Linear Programming Models for the One-Dimensional Cutting Stock Problem

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Abstract: We review some linear programming models for the cutting stock problem. We compare the models, pointing out their advantages and disadvantages.

Keywords: linear programming models, one-dimensional cutting stock problem.

1. INTRODUCTION

In this work some linear programming models for the One-Dimensional Cutting Stock Problem are presented. Many companies in the paper industry, furniture, glass, metal, textile etc. face daily the problem of cutting smaller pieces from larger ones. To make the best use of the available resources there has been an increasing interest in techniques that seek the optimization of the cutting process.

To evaluate different solutions of cutting large objects into smaller items, some criterion has to be used. Objectives functions such as waste minimization, production cost minimization and others have been observed in the literature.

In the next sections we present linear programming models suggested by Gilmore and Gomory (1961), Carvalho (2002), Dyckhoff (1981).

2. GILMORE AND GOMORY'S MODEL

The works of Gilmore and Gomory (1961, 1963) were very helpful in the search of a practical solution for the cutting stock problem. Gilmore and Gomory (1961) formulated the cutting stock problem as an integer programming problem.

Let

W be the length of the objects in stock;
w_d be the length of the items, $d = 1, 2, \ldots, m$;
b_d be the demand of item d, $d = 1, 2, \ldots, m$;
$A_j = (a_{1j}, \ldots, a_{dj}, \ldots, a_{mj})^T$ be a cutting pattern, $d = 1, 2, \ldots, n$, where a_{dj} is the number of items d in the pattern;
x_j be the frequency that the pattern j is cut.

The cutting stock problem can be modeled as

$$\text{Min} \sum_{j \in J} x_j \tag{1}$$

Subject to

$$\sum_{j \in J} a_{dj} x_j \geq b_d \qquad d = 1, 2, \ldots, m \tag{2}$$

$$\sum_{d=1}^{m} a_{dj} w_d \leq W \qquad \forall\, j \in J \tag{3}$$

$$x_j \geq 0 \text{ and integer} \qquad \forall\, j \in J \tag{4}$$

$$a_{dj} \geq 0 \text{ and integer} \qquad \forall\, j \in J,\ d = 1, 2, \ldots, m \tag{5}$$

The objective function (1) minimizes the number of objects used. Constraints (2) impose that the amount cut of each item is sufficient to fulfill its demand. Constraints (3) impose the condition of a pattern to be feasible; constraints (4) and (5) impose the integrality and non-negativity of the decision variables.

This model is non linear due to constraints (2). However, if all cutting patterns that satisfy constraints (3) are enumerated we obtain an integer linear problem composed of (1), (2) and (4) where the a_{dj} for all d and j are all parameters.

The difficulty in using this model is the large number of patterns to be enumerated in practical problems. The number of possible patterns increases exponentially as we increase the number of different items and the demands. If the number of patterns is large, so is the number of integer variables in the problem. The difficulty in solving integer programming problems usually increases with the number of integer variables involved.

Gilmore and Gomory (1961) propose to solve approximately this integer programming problem by using the linear programming relaxation of the problem and column generation. After obtaining the solution, if not integer, some rounding procedure can be used. This approximation usually provides good quality solutions (small errors compared with optimal solution values) when the demands are high, that is, when a large number of objects are used to cut the required items. However, the quality of the solution may be poor for instances when the demands are low.

A advantage of this model is that it presents the same structure for 1, 2, 3 or multidimensional cutting stock problems. The difficulty arises only in the generations of feasible patterns to the model.

3. BIN PACKING PROBLEM

The bin packing problem consists of allocating items to objects (bins) without exceeding the bin's capacity and minimizing the number of bins used. The mathematical model for this problem is

$$\text{Min } \sum_{i=1}^{n} y_i \tag{6}$$

Subject to

$$\sum_{j=1}^{n} w_j x_{ij} \leq W y_i \qquad \forall\, i \in I \tag{7}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall\, j \in J \tag{8}$$

$$y_i = 0 \text{ or } 1 \qquad \forall\, i \in I \tag{9}$$

$$x_{ij} = 0 \text{ or } 1 \qquad \forall\, i \in I, \forall\, j \in J \tag{10}$$

where

y_i is a binary variable that takes the value 1 if bin i is used and 0, otherwise

x_{ij} is a binary variable that takes the value 1 if item j is allocated to the bin i and 0, otherwise.

The objective function (6) minimizes the amount of objects used to allocate all the items; constraint (7) guarantees that the sum of the items in bin i does not exceed its capacity; constraint (8) guarantees that each item is allocated to a single object only; constraints (9) and (10) impose that the decision variables are binary.

In this model we have a variable for each item while in the Gilmore and Gomory's model of we have a variable for each item type.

The extension of this bin packing model to bi-dimensional or multi-dimensional problems is not immediate. Constraints regarding the generation of the patterns have to be included in the model.

4. ARC FLOW MODEL

Let $G = (V, A)$ be an acyclic directed graph with vertices $V = \{0, 1, 2, \ldots, W\}$ where W is the size of the object in stock and $A = \{(i, j)\colon 0 \le i < j \le W\}$ is the set of arcs. There exists a directed arc (i, j) in this graph G if there is an item d of size w_d and $j - i = w_d$.

Carvalho (1999) modeled the one-dimensional cutting stock problem as an arc flow model. In this model, a unit of flow from node 0 to node W corresponds to a cutting pattern since it defines a path from node 0 to node W where the addition of the sizes of the items in correspondence to the arcs in this path is smaller than the size of the object. The arc flow model has the following form:

$$\text{Min } z \tag{11}$$

Subject to

$$\left.\begin{aligned}
&\sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = -z && \text{if } j = 0\\
&\sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = 0 && \text{if } j = 1, 2, \ldots, W-1\\
&\sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = z && \text{if } j = W
\end{aligned}\right\} \tag{12}$$

$$\sum_{(k,k+w_d)\in A} x_{k,k+w_d} \ge b_d \qquad d = 1, 2, \ldots, m \tag{13}$$

$$x_{ij} \ge 0 \text{ and integer} \qquad \forall\ (i, j) \in A \tag{14}$$

where

z is the flow in a feedback arc, from vertex W to vertex 0

x_{ij} is the flow in arc (i, j)

b_d is the demand of item d, $d = 1, 2, \ldots, m$.

The objective function (11) minimizes the amount of flow from vertex W to vertex 0, that is, minimizes the number of objects used; (12) are flow conservation constraints; constraints (13) guarantee that the demand of each item is satisfied; (14) are the non-negativity and integrality conditions for the decision variables.

In Figure 1 we present a graph associated with an instance with bins of capacity $W = 5$ and items of sizes $w_1 = 3$ and $w_2 = 2$. In the same figure, a path is shown that corresponds to 2 items of size 2 and 1 unit of loss. Several paths exist between 0 and 5, that lead to different cutting patterns and that can be used to satisfy the demand of the given items.

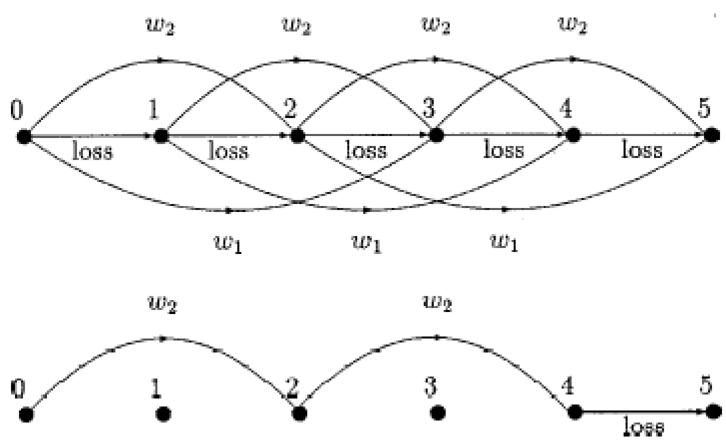

Figure 1. Graph and Cutting Pattern

This model can be used only to solve unidimensional cutting stock problems. It cannot be extended to 2, 3, or multidimensional problems, at least in a straightforward manner.

5. ONE-CUT MODEL

The one-cut model was proposed by Dyckhoff (1981). In this model a single cut in the object is considered at a time. This cut divides the object into two pieces and it is imposed that at least one of the pieces has the size of a demanded item (see figure 2).

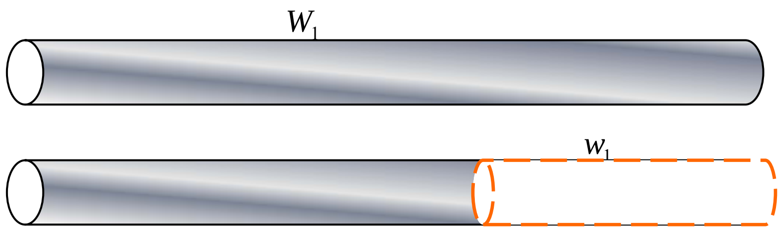

Figure 2: one-cut

Let

S	be the set of the (sizes of the) original objects, $p \in \{W_1, W_2, \ldots, W_K\} \subset N$
R	be the set of the (sizes of the) residual pieces obtained after cutting an item from an object or a residual piece itself
D	be the set of the (sizes of the) items $q \in \{w_1, w_2, \ldots, w_m\} \subset N$. We assume without loss of generality that $S \cap D = \phi$
$y_{p,q}$	be the number of pieces of size p that are divided into a piece of size q, and a residual piece of size p - q
z_k	be the number of objects of size W_k used
N_q	be the demand of items of size q
B_p	be the number of objects of of size p in stock, $p = 1, 2, \ldots, K$.

Dyckhoff's (1981) model is:

$$\text{Min} \sum_{k=1}^{K} W_k z_k \tag{15}$$

Subject to

$$z_k + \sum_{p\in D:(p+q)\in(S\cup R)} y_{p+q,p} \geq \sum_{p\in D:p<q} y_{q,p} \qquad \{\forall\, q = W_k\},\ k = 1, 2, \ldots, K \tag{16}$$

$$\sum_{p\in(S\cup R):p>q} y_{p,q} + \sum_{p\in D:(p+q)\in(S\cup R)} y_{p+q,p} \geq \sum_{p\in D:p<q} y_{q,p} + N_q \qquad \forall\, q \in (D\cup R)\backslash S \tag{17}$$

$$z_k \leq B_k \qquad k = 1, 2, \ldots, K \tag{18}$$

$$y_{p,q} \geq 0 \text{ and integer, } p \in S\cup R,\ q \in D,\ q < p \tag{19}$$

$$z_k \geq 0 \text{ and integer, } k = 1, 2, \ldots, K \tag{20}$$

The objective function (15) minimizes the sum of the sizes of the objects used; constraints (16) impose that the amount of each object is sufficient to fulfill its demand; (17) impose that the demand of each item must be satisfied; constraints (18) guarantee that the number of objects used does not exceed the available number of each size; constraints (19) and (20) impose the integrality and non-negativity of the decision variables.

To exemplify this model, let S = {9, 6, 5} be the sizes of the objects in stock, D = {4, 3, 2} be the sizes of the items, with demands 20, 10 and 20, respectively. Enumerating all possible one-cut operations, we obtain the following set of residual pieces: R = {7, 6, 5, 4, 3, 2}. In Figure 3 the coefficients of the set of constraints of the one-cut model are presented for this instance.

	z_1	z_2	z_3	y_{92}	y_{93}	y_{94}	y_{72}	y_{74}	y_{64}	y_{63}	y_{53}	y_{42}	
9	1			-1	-1	-1							$\geq$ 0
8													$\geq$ 0
7				1			-1	-1					$\geq$ 0
6		1			1				-1	-1			$\geq$ 0
5			1			1	1				-1		$\geq$ 0
4						1		1	1			-1	$\geq$ 20
3					1			1		2	1		$\geq$ 10
2				1			1		1		1	2	$\geq$ 20
W_k = 9	1												$\leq B_1$
6		1											$\leq B_2$
5			1										$\leq B_3$

Figure 3: One-cut model
Reproduced from Carvalho (2002)

The major disadvantage of this model is the increasing number of variables as the number of items increases. The possible sizes of residual pieces can be large and they need to be enumerated to solve the problem.

This model also cannot be extended, at least in a straightforward manner to bi-dimensional or multidimensional cutting stock problems.

6. ACKNOWLEDGMENTS

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