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ABSTRACT

An analytical approach for the residual magnetic torque acting in the spin-stabilized satellites is presented. It is assumed the inclined dipole model for the Earth´s magnetic field and the method of averaging the residual magnetic torque over each orbital period is applied to obtain the components of the torque in the satellite body frame reference system. The developments are presented in terms of the mean anomaly and contain terms of the second order in eccentricity. It is observed that the residual magnetic torque does not have component on the rotation axis. Then the residual torque does not affect the spin velocity magnitude, contributing only for the variations of the right ascension and declination of the rotation axis.

INTRODUCTION

The emphasis of this paper is placed on modeling the torques steaming from the residual magnetic moment associated with spin stabilized satellites (which has the spin axis along the geometric axis). A spherical coordinates system fixed in the satellite is used to located the spin axis of the satellite in relation to the terrestrial equatorial system. The direction of the spin axis are specified by the right ascension (α) and the declination (δ) and they are represented in the Fig. 1.

The magnetic residual torque occurs due to the interaction between the Earth magnetic field and the residual magnetic moment along the spin axis of the satellite. In spin stabilized satellites, equipped with nutation dumpers, such effect is usually the most perturbing torque. The torque analysis is performed through the modeling of the inclined Earth magnetic dipole, which orientation depends on the magnetic colatitude and on the ascending node of the magnetic plane. The developments are made in terms of the mean anomaly, considering second order on the eccentricity.

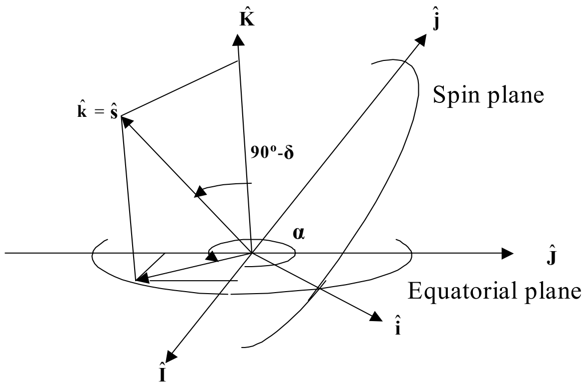

Figure 1- Orientation of the spin axis ($\hat{s}$): Equatorial System ($\hat{I}, \hat{J}, \hat{K}$), System of the Satellite ($\hat{i}, \hat{j}, \hat{k}$), right ascension ($\alpha$) and declination ($\delta$) of the spin axis.

To compute the average components of the residual magnetic torque in the satellite body frame reference system (satellite system) , a time average in terms of mean anomaly, which involves rotation matrices dependent on the orbit elements and right ascension and declination of the satellite spin axis, is utilized. The average torque already includes the main effects associated with the residual torque magnetic.

The results of this paper are useful to analytical attitude prediction and can shown that the residual torque does not affect the spin velocity magnitude but only the direction of the spin axis.

RESIDUAL TORQUE

Magnetic residual torques result from the interaction between the spacecraft's residual magnetic field and the Earth´s magnetic. If $\vec{m}$ is the magnetic moment of the spacecraft and $\vec{B}$ is the geomagnetic field, the residual magnetic torques is given by (Pisacane and Moore, 1994; Wertz,1978):

$$\vec{N}_r = \vec{m} \times \vec{B} \tag{1}$$

For the spin stabilized satellite the magnetic moment is along the spin axis and the residual torque can be expressed by:

$$\vec{N}_r = M_s \hat{k} \text{ x } \vec{B} \tag{2}$$

where M_s it is the satellite magnetic moment along its spin axis and $\hat{k}$ is the unit vector along the spin axis of the satellite.

GEOMAGNETIC FIELD

The inclined Earth magnetic dipole model is assumed in this paper. Its orientation depends on the magnetic colatitude (β) and on the ascending node of the magnetic plane (η). The magnetic system, which axis z_m is along the dipole vector, β and η are represented in the Fig. 2.

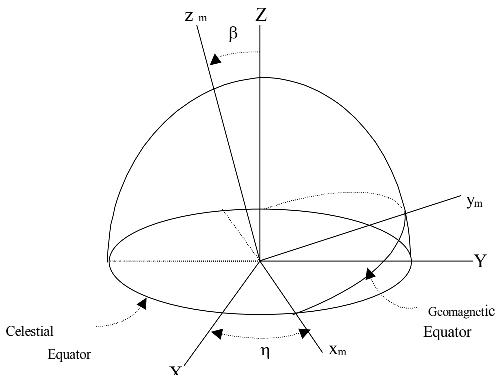

Figure 2- Magnetic System (O`x_m $y_m z_m$) and Equatorial System (O´XYZ).

It is well known that the Earth magnetic dipole model (Thomas and Capellari, 1964; Wertz,1978) may be expressed by:

$$\vec{B} = \frac{\ell}{4\pi\mu_o r^3}\left[\hat{k}_m - 3\left(\hat{i}_s \cdot \hat{k}_m\right)\hat{i}_s\right] \tag{3}$$

where ℓ is the magnetic moment of Earth's field magnitude, μ_o the permeability of free space, r the radius vector magnitude of the satellite, $\hat{k}_m$ the unit vector along the dipole vector and $\hat{i}_s$ the unit vector along radius vector of the satellite ($\vec{r}$).

The units vector $\hat{k}_m$ and $\hat{i}_s$ can be expressed in the satellite system through rotation matrices dependent on the orbit elements, right ascension and declination of the satellite spin axis and the angles β and η.

MEAN RESIDUAL TORQUE

In order to obtain the mean residual torque is necessary to integrate the instantaneous torque $\vec{N}_r$, given for (2), over one orbital period (T):

$$\vec{N}_{r_m} = \frac{1}{T}\int_{t_i}^{t_i+T} \vec{N}_r \, dt \tag{4}$$

where t is the time, t_i the initial time and T the orbital period.
In terms of the true anomaly, the mean residual torque can be gotten by (Thomas & Capellari,1964):

$$\vec{N}_{r_m} = \frac{1}{T}\int_{\upsilon_i}^{\upsilon_i+2\pi} \vec{N}_r \frac{r^2}{h}\, d\upsilon \tag{5}$$

where υ_i is the true anomaly at instant t_i and h is the specific angular moment.
Since the instantaneous torque is given by (2) and

$$r = \frac{a\left(1-e^2\right)}{1+e\,\mathrm{Cos}\,\upsilon} \quad , \qquad h = \frac{2\pi a^2\left(1-e^2\right)^{1/2}}{T}, \tag{6}$$

where a the semiaxis major and e the eccentricity of orbit., the mean residual torque (5) becomes:

$$\vec{N}_{r_m} = \Re\,\hat{k}\,x\int_{\upsilon_i}^{\upsilon_i+2\pi}\left[\hat{k}_m - 3\left(\hat{i}_s\cdot\hat{k}_m\right)\hat{i}_s\right]\left(1+e\,\mathrm{Cos}\,\upsilon\right)d\upsilon \tag{7}$$

with

$$\Re = \frac{M_s\,\ell}{8\pi^2\,\mu_o\,a^3\left(1-e^2\right)^{3/2}}\ . \tag{8}$$

To evaluate the integrals of (7) we will use the elliptic expansions of the true anomaly in terms of the mean anomaly M (Brouwer & Clemence, 1961), including terms in second order in the eccentricity (e). Then the present development can be applicable for elliptical orbits with precision. For simplification of the integrals we will consider the initial time for integration equal to the instant that the satellite passes through perigee.
The components of the unit vector $\hat{k}_m$ in the satellite system depend on the magnetic colatitude (β) and ascending node of the magnetic plane (η) and right ascension (α) and declination (δ) of the spin axis (Quirelli, 2002; Thomas & Capellari, 1964). In this paper, we will consider (Thomas and Capellari ,1964):

$$\eta = \eta_o + bM \tag{9}$$

where η_o it is the initial position of the ascending node of the geomagnetic equator in the instant of the satellite is in the perigee and if ω_e is the angular velocicity of the Earth, b is given by

$$b = \frac{\omega_e\,T}{2\pi} \tag{10}$$

The components of the unit vector $\hat{i}_S$ in the satellite system depend on ascending node orbit (Ω), orbital inclination (i), the true anomaly (ν) and right ascension (α) and declination (δ) of the spin axis (Quirelli, 2002). For one orbital period the angles Ω , i, ν, α, δ and β are constant.
Thus, using trigonometry properties and after exhausted but simple algebraic developments, the mean residual torque can be expressed by (Quirelli, 2002):

$$\vec{N}_{rm} = \Re \left\{ N_{rx}\, \hat{i} + N_{ry}\, \hat{j} \right\} \tag{11}$$

where:

$$\begin{aligned} N_{rx} = \; & \mathrm{Sen}\,\beta \{ A\,\mathrm{Cos}\,\alpha\,\mathrm{Sen}\,\delta - B\,\mathrm{Sen}\,\alpha\,\mathrm{Sen}\,\delta + 3[\mathrm{Sen}\,\delta\,\mathrm{Cos}(\Omega-\alpha)(D_1\,\mathrm{Sen}\,\Omega + \\ & + D_3\,\mathrm{Cos}\,\Omega\,\mathrm{Cos}\,i - C_1\,\mathrm{Cos}\,\Omega + C_3\,\mathrm{Sen}\,\Omega\,\mathrm{Cos}\,i) + \left(\mathrm{Sen}\,i\,\mathrm{Cos}\,\delta + \mathrm{Cos}\,i\,\mathrm{Sen}\,\delta\,\mathrm{Sen}(\Omega-\alpha)\right) \\ & \left[D_2\,\mathrm{Cos}\,\Omega\,\mathrm{Cos}\,i + D_3\,\mathrm{Sen}\,\Omega + C_2\,\mathrm{Sen}\,\Omega\,\mathrm{Cos}\,i - C_3\,\mathrm{Cos}\,\Omega \right]] \} \\ & + \pi\,\mathrm{Cos}\,\beta \{ -2\,\mathrm{Cos}\,\delta + 3\,\mathrm{Sen}\,i\,\mathrm{Cos}\,i\,\mathrm{Sen}\,\delta\,\mathrm{Sen}(\Omega-\alpha) + 3\,\mathrm{Sen}^2 i\,\mathrm{Cos}\,\delta \} \end{aligned} \tag{12}$$

$$\begin{aligned} N_{ry} = \; & \mathrm{Sen}\,\beta \; \{ A\,\mathrm{Sen}\,\alpha - B\,\mathrm{Cos}\,\alpha + 3\,[\,\mathrm{Sen}(\Omega-\alpha)(D_1\,\mathrm{Sen}\,\Omega + \\ & + D_3\,\mathrm{Cos}\,\Omega\,\mathrm{Cos}\,i - C_1\,\mathrm{Cos}\,\Omega + C_3\,\mathrm{Sen}\,\Omega\,\mathrm{Cos}\,i) + \left(\mathrm{Cos}\,i\,\mathrm{Cos}(\Omega-\alpha)\right) \\ & \left[D_2\,\mathrm{Cos}\,\Omega\,\mathrm{Cos}\,i + D_3\,\mathrm{Sen}\,\Omega + C_2\,\mathrm{Sen}\,\Omega\,\mathrm{Cos}\,i - C_3\,\mathrm{Cos}\,\Omega \right]] \} + \\ & + \pi\,\mathrm{cos}\,\beta \left[3\,\mathrm{Sen}\,i\,\mathrm{Cos}\,i\,\mathrm{Cos}(\Omega-\alpha) \right] \} \end{aligned} \tag{13}$$

with:

$$A = \mathrm{Sen}\,\eta_o \left(A_1 + 3e A_3 + e^2 (2A_4 + 7/2\,A_5 - A_1) \right) + \mathrm{Cos}\,\eta_o \left(A_2 + 3e A_6 + e^2 (2A_7 + 7/2\,A_8 - A_2) \right)$$

$$B = \mathrm{Cos}\,\eta_o \left(A_1 + 3e A_3 + e^2 (2A_4 + 7/2\,A_5 - A_1) \right) - \mathrm{Sen}\,\eta_o \left(A_2 - 3e A_6 - e^2 (2A_7 - 7/2\,A_8 + A_2) \right)$$

$$\begin{aligned} C_1 = & \left\{ \mathrm{Cos}^2 \omega \left(F_1\,\mathrm{Sen}\,\eta_o + F_2\,\mathrm{Cos}\,\eta_o \right) \right\} + \left\{ \frac{\mathrm{Cos}\,(2\omega)}{2} \left(F_3\,\mathrm{Sen}\,\eta_o + F_4\,\mathrm{Cos}\,\eta_o \right) \right\} - \\ & - \left\{ \frac{\mathrm{Sen}\,(2\omega)}{2} \left(F_5\,\mathrm{Sen}\,\eta_o + F_6\,\mathrm{Cos}\,\eta_o \right) \right\} \end{aligned}$$

$$\begin{aligned} C_2 = & \left\{ \mathrm{Sen}^2 \omega \left(F_1\,\mathrm{Sen}\,\eta_o + F_2\,\mathrm{Cos}\,\eta_o \right) \right\} - \left\{ \frac{\mathrm{Cos}\,(2\omega)}{2} \left(F_3\,\mathrm{Sen}\,\eta_o + F_4\,\mathrm{Cos}\,\eta_o \right) \right\} - \\ & - \left\{ \frac{\mathrm{Sen}\,(2\omega)}{2} \left(F_5\,\mathrm{Sen}\,\eta_o + F_6\,\mathrm{Cos}\,\eta_o \right) \right\} \end{aligned}$$

$$\begin{aligned} C_3 = & \left\{ \mathrm{Sen}\,\omega\,\mathrm{Cos}\,\omega \left((F_7 + F_{13})\,\mathrm{Sen}\,\eta_o + (F_8 + F_{14})\,\mathrm{Cos}\,\eta_o \right) \right\} - \\ & - \left\{ \mathrm{Sen}^2 \omega \left(F_9\,\mathrm{Sen}\,\eta_o + F_{10}\,\mathrm{Cos}\,\eta_o \right) \right\} + \left\{ \mathrm{Cos}^2 \omega \left(F_{11}\,\mathrm{Sen}\,\eta_o + F_{12}\,\mathrm{Cos}\,\eta_o \right) \right\} \end{aligned}$$

$$D_1 = \left\{ \mathrm{Cos}^2\omega\left(F_1 \mathrm{Cos}\,\eta_o + F_2 \mathrm{Sen}\,\eta_o \right) \right\} + \left\{ \frac{\mathrm{Cos}\,2\omega}{2} \left(F_3 \mathrm{Cos}\,\eta_o + F_4 \mathrm{Sen}\,\eta_o \right) \right\} -$$
$$- \left\{ \frac{\mathrm{Sen}\,2\omega}{2} \left(F_5 \cos\eta_o + F_6 \mathrm{Sen}\,\eta_o \right) \right\}$$
$$D_2 = \left\{ \mathrm{Sen}^2\,\omega \left(F_1 \mathrm{Cos}\,\eta_o + F_2 \mathrm{Sen}\,\eta_o \right) \right\} - \left\{ \frac{\mathrm{Cos}\,(2\omega)}{2} \left(F_3 \mathrm{Cos}\,\eta_o + F_4 \mathrm{Sen}\,\eta_o \right) \right\} -$$
$$- \left\{ \frac{\mathrm{Sen}\,(2\omega)}{2} \left(F_5 \mathrm{Cos}\,\eta_o - F_6 \mathrm{Sen}\,\eta_o \right) \right\}$$

$$D_3 = \left\{ \mathrm{Sen}\,\omega\, \mathrm{Cos}\,\omega \left((F_7 - F_{13}) \mathrm{Cos}\,\eta_o + (F_8 - F_{14}) \mathrm{Sen}\,\eta_o \right) \right\} -$$
$$- \left\{ \mathrm{Sen}^2\,\omega \left(F_{15} \mathrm{Cos}\,\eta_o - F_{16} \mathrm{Sen}\,\eta_o \right) \right\} + \left\{ \mathrm{Cos}^2\,\omega \left(F_{11} \cos\eta_o + F_{12} \mathrm{Sen}\,\eta_o \right) \right\}$$

The coefficients F_j, j= 1,2,...16, are given in the appendix A and the coefficients A_i, i = 1,2,...,8, are given in the appendix B.
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APPLICATIONS

The variations of the angular velocity, the declination and the ascension right of the spin axis are given by the Euler equations (Kuga et al, 1987) :

$$\dot{W} = \frac{1}{I_z} N_z , \qquad \dot{\delta} = \frac{1}{I_z\ W} N_y , \qquad \dot{\alpha} = \frac{1}{I_z\ W\ \mathrm{Cos}\,\delta} N_x \qquad (14)$$

where I_z is the moment of inertia along spin axis, N_x , N_y e N_z are component of external torques in the satellite body frame reference system (satellite system).
Therefore the residual magnetic torque does not affect the angular velocity, but it can produce the precession and nutation of the spin axis of the satellite.
The magnitude of residual torque for the Brazilian Satellite Data, show in the table (1), is $1{,}5718\text{x}10^{-5}$ for SCD1 and of $9{,}6649\text{x}10^{-7}$ for SCD2.

	SCD 1 (24/07/1993)	**SCD 2** (16/04/2002)
a (meter)	7139615,83	7133679,1
E	0,00453	0,00174
I (degrees)	25,002	24,99
Ω (degrees)	260,429	353,304
ω (degrees)	260,32	346,341
I_z (Kg m^2)	13	14,5
W(rpm)	90,76	32,96

α_o (degrees)	233,94	256,36
δ_o (degrees)	77,43	58,29
β (degrees)	11,4	11,4
μ_o (Weber/A m)	$4\ \pi\ .\ 10^{-7}$	$4\ \pi\ .\ 10^{-7}$
ℓ (Weber m)	10^{17}	10^{17}
M_s (A m^2)	-0,63	0,11

Table 1- Brazilian Satellites Data (SCD 1 and SCD 2).

SUMMARY

The residual magnetic torque model was discussed in this paper, considering the spin-stabilized satellite. The mean components of this torque in the satellite body frame reference system were obtained and show that residual torque component in z-axis is equal zero. Therefore this torque not affect the spin velocity magnitude. The results agree with the results presented in Thomas & Capellari (1964), which one were gotten through other developments of the instantaneous torque.

The results of this paper are useful to predict the satellite attitude.

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APPENDIX A

$F_1=A_1 + 3\ e\ A_3 + e^2\ (2\ A_4 + 7/2\ A_5 - A_1)$

$F_2=A_2 + 3\ e\ A_6 + e^2\ (2\ A_7 + 7/2\ A_8 - A_2)$

$F_3=2\ A_{13} + 3\ e\ (2\ A_9 + 4\ A_{13}) + e^2\ (4\ A_{10} + 7\ A_{11} - 2\ A_{12} + 4\ A_{14} + 2\ A_{15} + 1/4\ A_{16})$

$F_4=2\ A_{17} + 3\ e\ (2\ A_{18} + 4\ A_{21}) + e^2\ (4\ A_{19} + 7\ A_{20} - 2\ A_{17} + 4\ A_{22} + 2\ A_{23} + 1/4\ A_{24})$

$F_5=2\ A_{25} + 3\ e(\ 2\ A_{26} + 2\ A_{29} + 2\ A_{32})$

F_6=A_{35} +3 e (2 A_{36} + 2 A_{39} +2 A_{42}) + e^2 (2 A_{37} + 7 A_{38} – 2 A_{35} + 2 A_{40} + 1/4 A_{41} +
+ 2 A_{43} + 2 A_{44} + 18/8 A_{45})

F_7=A_4 +3 e (A_{46} + 2 A_{50}) + e^2 (2 A_{47} + A_{48} - A $_4$ + A_{49} + 2 A_{51} + 18/8 A_{52})

F_8=A_7 +3 e(A $_{53}$ + 2 A_{57}) + e^2 (2 A_{54} + A_{55} -A_7 + A_{56} + 2 A_{58}+ 18/8 A_{59})

F_9=A_{25} +3 e (A_{26} + A_{29} +A_{32}) +e^2 (2 A_{27} + +A_{28} – A_{25} + A_{30} + 1/8 A_{31} + A_{60} + A_{50} +
+9/8 A_{34})

F_{10}=A_{35} +3 e (A_{36} + A_{39} + A_{42} + A_{60}) + e^2 (2 A_{37} +7/2 A_{38} – A_{35} + A_{40} + 1/8 A_{41} + A_{43} +
A_{44}++A_{57} + 9/8 A_{45})

F_{11}=A_{25} +3 e (A_{26} + A_{32} + A_{29}) + e^2 (2 A_{27} + +A_{28} – A_{25} + A_{33} + 9/8 A_{34} + A_{30} +A_{61}+
+1/8 A_{31})

F_{12}=A_{35} + 3 e (A_{36} + A_{42}) + e^2 (2 A_{37} + A_{38}- -A_{35} + 9/8 A_{45} + A_{40} + A_{44} + 1/8 A_{41})

F_{13}=A_{12} + 3 e (A_9 + 2 A_{13}) + e^2 (2 A_{10} + 7/2 A_{11} – A_{12} +A_{15} + 2 A_{14} + 1/4 A_{16})

F_{14}=A_{17} +3 e (A_{18} + 2 A_{21}) + e^2 (2 A_{19} + 7/2 A_{20} – A_{17} + A_{23} + 2 A_{22} + 1/4 A_{24})

F_{15}=A_{25} +3 e (A_{26} + A_{29} + A_{32}) + e^2 (2 A_{27} + +A_{28} – A_{25} + A_{30} + 1/8 A_{31} + A_{33} +A_{61}+
+ A_{50} + 9/8 A_{35})

F_{16}=A_{35} – 3 e (A_{39} + 2 A_{42}) + e^2 (A_{35} – A_{40} – -A_{41} – A_{57} – A_{44} – A_{57} –9/8 A_2)

APPENDIX B

$$A_1=\frac{1}{b}\mathrm{Sin}(2\pi b) \qquad A_2=\frac{1}{b}[1-\mathrm{Cos}(2\pi b)] \qquad A_3=\frac{b\,\mathrm{Sin}(2\pi b)}{b^2-1}$$

$$A_4=\frac{(b^2-2)\mathrm{Sin}(2\pi b)}{b(b^2-4)} \qquad A_5=\frac{b\,\mathrm{Sin}(2\pi b)}{b^2-4} \qquad A_6=\frac{b[1-\mathrm{Cos}(2\pi b)]}{b^2-1}$$

$$A_7=\frac{(b^2-2)[1-\mathrm{Cos}(2\pi b)]}{b(b^2-4)} \qquad A_8=\frac{b\,[1-\mathrm{Cos}(2\pi b)]}{4-b^2} \qquad A_9=\frac{-2b\,\mathrm{Sin}(2\pi b)}{(1-b^2)(9-b^2)}$$

$$A_{10}=\frac{2\,\mathrm{Sin}(2\pi b)}{b(16-b^2)} \qquad A_{11}=\frac{-2(b^2+8)\ \mathrm{Sin}(2\pi b)}{b(b^2-4)(b^2-16)} \qquad A_{12}=\frac{2\mathrm{Sin}(2\pi b)}{b(4-b^2)}$$

$$A_{13} = \frac{-4b\ \mathrm{Sin}(2\pi b)}{(b^2-1)(b^2-9)} \qquad A_{14} = \frac{4\ \mathrm{Sin}(2\pi b)}{b(16-b^2)} \qquad A_{15} = \frac{8\ \mathrm{Sin}(2\pi b)}{b(16-b^2)}$$

$$A_{16} = \frac{-8(5b^2+28)\mathrm{Sin}(2\pi b)}{b(b^2-4)(b^2-16)} \qquad A_{17} = \frac{2[1-\mathrm{Cos}(2\pi b)]}{b(4-b^2)} \qquad A_{18} = \frac{-2b\ [1-\mathrm{Cos}(2\pi b)]}{(1-b^2)(9-b^2)}$$

$$A_{19} = \frac{2[1-\mathrm{Cos}(2\pi b)]}{b(16-b^2)} \qquad A_{20} = \frac{-2(b^2+8)[1-\mathrm{Cos}(2\pi b)]}{b(b^2-4)(b^2-16)} \qquad A_{21} = \frac{-4b\ [1-\mathrm{Cos}(2\pi b)]}{(b^2-1)(b^2-9)}$$

$$A_{22} = \frac{4[1-\mathrm{Cos}(2\pi b)]}{b(16-b^2)} \qquad A_{23} = \frac{8[1-\mathrm{Cos}(2\pi b)]}{b(16-b^2)} \qquad A_{24} = \frac{-8(5b^2+28)\ [1-\mathrm{Cos}(2\pi b)]}{b(b^2-4)(b^2-16)}$$

$$A_{25} = \frac{[1-\mathrm{Cos}(2\pi b)]}{(4-b^2)} \qquad A_{26} = \frac{(3-b^2)[1-\mathrm{Cos}(2\pi b)]}{(1-b^2)(9-b^2)} \qquad A_{27} = \frac{(10-b^2)\ [1-\mathrm{Cos}(2\pi b)]}{(4-b^2)(16-b^2)}$$

$$A_{28} = \frac{[1-\mathrm{Cos}(2\pi b)]}{(16-b^2)} \qquad A_{29} = \frac{2(3-b^2)\ [1-\mathrm{Cos}(2\pi b)]}{(1-b^2)(9-b^2)} \qquad A_{30} = \frac{2(10-b^2)\ [1-\mathrm{Cos}(2\pi b)]}{(4-b^2)(16-b^2)}$$

$$A_{31} = \frac{4(26-5b^2)\ [1-\mathrm{Cos}(2\pi b)]}{(4-b^2)(16-b^2)} \qquad A_{32} = \frac{-12\ \ [1-\mathrm{Cos}(2\pi b)]}{(b^2-1)(b^2-9)}$$

$$A_{33} = \frac{-12\ [1-\mathrm{Cos}(2\pi b)]}{(b^2-4)(b^2-16)} \qquad A_{34} = \frac{-24[1-\mathrm{Cos}(2\pi b)]}{(b^2-4)(b^2-16)}$$

$$A_{35} = \frac{b(b^4-30b^2+149)\ \mathrm{Sin}(2\pi b)}{b^6-35b^4+259b^2-225} \qquad A_{36} = \frac{(b^2-3)\mathrm{Sin}(2\pi b)}{(9-b^2)(1-b^2)}$$

$$A_{37} = \frac{(b^2-10)\mathrm{Sin}(2\pi b)}{(16-b^2)(4-b^2)} \qquad A_{38} = \frac{\mathrm{Sin}(2\pi b)}{(b^2-16)}$$

$$A_{39} = \frac{2(b^2-3)\mathrm{Sin}(2\pi b)}{(b^2-1)(b^2-9)} \qquad A_{40} = \frac{2(b^2-10)\mathrm{Sin}(2\pi b)}{(b^2-4)(b^2-16)}$$

$$A_{41} = \frac{4(5b^2-26)\mathrm{Sin}(2\pi b)}{(b^2-4)(b^2-16)} \qquad A_{42} = \frac{12\mathrm{Sin}(2\pi b)}{(b^2-1)(b^2-9)}$$

$$A_{43} = \frac{12}{(b^2-4)(b^2-16)}\mathrm{Sin}(2\pi b) \qquad A_{44} = \frac{24\mathrm{Sin}(2\pi b)}{(b^2-4)(b^2-16)}$$

$$A_{45} = \frac{24\mathrm{Sin}(2\pi b)}{(-2+b)(2+b)(-16+b^2)} \qquad A_{46} = \frac{-b(7-b)^2\ \mathrm{Sin}(2\pi b)}{(1-b^2)(9-b^2)}$$

$$A_{47} = \frac{(24-16b^2+b^4)\mathrm{Sin}(2\pi b)}{(-4+b)(-2+b)b(2+b)(4+b)}$$

$$A_{48} = \frac{(16-14b^2+b^4)\mathrm{Sin}(2\pi b)}{(-4+b)(-2+b)b(2+b)(4+b)}$$

$$A_{49} = \frac{8\mathrm{Sin}(2\pi b)}{b(-16+b^2)}$$

$$A_{50} = \frac{4b\mathrm{Sin}(2\pi b)}{(-3+b)(-1+b)(1+b)(3+b)}$$

$$A_{51} = \frac{4}{b(-16+b^2)}\mathrm{Sin}(2\pi b)$$

$$A_{52} = \frac{8\mathrm{Sin}(2\pi b)}{b(-16+b^2)}$$

$$A_{53} = \frac{-b(7-b^2)[1-\mathrm{Cos}(2\pi b)]}{(1-b^2)(9-b^2)}$$

$$A_{54} = \frac{(-24+16b^2-b^4)[\mathrm{Cos}(2\pi b)-1]}{(-4+b)b(4+b)(-4+b^2)}$$

$$A_{55} = \frac{(-16+14b^2-b^4)[\mathrm{Cos}(2\pi b)-1]}{(-4+b)b(4+b)(-4+b^2)}$$

$$A_{56} = \frac{8[1-\mathrm{Cos}(2\pi b)]}{b(-16+b^2)}$$

$$A_{57} = \frac{4b[1-\mathrm{Cos}(2\pi b)]}{(-3+b)(-1+b)(1+b)(3+b)}$$

$$A_{58} = \frac{4[1-\mathrm{Cos}(2\pi b)]}{b(-16+b^2)}$$

$$A_{59} = A_{56}$$

$$A_{60} = \frac{4\mathrm{Sin}(2\pi b)}{b(-4+b^2)}$$

$$A_{61} = \frac{24[\mathrm{Cos}(2\pi b)-1]}{(-2+b)(2+b)(-16+b^2)}$$