SAE 0187

Stability/Instability Regions for Sampled-Data Control Systems as a function of the Sampling Period and the Plant Obtained from a Liapunov Function

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ABSTRACT

In this work we developed a comparative study between many resolution methods of the exponential matrix from an inequality extracted from a specific Liapunov function to obtain information about some stability regions for sampled-data control systems as a function of the sampling-time and the plant used. We used many methods from the literature like: power series, Cayley-Hamilton theorem, Lagrange-Sylvester theorem, eigenvalues/ eigenvectors decomposition and Padé rational approximation. The main objective of this study is to choose the most appropriate method for the resolution of this important problem and to use the results found as a basis for future works.

MATRIX EXPONENTIAL MCLAURIN-TAYLOR (MT) AND PADÉ APPROXIMANTS

Here we will do a simple explanation about the MT and Padé approximation techniques starting with the MT series and showing how is the Padé algorithm with examples. Basically, a continuous function f(t) can be expressed as a MT series like expressed in Eq.1.

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{[k]}(0)}{k!}.t^k = f(0) + \frac{df(0)}{dt}.t + \frac{d^2 f(0)}{dt^2}.\frac{t^2}{2} + \ldots \qquad \textbf{Eq 1}$$

The N$^{\underline{th}}$ order MT approximation of f(t) if clearly given by,

$$f(t) \approx \sum_{k=0}^{N} \frac{f^{[k]}(0)}{k!}.t^k \qquad \textbf{Eq 2}$$

The Padé rational approximation is a natural extension of the MT and Taylor series that make use of the MT approximation in its calculations. The Padé method uses rational polynomials and it is much more accurate than the MT.
Basically, the Padé algorithm is summarized in the following expression making use of Eq.2:

$$\left[\sum_{k=0}^{N} \frac{f^{[k]}(0)}{k!}.t^k\right].\left(1 + q_1 t + q_2 t^2 + \ldots + q_M t^M\right) = p_0 + p_1 t + p_2 t^2 + \ldots + p_L t^L \qquad \textbf{Eq 3}$$

Where the objective is to find a linear system that solve the q_i and p_i coefficients, having N=L+M, and finally express the Padé approximation of degree [L/M] of f(t) as,

$$f(t)_{[L/M]} \approx \frac{p_0 + p_1 t + p_2 t^2 + \ldots + p_L t^L}{1 + q_1 t + q_2 t^2 + \ldots + q_M t^M} \qquad \textbf{Eq 4}$$

<u>Example 1</u>: if $f(t) = e^{-t}$ evaluate the Padé approximation $f_{[3/2]}$.
The MT expansion will be of order N=L+M=5,

$$\left[1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120}\right].\left(1 + q_1 t + q_2 t^2\right) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

that gives,

$$.1+q_1t+q_2t^2-t-q_1t^2-q_2t^3+\frac{t^2}{2}+\frac{q_1t^3}{2}+\frac{q_2t^4}{2}+$$
$$-\frac{t^3}{6}-\frac{q_1t^4}{6}-\frac{q_2t^5}{6}+\frac{t^4}{24}+\frac{q_1t^3}{24}+\frac{q_2t^6}{24}+$$
$$-\frac{t^5}{120}-\frac{q_1t^6}{120}-\frac{q_2t^7}{120}=p_0+p_1t+p_2t^2+p_3t^3$$

that results the following linear system of N=5 equations:

$$\begin{array}{l} t^0: \\ t^1: \\ t^2: \\ t^3: \\ t^4: \\ t^5: \end{array}\begin{cases} p_0=1 \\ p_1=q_1-1 \\ p_2=q_2-q_1+1/2 \\ p_3=-q_2+q_1/2-1/6 \\ q_2/2-q_1/6+1/24=0 \\ -q_2/6+q_1/24-1/120=0 \end{cases}$$

That gives,

$$q_1=2/5;\, q_2=1/20;\, p_0=1;\, p_1=-3/5;\, p_2=3/20;\, p_3=-1/60.$$

Finally, the Padé approximation will expressed by,

$$e^{-t}\approx e^{-t}{}_{[3/2]}=\frac{1-3/5\,t+3/20\,t^2-1/60\,t^3}{1+2/5\,t+1/20\,t^2}$$

■

Some comparative results of the MT and Padé approximations for the exponential e^{-t} are shown in figure 1 for 5th MT approximation and Padé's [1/1],[3/2] and [3/3]. The approximation errors between these methods and e^{-t} are disposed in figure 2. From these two figures can we see that how is inaccuracy and inappropriate the MT method and how better is the Padé technique.

Example 2: given $f(s)=e^{sT}$ show that $f_{[1/1]}$ corresponds to the Tustin (bilinear or trapezoidal) extrapolate mapping. For $f_{[1/1]}$ L=M=1, N=2. The "T" here is the sampling-period of A/D and D/A converters.

The 2nd order MT for this case is: $1+Ts+\frac{T^2s^2}{2}$. Applying this on the Padé algorithm,

$$\left[1+Ts+\frac{T^2s^2}{2}\right].(1+q_1t)=p_0+p_1t$$

$$1+Ts+\frac{T^2s^2}{2}+q_1s+q_1Ts^2+q_1\frac{T^2s^3}{2}=p_0+p_1s$$

that produces the linear system,

$$\begin{array}{l} s^0: \\ s^1: \\ s^2: \end{array}\begin{cases} p_0=1 \\ p_1=T+q_1 \\ \frac{T^2}{2}+q_1T=0 \end{cases}$$

That gives,

$$q_1=-T/2;\, p_0=1;\, p_1=T/2$$

Finally, the Padé approximation is,

$$e^{sT}\approx e^{sT}{}_{[1/1]}=\frac{p_0+p_1s}{1+q_1s}=\frac{1+T/2\,s}{1-T/2\,s}$$

Calling $z:=e^{sT}$ we have,

$$z\approx\frac{1+T/2\,s}{1-T/2\,s}$$

The Tustin extrapolate mapping is found isolating "s" above,

$$s\approx\frac{2}{T}\frac{z-1}{z+1}$$

That represents the analog-discrete mapping as explained in [11].

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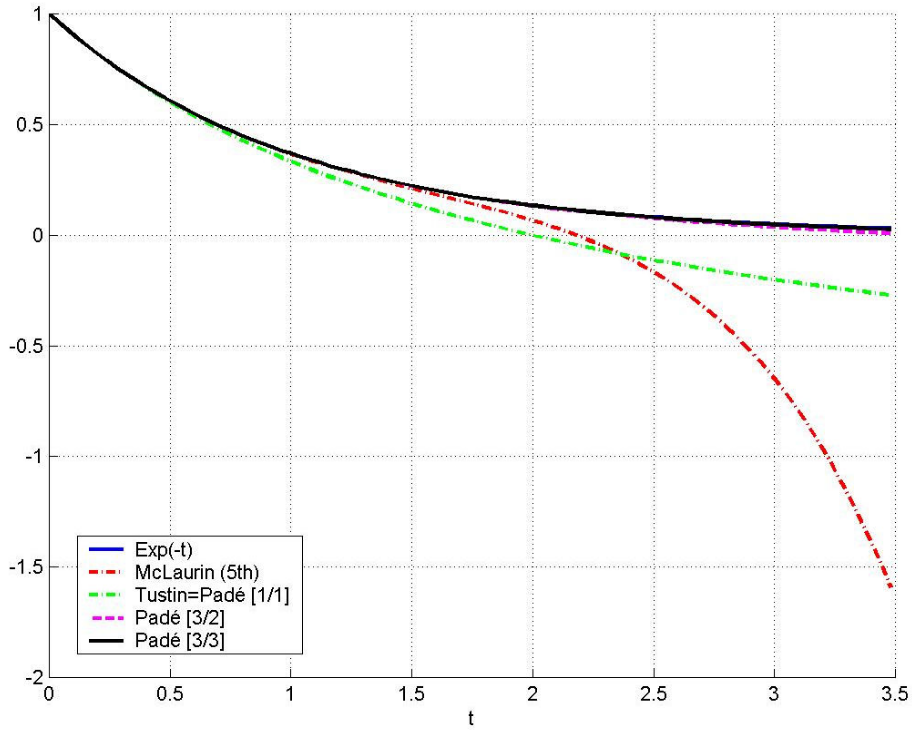

Figure 1: approximations for e^{-t}.

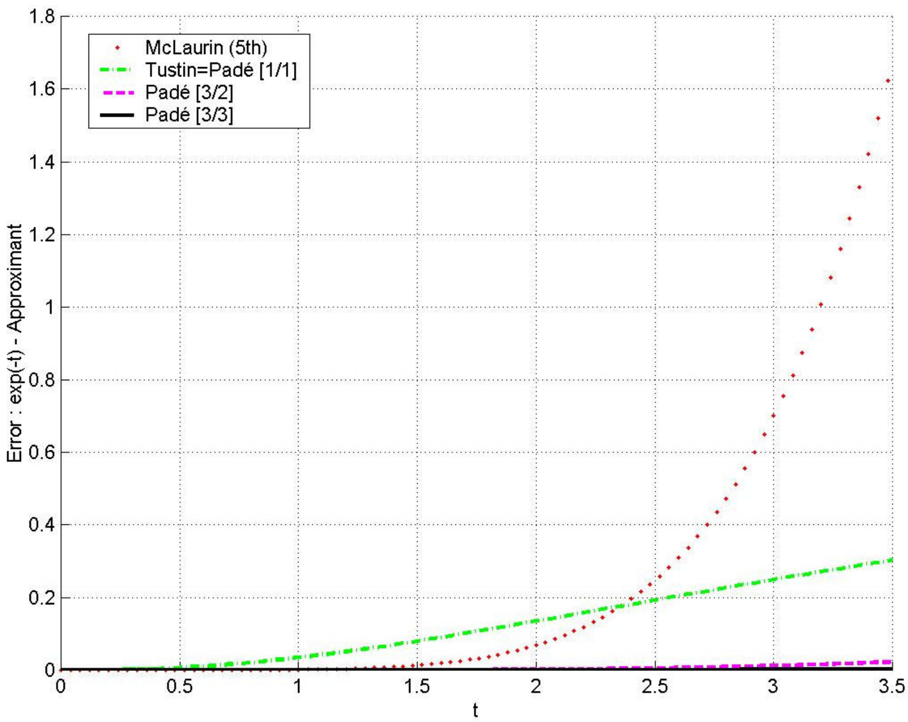

Figure 2: error on approximations of exponential.

NUMERICAL INVESTIGATIONS OF STABILITY

In the later SAE 2004 work of these authors [1] it was presented a theorem that determinates a stability rule for a general sampled-data system through its dynamics matrix, being the control action function of the state velocity feedback vector.
The main result was,

$$\min_{\lambda} P > \max_{\lambda} (e^{A.T})' P e^{A.T} \qquad \textbf{Eq 5}$$

Where P is a positive-definite matrix.
That is the same of,

$$\rho\left[(e^{A.T})' P e^{A.T}\right] < \min_{\lambda} P \qquad \textbf{Eq 6}$$

where ρ is the spectral radius of the matrix between brackets.
As the first part of this present work we understood how is accurate the Padé method to calculate approximations for the exponential. For this manner we will explore approximations of this nature to present numerical results for the inequality presented.

In this work we have used Padé [3/3] (third order) and Padé [8/8] (eighth order) approximations of the matrix exponential e^{AT} and it was sufficient to shown the presence of stability/instability regions of the sampled-data control system as a function of the sampling-period T. The dynamics matrix A in companion form used was

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -5 \end{pmatrix} \qquad \textbf{Eq 7}$$

To calculate the positive-definite matrix P we made use of the Liapunov equation AP+PA'=Q; where Q is other positive-definite matrix given by,

$$Q = \begin{pmatrix} 50 & 100 \\ 40 & 100 \end{pmatrix} \qquad \textbf{Eq 8}$$

From this Liapunov equation we have,

$$P = \begin{pmatrix} 1{,}4667 & -0{,}38 \\ -0{,}62 & 0{,}5 \end{pmatrix} \qquad \textbf{Eq 9}$$

The numerical results are shown in the Figure 3 with T varying until 50 seconds..

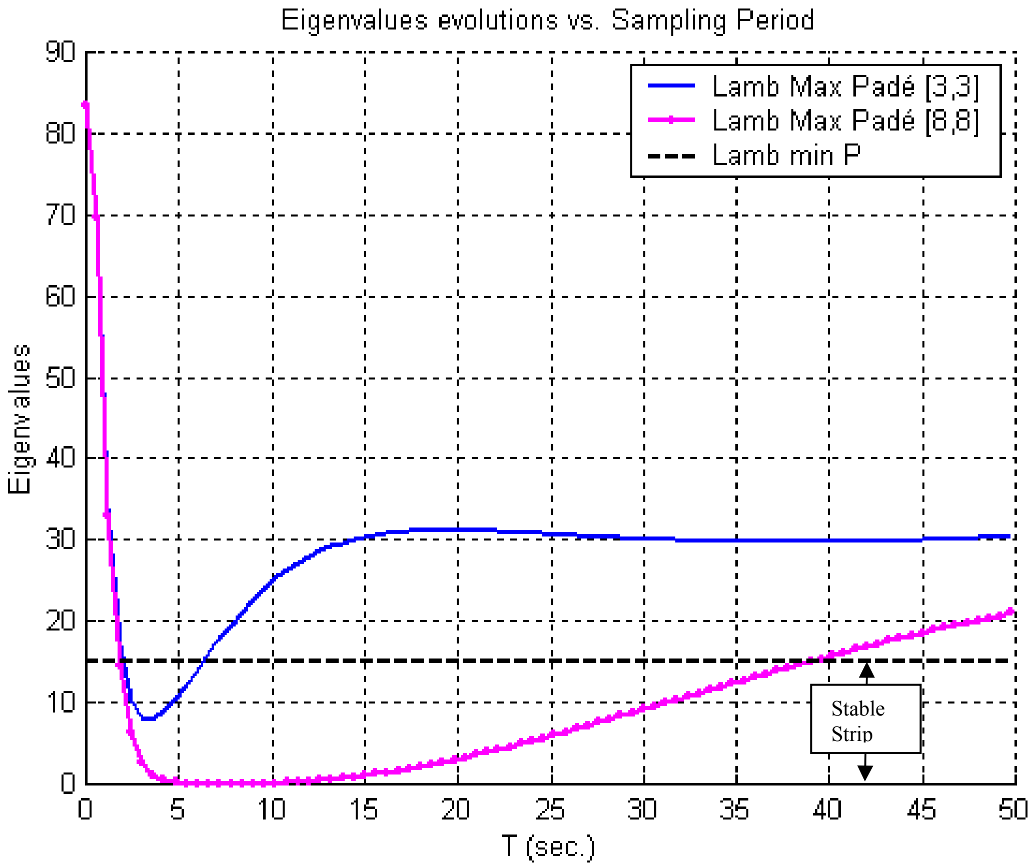

Figure 3: approximate stability/instability regions as a function of sampling period.

These accurate approximations are,

$$e^{AT}{}_{[3/3]} = (120.I\text{-}60TA + 12T^2A^2\text{-}T^3A^3)^{-1}(120.I + 60TA + 12T^2A^2 + T^3A^3)$$

$$e^{AT}{}_{[8/8]} = (518918400.I\text{-}259459200.TA + 60540480.T^2A^2\text{-}8648640.T^3A^3 + 831600T^4A^4\text{-}55440T^5A^5 +$$
$$+ 2520T^6A^6\text{-}72T^7A^7 + T^8A^8)^{-1}(518918400.I + 259459200.TA + 60540480.T^2A^2 +$$
$$+ 8648640.T^3A^3 + 831600.T^4A^4 + 55440.T^5A^5 + 2520.T^6A^6 + 72.T^7A^7 + T^8A^8)$$

From figure 3 we can see that precisely are a stable region for $T \in (1.8, 38.7)$ and the instable regions are: $T \in (0, 1.8]$ and $T \in [38.7, \infty)$, with T in seconds.
Thought the Eq.6 we have a geometrical disposal of this theory by the use of spectral radius as shown in Figure 4.

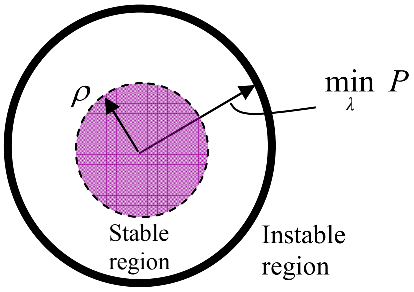

Figure 4: a geometrical representation of the stability rule using the spectral radius.

CONCLUSION

In this work we have presented accurate numerical solutions with the theorem early presented and we have proved that can exist regions for stability and instability in a sampled-data control system in function of the sampling-period. For future works we will extent these results and explore more industrial applications.

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