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Quenched disorder in Josephson-junction arrays in a transverse magnetic field

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A simple model for weak quenched disorder in Josephson-junction arrays is considered and the effects of two kinds of disorder on the resistive behavior of the array is determined. For the case of random plaquette areas, the lower envelope of resistance increases quadratically with the magnetic field and no superconducting phase exists, whereas positional disorder leads to a reentrant phase at low temperatures in finite field and a critical value of the field above which there is no superconducting phase. The consequences of these results for experiments reported recently are also presented.

Experiments on two-dimensional arrays of Josephson junctions,[1,2] proximity-coupled grains,[3–5] and superconducting wires[6] in a transverse magnetic field have stimulated a great deal of theoretical interest[7–12] recently. They provide an interesting model system in which the resistive transition can be studied without the high degree of randomness and inhomogeneity inherent in granular films. Novel periodic behavior of the resistance of the array as a function of the magnetic field has also been observed. These systems have been studied in the context of uniformly frustrated XY models where the frustration defined by $f=Ha^2/\Phi_0$ is the number of flux quanta $\Phi_0=hc/2e$ per plaquette of area a^2 introduced by the external field. These models explain the main features of the experimental data, in particular the basic periodicity in f and also the subsidiary minima at rational values[12] of f observed experimentally.

There are, however, certain features of the experiments which have not been explained. The frustrated XY model used predicts that the resistance minima at integer values of the flux should be zero at low temperatures (except for finite size effects which are assumed small in this paper) and should remain zero until the applied field reaches its critical value, at which the superconducting grains or wires making up the array go normal. However, all experiments seem to disagree with this in two respects except at $H=0$. First, the resistance minima never reach zero except at zero field and also oscillate with a rather long period. Second, the resistance minima start rising monotonically and cross over into a more or less linear increase at fields well below the critical value for the loss of superconductivity in the individual grains or wires. This Rapid Communication addresses the second point where the rise in the lower envelope of the resistance is ascribed to randomness in the array. Although a regular array minimizes the effects of disorder, some are inevitably present and one should include these effects. Weak disorder is irrelevant to the critical behavior of the XY model which corresponds to the zero field case and is also expected to be irrelevant in finite field provided the disorder does not couple to the field. However, the types of randomness we envisage here are (a) variations in the area of the superconducting elements of the array which is presumably least serious in the IBM arrays[1,2] consisting of junctions connected by niobium wires, and most serious in the arrays of superconducting squares studied by Refs. 3 and 4, and (b) randomness in the positions of the nodes or superconducting grains of the array which will always be present. The first leads to uncorrelated variations in the area of each plaquette, provided the penetration depth in the grain is small compared to the grain size and hence to random variations in the flux per plaquette, and the second to highly correlated variations in the flux per plaquette.

In this paper we present a simple model of a Josephson-junction array in a perpendicular magnetic field with disorder, and map the model into a Coulomb gas of fractional charges perturbed by a quenched distribution of random charges in the uncorrelated random area situation and, in the random-position case, a random distribution of dipoles. Here we are interested only in the effects of disorder on the lower envelope of resistance as a function of the average number of flux quanta per plaquette which corresponds to integer f. In this case the model reduces to a Coulomb gas of integer charges perturbed by the random distribution of charges or dipoles. This latter case has been studied in another context by Rubinstein, Shraiman, and Nelson,[13] and the results of their analysis leads to a reentrant phase at low temperature for values of the magnetic field less than some disorder-dependent critical value. For larger fields and higher temperatures there is no superconducting phase. In the random-plaquette-area situation there is no superconducting phase for any finite field, but there is a remnant of it provided the disorder is small enough.

Consider a Josephson-junction array in a transverse magnetic field described by the Hamiltonian

$$H=-J\sum_{\langle rr'\rangle}\cos(\theta_r-\theta_{r'}-A_{rr'}) \ , \tag{1}$$

where J is the Josephson coupling for a single junction. θ_r is the phase of the superconducting order parameter $\psi_r=|\psi_r|e^{i\theta_r}$ at the sites $\mathbf{r}$ of a square lattice, and

$$A_{rr'}=\frac{2\pi}{\Phi_0}\int_r^{r'} A\cdot dl$$

are constrained to $\sum_R' A_{rr'}=2\pi f$. Here $\sum_R'$ means a discrete curl around a plaquette with dual site $\mathbf{R}$. Standard duality transformations lead to a Coulomb gas representation of the partition function given by[14]

$$Z=\sum_{\{M_R\}}\exp\left[-2\pi^2K\sum_{R\neq R'}(M_R-f_R)G(R-R')(M_{R'}-f_{R'})\right], \tag{2}$$

where $M_R=0, \pm1, \pm2 \ldots$, $K=J/k_BT$, and $G(R-R')$ is the lattice Green's function associated with the dual sites $\mathbf{R}$.

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Consider first the random plaquette area case. We assume that the areas are randomly distributed (Gaussian) about some mean a_0^2 so that the distribution for the f is given by

$$P(f_R) \propto \exp\left[-\frac{\Phi_0^2}{2H^2\Delta^2}(f_R - f_0)^2\right] , \tag{3}$$

where Δ^2 is the variance of the area distribution. This problem has already been investigated in Ref. 11, in a particular gauge. Here we give a simple gauge-invariant argument. If we consider only resistance minima where f_0 is an integer, we can shift $M_R \rightarrow M_R + f_0$ so that we have the problem of a set of integer charges in a background of random charges with mean zero. Now we consider a test charge in a region of size L in which there will be fluctuations of random charge $\delta f \approx H\Delta L/\Phi_0$. So when this region is chosen of size larger than $\xi \approx \Phi_0/H\Delta$, the test charge will be swamped and charges separated by distances larger than ξ will be unbound.[15] We can now identify ξ as the correlation length[16] of the free vortices contributing to the resistance of the array. Using standard arguments, the resistance is given by $R(H) \propto n_f \propto \xi^{-2}$ so $R(H) \propto H^2$ and the resistance will oscillate with a period corresponding to one flux quantum per mean area and the lower envelope will rise quadratically for small H. We have been unable to locate any published experimental data which may test this prediction.[17] We expect this to be valid for arrays of lead squares since there is bound to be randomness in the individual square size and provided the transverse penetration depth (on the order of a few angstroms for the grain is less than the grain size on the order of a few micrometers). Of course this effect will always be present, but to a smaller degree even when the penetration depth is comparable with the grain size, because there is a tendency for the field to be excluded via a partial Meissner effect and there will be more flux going through a large rather than a small area between the grains. In this argument we have ignored the fact that in arrays of squares or disks the junction size is quite large and $T_c(H)$ will itself decrease as H^2 in a nonrandom array. Consequently, at any fixed temperature, one expects an increase of the resistance at sufficiently large H. The arguments presented here are low-temperature ones and lead one to expect quadratic increase in the resistance even at temperatures well below the critical temperature of the corresponding pure system.

In the limit where the grain size is small relative to the areas between grains or a network of superconducting wires of constant or very small cross section interspersed with weak links, the only other source of randomness is in the positions of the nodes of the network. For example, suppose that a superconducting wire separating two areas is displaced from its ideal position, increasing one area and decreasing the other. This intuitively leads to two equal and opposite neighboring charges in the equivalent Coulomb gas, or to a quenched distribution of random dipoles. The same will happen for a displacement of the nodes of the array.

We now introduce positional disorder which consists in allowing displacements of the sites from their average position $\mathbf{r}$ by an amount $\mathbf{u}_r$ with a probability distribution

$$P(\mathbf{u}_r) \propto \exp\left[-\frac{1}{2\Delta^2}\mathbf{u}_r^2\right] . \tag{4}$$

In practical systems this kind of disorder will also induce disorder in the couplings between nodes $J_{rr'}$ but this can be shown to be irrelevant for small disorder by use of the replica trick.[18] We do not consider granular superconductors because the disorder is very large and presumably dominates the behavior. We now take the continuum limit of the lattice model in order to evaluate the change of the area of the plaquette. In this limit we do not distinguish between a lattice point $\mathbf{r}$ in the original lattice and a lattice point $\mathbf{R}$ in the dual lattice, and we obtain $S \approx S_0 + S_0 \nabla_R \cdot \mathbf{u}_R$, where S_0 is the area of the plaquette in the undisplaced lattice. This gives

$$f = HS/\Phi_0 = f_0 + f_0 \nabla_R \cdot \mathbf{u}_R .$$

Consistent with the linear approximation for the change in the area we then obtain from (2)

$$Z = \sum_{\{M_R\}} \exp\left[-2\pi^2 K \sum_{R \neq R'} (M_R - f_0) G(R-R')(M_{R'} - f_0) - 4\pi^2 K f_0 \sum_{R'} \int \frac{d\mathbf{R}}{a^2} \nabla_R \cdot \mathbf{u}_R G(R-R')(M_{R'} - f_0)\right] . \tag{5}$$

A partial integration in the last term finally gives

$$Z = \sum_{\{M_R\}} \exp\left[\pi K \sum_{R \neq R'} (M_R - f_0) \ln\frac{|\mathbf{R}-\mathbf{R}'|}{a}(M_{R'} - f_0) + \ln y \sum_R (M_R - f_0)^2 + 2\pi K f_0 \sum_{R'} \int \frac{d\mathbf{R}}{a^2} \mathbf{u}_R \cdot \frac{(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^2}(M_{R'} - f_0)\right] , \tag{6}$$

where y is the vortex fugacity $y \simeq e^{-\pi^2 K/2}$, and the vortices satisfy the neutrality condition $\sum_R (M_R - f_0) = 0$.

This derivation was rather sloppy but a more careful derivation taking into account the differences between the original and dual lattices gives the same result to linear order in the displacements. Higher-order terms in fact can be shown to make no difference to the final result.

This can be viewed as a fractional Coulomb gas perturbed by a random distribution of dipoles $\mathbf{p}_R \propto \mathbf{u}_R$. As in the uniformly frustrated case one would be very much interested in the behavior for rational f_0, but here we concentrate only on the case where f_0 is an integer. In this case shifting $M_R \rightarrow M_R + f_0$, we obtain a Coulomb gas of integer charges perturbed by a random distribution of dipoles. This problem has been previously studied in Ref. 13 in another context so we merely state the relevant results. The recursion relations for $f_0 =$ integer are

$$\frac{dK(l)}{dl} = -4\pi^3 K^2 y(l)^2 ,$$
$$\frac{dy(l)}{dl} = [2 - \pi K(l) + 4\pi^3 f_0^2 \Delta^2 K(l)^2] y(l) , \tag{7}$$

where lengths have been rescaled by a factor e^l. Note that f_0 and Δ appear in the combination $f_0\Delta$, which can be regarded as an effective measure of the disorder which increases linearly with applied field. For a given sample Δ is fixed and the degree of disorder is varied by changing f_0.

These recursion relations have been analyzed in detail in Ref. 13 and lead to the phase diagram of Fig. 1. For sufficiently small f_0 there is the usual resistive transition at a temperature $T_c = T^+(f_0)$. In contrast with the pure model one also finds a reentrant unbinding transition at a lower temperature $T^-(f_0)$. These temperatures are determined by the intersection of the curve of initial values $y = e^{-\pi^2 K/2}$ with the phase boundary of the renormalization-group flows determined by (7) which separates the two regions where y is relevant and irrelevant. The low-temperature transition can be understood as an unbinding of vortex pairs by the random dipole background because of an insufficient thermal vortex density to screen this. At higher temperatures, although the fugacity is relevant for $y \simeq 0$, at larger length scales the density becomes sufficiently large to screen the random background, and the fugacity eventually scales to zero. The high-temperature transition is the usual thermal unbinding of vortices and the temperature range of the superconducting phase decreases as the degree of disorder $f_0\Delta$ increases.[13]

In the superconducting phase there is phase coherence in the sense that the correlation function $\langle e^{i\theta_r - i\theta_{r'}}\rangle$ decays algebraically to zero as $|\mathbf{r}-\mathbf{r}'|^{-\eta(t,f\Delta)}$. Note that since we are only considering integer f_0, this correlation function is in fact gauge invariant since $\int_\Gamma \mathbf{A}_{rr'} \cdot d\mathbf{l} = 2\pi \times$ integer for any path Γ along links. The exponent η approaches the value

$$\eta^*(f\Delta) = \tfrac{1}{8}[1 + (1 - 32\pi\Delta^2 f_0^2)^{1/2}] + 2\pi\Delta^2 f_0^2$$

at the two phase boundaries $T^\pm(\Delta f_0)$. In the normal phase, there is the usual exponential decay. As f_0 approaches the critical value $f_M = (32\pi)^{-1/2} 1/\Delta$, T^+ and T^- merge at M and the superconducting phase shrinks to zero. For $f_0 > f_M$ the system is always unstable to vortices.

Near $T^\pm$, ξ diverges exponentially like $\xi \approx e^{C|T-T_c^\pm|^{-1/2}}$, where C is a constant for constant f_0 and at constant T like $\xi \approx e^{C|f_0 - f_c(T)|^{-1/2}}$. The renormalized spin-wave stiffness constant approaches

$$K_R = \frac{1}{8\pi^2\Delta^2 f_0^2}[1 - (1 - 32\pi\Delta^2 f_0^2)^{1/2}]$$

on the boundaries of the superconducting phase. Thus, in contrast with the pure systems, the superfluid density jump is not universal but depends on f_0.

These results have direct experimental consequences. For $f \gtrsim f_c$ the density of free vortices where $M_R = f_0 \pm 1$ is approximately $n_v \approx \xi^{-2}$ and so the resistance increases very slowly. Note that only these excitations from the ground state will contribute to the dissipation of the supercurrent. More important, we have shown that the resistance increases exponentially slowly at constant temperature as f_0 is increased beyond f_c. At higher fields, one expects that this will crossover to the usual flux-flow resistance $R(H) \propto H$.

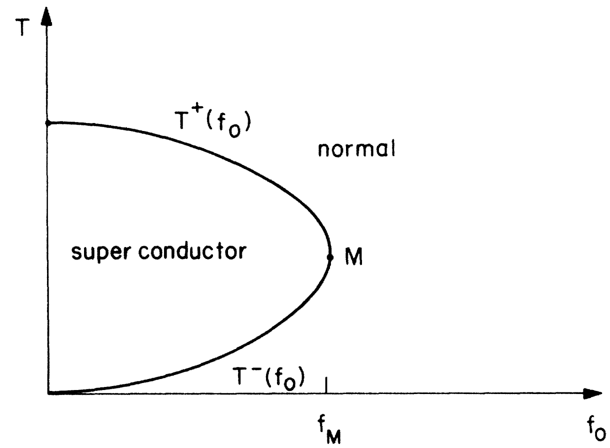

FIG. 1. Qualitative phase diagram as a function of the temperature T and average number of flux quanta per plaquette f_0. In the case we are considering here f_0 is an integer.

This seems to be in qualitative agreement with the experiments performed by Webb, Voss, Grinstein, and Horn.[2] They observe a sample-dependent critical value of the field above which the lower envelope of the resistance begins to rise very slowly and becomes linear for larger fields. This picture is, however, complicated by oscillations in the lower envelope of resistance which are outside this simple model. We believe that these oscillations can be explained by a more careful analysis of the array used in these experiments in which there are in fact two different areas which can lead to oscillations of two different periods. The fact that the critical field is also associated with the development of a minimum in the resistance may be attributed to a coincidence, but discussion of this is deferred to later publication. The reentrant transition predicted for random positional disorder has not been observed experimentally, but this will not be an easy effect to observe. This reentrant phase predicted here is a separate issue to that predicted to be due to charging effects.[19]

In summary, we consider a simple model for weak quenched disorder in Josephson-junction arrays and discuss the effects of the two kinds of disorder that we believe to be of relevance. For the case of random areas we predict that $R(H) \propto H^2$ and no superconducting phase, whereas positional disorder leads to a reentrant phase at low temperatures in finite field and a critical field above which there is no superconducting phase. We discuss the consequences of the results for experiments and suggest this may be responsible for some of the effects reported recently.

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