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Robust RA Estimators in AR-2D Models for Images

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Abstract. Usually it is adopted for processing and analysis of images SAR, the multiplicative model $Z_{(m,n)} = X_{(m,n)}Y_{(m,n)}$, where Z are the observations, X is the backscatter process and Y the noise speckle. In Frost et al (1982) (mentioned by Sant'Anna (1995)) a model AR-2D is suggested for the process X. Because the noise speckle is not cleaned totally, the robust methods are the most appropriate alternative to estimate this model's parameters. The purpose of this paper is to present a proposal of robust estimation in the AR-2D contaminated model: The RA estimators. They conform the bidimensional version of the estimators based on the residual covariances for the ARMA unidimensional models, introduced by Bustos and Yohai (1986).

Keywords: Robustness

1- Models for Images and Robustness

Among the models of mayor importance in the treatment and processing of images are very important the ARMA-2D models and particularly the AR-2D ones; in these models, it is very common to assume that the intensity matrix of the image has got a multivariate gaussiana distribution; however it is well known that in many applications the gaussiana supposition is not the appropriate one. As a result it comes out a difficult hypothesis to hold. A very much realistic supposition consists of considering the presence of a contaminated gaussiano noise.

Unfortunately the least square estimators and maximum likelihood estimators are very sensitive when the gaussian supposition is not fulfilled. So the development of robust techniques and the proposal a new robust estimators are very important for the estimation of the parameters in models for images. Kashyap and Eom (1988) present the M robust estimator in a model AR-2D, which is contaminated by two different processor of outliers. These estimators are very highly superior to the classic estimators.

The proposal of RA estimators make up a robust estimation alternative in the AR-2D contaminated model. The basic idea to build the RA estimators consists of showing the usual least square estimators so that they involve the so-called residual covariances and finally the least square estimators are strengthened by making this covariances robust.

2- AR-2D Model

Before referring to this model it will be necessary to present some concepts:

First we will call Z the set of integer numbers and we will call C the set of complex numbers. So Z^2 tells us the set of all the pairs (m,n) the integers numbers, and C^2 will show the set of all pairs (z,w) of complex numbers.

Let S is a configuration of pixels so that $X(s)$ con $s \in S$ represent a random variable such as color, intensity of energy, light or grey level, etc. in the pixel or s place.

Considering that $S = Z^2$, the result is that $\left(X(s): s \in S\right)$ is a random process. Thus, when this process is performed, an image is obtained. If the random variables $X(s)$ are independent, and $E(X(s)) = E(X(\tilde{s})$ for all $s, \tilde{s} \in S$, we obtain the so-called white noise.

Let $X = \left(X(s): s \in S\right)$ a week stationary random process (i.e. $E(|X(s)|^2)$ is a finite number for all $s \in S$, $E(X(s)) = E(X(\tilde{s}) = 0$ for all $s, \tilde{s} \in S$, and the autocovariance function of X process depends only on the difference of its arguments), and let $e = \left(e(s): s \in S\right)$ a white noise process (Whittle, (1954)). So we say that $X = \left(X(s): s \in S\right)$ is a process $\mathrm{AR}(P,e)$ if :

$$X(m,n) - \sum_{(k,l)\in T} a(k,l) X(m-k, n-l) = e(m,n)$$

Where T is a finite subset of S and

$$P(z,w) = 1 - \sum_{(k,l)\in T} a(k,l) z^k w^l$$

is a polinomic function with real coefficients, which is not annulled in the set $\{(z,w) \in C^2 : |z| = |w| = 1\}$, and T is contained in positive quadrant of the plan. (Guyon, (1982)).

The modelling of images using AR-2D models very often used. They not only have a good performance for modelling images but besides they have a structure much more simple than other models for images. That´s why the efficient methods of estimation of its parameters are so important to develop.

3- The Least Square Estimators In The AR-2D Model

Let X an AR (P, e) model with zero mean observed a window strongly causal W, where:

$$P(z,w) = 1 - \sum_{(k,l)\in T} a(k,l) z^k w^l$$

and $T = \{(k,l) \in S{:}0 \le k,l \le L\}$ being L a natural number. Then:

$$a = \begin{bmatrix} 1 & a(0,1) & a(0,2) & \dots & a(0,L) \\ a(1,0) & a(1.1) & \vdots & \vdots & a(1,L) \\ a(2,0) & a(2,1) & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ a(L,0) & a(L,1) & \dots & \dots & a(L,L) \end{bmatrix}$$

is the matrix of parameters to be estimated in the referred model AR(P,e).

Let $B^{(k,l)}$ the operator which lets $X(m,n)$ “goes back” k units in the first coordinate and l units in the second one; i.e.:

$$B^{(k,l)}\left(X(m,n)\right) = X(m-k, n-l) \quad \forall (k,l) \in S, \forall (m,n) \in S$$

The least square estimator $\hat{a}$ of a, minimizes the expression:

$$\sum_{(k,l)\in W \approx T} r^2_{(k,l)}(a) \qquad (3.1)$$

where $W \approx T$ is a subset of window W. Precisely:

$$W \approx T = \left\{(k,l) \in W{:}\, B^{(k,l)}\left(X(m,n)\right) \in W\right\}$$

and $r_{(k,l)}(a)$ is the residual in the pixel or place s. It is defined by:

$$r_{(m,n)}(a) = \begin{cases} -\sum_{(k,l)\in T} a(k,l)B^{(k,l)}X(m,n) & (m,n)\in (W \approx T) \\ 0 & c.c. \end{cases} \qquad (3.2)$$

The estimator $\hat{a}$ observes a good behaviour supposing that for all $s \in S$, $e(s)$ has a normal distribution; but it is not robust in the sense that it is badly affected by the presence of observations that move slightly away from the normal hypothesis.

In the next section a robust estimator will be proposed to estimate the parameters in the AR-2D model contaminated by innovation outliers. This distortion appears when the process e has a normal distribution contaminated given by:

$$F = (1-d)N\left(0, s^2\right) + dG$$

where $0 < d < 0.5$, $N\left(0, s^2\right)$ represents the normal distribution with mean zero and variance s^2 and G is an unknown arbitrary distribution with dispersion $t^2 \geq s^2$. Finally, the process e has distribution $N\left(0, s^2\right)$ with probability $(1-d)$ and has distribution G with probability d. The random variables $e(k,l)$ are considered outliers when they respond to G distribution.

4- RA Robust Estimators in the AR-2D Models contaminated by Innovation Outliers

Let X the AR (P, e) model of the previous section distorted by the presence of innovation outliers and observed in the window strongly causal W. Let the least square estimator $\hat{a}$ of a, which is obtained by minimizing in equation **3.1**. Differentiating this expression we obtained the following equations:

$$\sum_{(k,l)\in W \approx T} r_{(k,l)}(\hat{a}) \left(¶r_{(k,l)}(\hat{a}) \big/ ¶\hat{a}_{(i,j)} \right) = 0 \quad \forall (i,j) \in T \qquad (4.1)$$

An auxiliary calculus let us demonstrate that:

$$\left(¶r_{(k,l)}(\hat{a}) \big/ ¶\hat{a}_{(i,j)} \right) = -P^{-1}(B) r_{(k-i,l-j)}(\hat{a}) \qquad \forall (i,j) \in T,\ \forall (k,l) \in W \approx T \qquad (4.2)$$

where:

$$P^{-1}(z,w) = \sum_{(s,t)\in S} p(s,t) z^s w^t$$

and so:

$$P^{-1}(B) = \sum_{(s,t)\in S} p(s,t) B^{(s,t)} \qquad (4.3)$$

Then, we can write:

$$\sum_{(k,l)\in W\approx T} r_{(k,l)}(\hat{a}) \left(\sum_{(s,t)\in S} p(s,t) r_{(k-i-s,l-j-t)}(\hat{a}) \right) = 0 \quad \forall (i,j) \in T \qquad (4.4)$$

Due to equation **3.1**, the residual in $(k-i-s, l-j-t)$ is null except over a finite subset D of S (the exact expression of D depends on $L,(i,j)$ and the size of the window W). Thus, we rewrite equation **4.4** as:

$$\sum_{(k,l)\in W\approx T} r_{(k,l)}(\hat{a}) \left(\sum_{(s,t)\in D} p(s,t) r_{(k-i-s,l-j-t)}(\hat{a}) \right) = 0 \quad \forall (i,j) \in T \qquad (4.5)$$

Interchanging the order of summations, we obtain:

$$\sum_{(s,t)\in D} p(s,t) \left(\sum_{(k,l)\in W\approx T} r_{(k-i-s,l-j-t)}(\hat{a}) r_{(k,l)}(\hat{a}) \right) = 0 \quad \forall (i,j) \in T \qquad (4.6)$$

For each $(u,v) \in S$, we define:

$$(W \approx T)_{(u,v)} = \{ (m-u, n-v) : (m,n) \in (W \approx T) \}$$

So, equation **4.6** is equivalent as:

$$\sum_{(s,t)\in D} p(s,t) \left(\sum_{(k,l)\in (W\approx T)_{(s,t)}} r_{(k+i+s,l+j+t)}(\hat{a}) r_{(k,l)}(\hat{a}) \right) = 0 \quad \forall (i,j) \in T \qquad (4.7)$$

Now, for each $(u,v) \in S$ we define a residual covariance in (u,v) to:

$$g_{(u,v)}(\hat{a}) = \sum_{(k,l)\in (W\approx T)_{(u,v)}} r_{(k+u,l+v)}(\hat{a}) r_{(k,l)}(\hat{a}) \qquad (4.8)$$

So, according to equations **4.8** and **4.7** becomes to:

$$\sum p(s,t)g_{(s+i,t+j)}(\hat{a}) = 0 \quad \forall(i,j) \in T \quad (4.9)$$

As we have said at the end of section 2, the least square estimators of a will be strengthened by making robust the residual covariances. For doing this we will replace equation **4.8** by:

$$j_{(u,v)}(\hat{a}) = \sum_{(k,l)\in(W\approx T)_{(u,v)}} h\left(\frac{r_{(k,l)}(\hat{a})}{\hat{s}}, \frac{r_{(k-u,l-v)}(\hat{a})}{\hat{s}}\right) \quad \forall(u,v) \in S \quad (4.10)$$

where h is a continual, bounded and odd function in each variable and $\hat{s}$ is a robust estimator of the scale factor of e process. Thus, we define the RA robust estimator of a, by means of the following equations:

$$\sum_{(s,t)\in D} p(s,t)j_{(s+i,t+j)}(\hat{a}) = 0 \quad \forall(i,j) \in T \quad (4.11)$$

where $\hat{s}$ is simultaneously calculated by means of the expression:

$$\hat{s} = \frac{Med\left(\left|r_{(k,l)}(\hat{a})\right| : (k,l) \in W \approx T\right)}{0.6745}$$

where 0,6745= $Med(|X|)$ and X has distribution $N(0,1)$.

5- Further Investigations

The AR (1) is unidimentional model is generalized by AR-2D Models which has a good performance to model images; besides its structure is very simple. Kashyap and Eom (1988) introduce the M robust estimators for this contaminated model, and Nasburg and Kashyap (1975) prove the consistency and asymptotic normality of these estimators. So it is interesting to study the properties of R.A. estimators such as consistency and asymptotic normality in AR-2D model.

On the other hand, the RA estimators in the ARMA unidimentional models contain the class of M estimators and they are asymptotically equivalents to M estimators (Bustos and Yohai (1986)). We should analyze if the RA estimators, we have already introduced for the AR-2D models, whether contain or not to the class of M estimators.

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