 1000$ m.

The role of Ω is detected when it is noticed that for the same $\bar{L}_1$ one finds

$$\theta_{\Omega+i} = \theta_\Omega/(R_L)^i \qquad (42)$$

$$k_{\Omega+i} = k_\Omega \cdot (R_L)^i \qquad (43)$$

Notice that while Ω is dependent on map scale and subjective judgment, the Horton numbers are not, and thus (36) and (37) yield the same values of θ and k for a basin that two hydrologists may have identified with different Ω. This is a convenient and necessary feature for the framework to have practical value.

Equations (42) and (43) hold extremely well for all the individual cases. One may then rewrite (40) and (41) for $\Omega = 3$ as

$$\theta = \frac{1.31}{L_\Omega R_L^{1-\Omega}} R_L^{-1.57}$$

$$k = 0.44 L_\Omega R_L^{1-\Omega} R_B^{0.55} R_A^{-0.55} R_L^{1.62}$$

and for any Ω and any $\bar{L}_1$ one has

$$\theta = \frac{1.31}{L_\Omega R_L^{1-\Omega}} R_L^{-1.57} \frac{1}{R_L^{\Omega-3}}$$

$$k = 0.44 L_\Omega R_L^{1-\Omega} R_B^{0.55} R_A^{-0.55} R_L^{1.62} R_L^{\Omega-3}$$

which simplify to

$$\theta = 1.31/L_\Omega \cdot R_L^{0.43} \qquad (44)$$

$$k = 0.44 L_\Omega R_B^{0.55} R_A^{-0.55} R_L^{-0.38} \qquad (45)$$

Equations (44) and (45) are the basic general equations which allow the estimation of the peak and time to peak of the IUH through the relations

$$q_p = \theta \cdot v \qquad t_p = k/v$$

In (44), θ represents the slope of the line q_p (h^{-1}) versus v (m/s); thus with L_Ω in kilometers one estimates θ by means of (44) and multiplies its value by the velocity in meters per second to obtain q_p (h^{-1}). Similarly, the k obtained by (45), when divided by v (m/s), gives the estimate of t_p in hours.

It is interesting to notice that the product $q_p \cdot t_p$ is independent of the velocity v and the scale variable L_Ω. Calling this dimensionless product IR, one may write

$$IR = q_p \cdot t_p = 0.58 (R_B/R_A)^{0.55} \cdot R_L^{0.05}$$

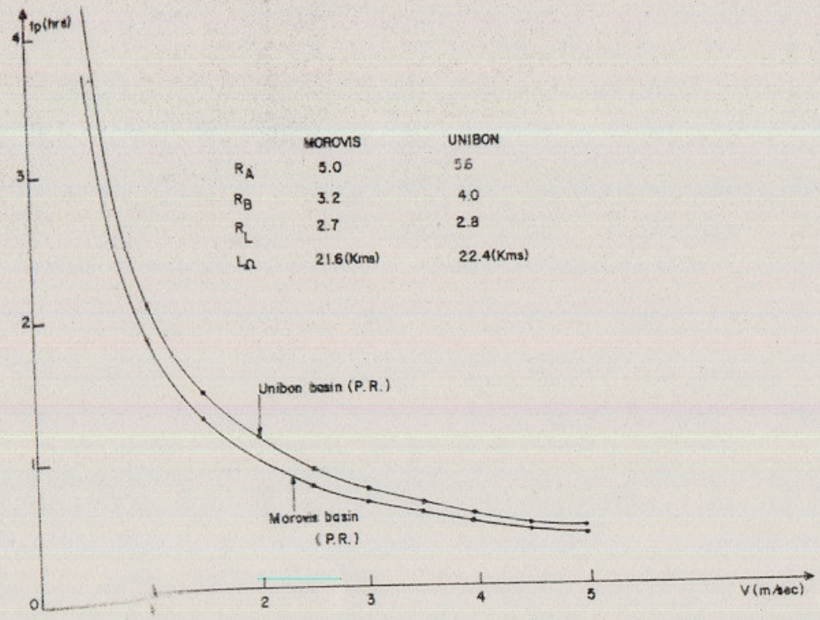


Fig. 7. Examples of the variation of the time to peak of the IUH as function of the flow velocity.

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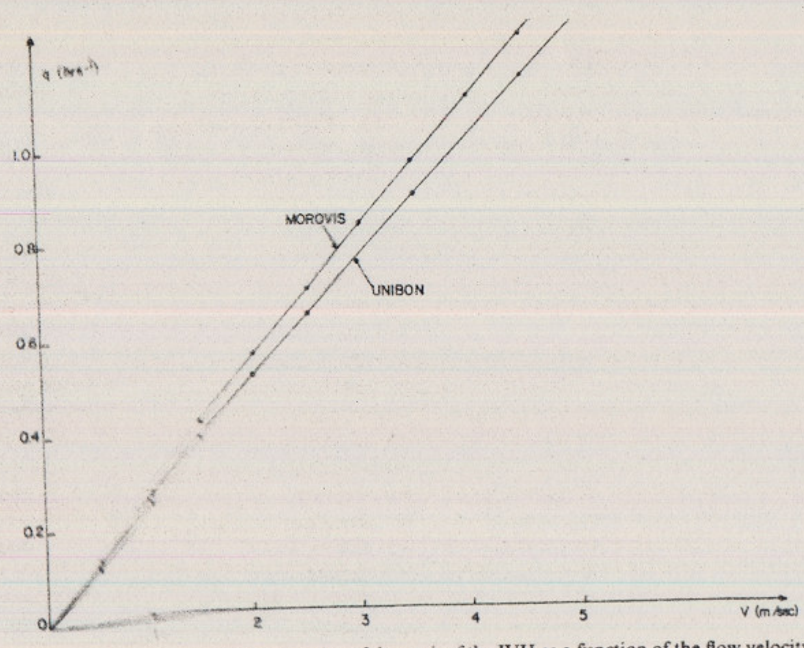

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The functional dependence of q_p and t_p on v contained in (34) and (35) is somewhat expected: if one approximates the IUH with a triangle, then

$$(q_p \cdot t_b)/2 = 1$$

where t_b stands for the base time or total duration of the IUH. The t_b is the time that it takes the last drop of the unit impulse rainfall to reach the outlet of the basin. Thus t_b is some length over a certain velocity, and q_p then will be a velocity over a length. Therefore θ and k have dimensions of L^{-1} and L, respectively.

The task is now to find the geomorphologic dependence of θ and k. With fixed $\bar{L}_1$ and Ω a regression analysis was performed between the 126 combinations of R_B, R_A, and R_L versus θ and k. The regressions giving a better fit are of multiplicative form, for example,

$$k = \alpha R_B^{\beta_1} R_A^{\beta_2} R_L^{\beta_3}$$

With all the R^2 above 0.97 and most of them above 0.99 they are shown in detail in the report by *Rodríguez-Iturbe et al.* [1979].

It is crucial to understand that the regression analyses performed here are not empirical: we knew the functional relationship of the geomorphologic IUH, and thus the regressions have to yield excellent fits. Their only purpose is of an operational character in order to present general results which are very difficult to obtain with straight mathematics from IUH equations because their form, sum of exponentials, does not lend itself to clean mathematics.

The generalization of the results may be better understood in terms of an example taken from the computations. For a third-order basin ($\Omega = 3$) and a size parameter $\bar{L}_1 = 500$ m the following regression equations are obtained for θ and k:

$$\theta = 2.61\ R_L^{-1.57} \qquad R^2 = 0.997 \tag{36}$$

$$k = 0.22 R_B^{0.56} R_A^{-0.55} R_L^{1.62} \qquad R^2 = 0.993 \tag{37}$$

For $\Omega = 3$ and $\bar{L}_1 = 1000$ m the equations are

$$\theta = 1.31 R_L^{-1.57} \qquad R^2 = 0.997 \tag{38}$$

$$k = 0.44 R_B^{0.56} R_A^{-0.55} R_L^{1.62} \qquad R^2 = 0.992 \tag{39}$$

The important point is that for fixed Ω the exponents of R_B, R_A, and R_L variables remain practically the same for all values of $\bar{L}_1$. The coefficient in front of the equation for both θ and k is in almost exact proportion to the size of $\bar{L}_1$ in all the analyzed cases. In this manner, for $\Omega = 3$ we can write the general equations

$$\theta = 1.31/\bar{L}_1 \cdot R_L^{-1.57} \tag{40}$$

$$k = 0.44 \bar{L}_1 R_B^{0.55} R_A^{-0.55} R_L^{1.62} \tag{41}$$

where $\bar{L}_1$ is expressed in kilometers, since we have used the coefficients obtained for $\bar{L}_1 = 1000$ m.

The role of Ω is detected when it is noticed that for the same $\bar{L}_1$ one finds

$$\theta_{\Omega+i} = \theta_\Omega/(R_L)^i \tag{42}$$

$$k_{\Omega+i} = k_\Omega \cdot (R_L)^i \tag{43}$$

Notice that while Ω is dependent on map scale and subjective judgment, the Horton numbers are not, and thus (36) and (37) yield the same values of θ and k for a basin that two hydrologists may have identified with different Ω. This is a convenient and necessary feature for the framework to have practical value.

Equations (42) and (43) hold extremely well for all the individual cases. One may then rewrite (40) and (41) for $\Omega = 3$ as

$$\theta = \frac{1.31}{L_\Omega R_L^{1-\Omega}} R_L^{-1.57}$$

$$k = 0.44 L_\Omega R_L^{1-\Omega} R_B^{0.55} R_A^{-0.55} R_L^{1.62}$$

and for any Ω and any $\bar{L}_1$ one has

$$\theta = \frac{1.31}{L_\Omega R_L^{1-\Omega}} R_L^{-1.57} \frac{1}{R_L^{\Omega-3}}$$

$$k = 0.44 L_\Omega R_L^{1-\Omega} R_B^{0.55} R_A^{-0.55} R_L^{1.62} R_L^{\Omega-3}$$

which simplify to

$$\theta = 1.31/L_\Omega \cdot R_L^{0.43} \tag{44}$$

$$k = 0.44 L_\Omega R_B^{0.55} R_A^{-0.55} R_L^{-0.38} \tag{45}$$

Equations (44) and (45) are the basic general equations which allow the estimation of the peak and time to peak of the IUH through the relations

$$q_p = \theta \cdot v \qquad t_p = k/v$$

In (44), θ represents the slope of the line q_p (h^{-1}) versus v (m/s); thus with L_Ω in kilometers one estimates θ by means of (44) and multiplies its value by the velocity in meters per second to obtain q_p (h^{-1}). Similarly, the k obtained by (45), when divided by v (m/s), gives the estimate of t_p in hours.

It is interesting to notice that the product $q_p \cdot t_p$ is independent of the velocity v and the scale variable L_Ω. Calling this dimensionless product IR, one may write

$$IR = q_p \cdot t_p = 0.58 (R_B/R_A)^{0.55} \cdot R_L^{0.05}$$

For the range of values one may possibly find in nature, IR simplifies to

$$IR = 0.58(R_B/R_A)^{0.55} \tag{46}$$

The ratio IR is a constant for each basin and indicates that the IUH description can be acomplished in practical terms with only one parameter (in this case, either t_p or q_p). This observation has been made in the past in empirical terms by many hydrologists. It also appears that IR could play an interesting role when trying to approach the elusive and difficult problem of hydrologic similarity or, in other words, when trying to make inferences about the structure of the hydrologic response of different basins.

Conclusions

1. The structure of the hydrologic response is intimately linked to the geomorphologic parameters of a basin. When the hydrologic response is represented by the IUH, it is found that it can be expressed in a general manner dependent on R_A, R_B, R_L, a scale variable L_Ω, and a dynamic parameter v. Thus the IUH varies from storm to storm and throughout the same storm as a function of the velocity v which occurs in the different instances of time throughout the basin.

2. Equations (44) and (45) combined with (34) and (35) represent a general relationship which allows the estimation of the peak and time to peak of the IUH of a watershed.

3. The dimensionless ratio IR is a characteristic variable constant for each basin which is independent of the storm characteristics and which is intimately linked to the geomorphology of the watershed and to its hydrologic response structure.

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