Calculation of Eddy Currents in the ETE Spherical Torus

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Abstract

A circuit model based on a Green's function method was developed to evaluate the currents induced during startup in the vacuum vessel of ETE (Experimento Tokamak Esférico). The eddy currents distribution is calculated using a thin shell approximation for the vacuum vessel and local curvilinear coordinates. The results are compared with values of the eddy currents measured in ETE.

1. Introduction

The study of breakdown and current buildup conditions in ETE (Experimento Tokamak Esférico) requires careful calculations of the error fields and flux consumption. In low aspect ratio devices such as ETE the center column must have a diameter as small as possible, with minimum space reserved for conductors to carry the toroidal field current as well as to fit a high-performance ohmic heating solenoid. This space restriction resulted in the construction of a robust vacuum vessel for ETE without any insulating toroidal break, which implies small toroidal electrical resistance notwithstanding the use of a high-resistivity alloy (Inconel). The small toroidal resistance introduces limitations in the electric field that may be applied inductively and affects the poloidal field system by way of eddy currents induced in the vessel.

This paper presents magnetostatic calculations used to evaluate the currents induced in the vacuum vessel of ETE during startup. The distribution of eddy currents is modeled using a thin shell approximation for the vacuum vessel. The equation governing the surface current induced on the thin shell is derived using a Green's function method. This three-dimensional problem in space can be reduced to one dimension due to symmetry, and by the adoption of local curvilinear coordinates and a spectral representation for the contour of the vacuum vessel. The resulting one-dimensional integral equation for the surface current can be solved expanding the current in a Fourier series in the poloidal angle. Introducing Laplace transformation in time, the problem for the set of Fourier components of the surface current is reduced to a circuit model that can be solved by matrix procedures. The results are compared with preliminary measurements of the eddy currents in ETE.

2. Formulation of the magnetostatic problem

The surface current density in a thin shell of thickness δ is given in terms of the current density $\vec{j}$

by[1]

$$\overrightarrow{K} = \delta \overrightarrow{j},$$

where the current density is related to the electric field by Ohm's law, $\overrightarrow{j} = \sigma \overrightarrow{E}$. Application of Faraday's law for a constant conductivity σ leads to

$$\nabla \times \overrightarrow{K} = \sigma\delta\nabla \times \overrightarrow{E} = -\sigma\delta\frac{\partial \overrightarrow{B}}{\partial t},$$

where $\overrightarrow{B}$ corresponds to the total induction. The condition of current continuity gives

$$\nabla \cdot \overrightarrow{K} = \sigma\delta\nabla \cdot \overrightarrow{E} = 0.$$

For an axisymmetric configuration there is no dependence on the toroidal angle ζ. Furthermore, the variation of the toroidal flux in time, $\partial\Phi_T/\partial t$, is neglected during startup, so that no poloidal currents are induced on the vacuum vessel. In this case, axisymmetry and the solenoidal property of the magnetic field, $\nabla \cdot \overrightarrow{B} = 0$, imply a single toroidal component of the vector potential. In vector form the potential is given in terms of the poloidal flux Φ_P by

$$\overrightarrow{A} = \frac{\Phi_P}{2\pi}\nabla\zeta.$$

In the same way, the surface current vector is expressed in terms of the single toroidal component K_T

$$\overrightarrow{K} = h_\zeta K_T \nabla\zeta,$$

where the scale factor $h_\zeta = |\partial\overrightarrow{r}/\partial\zeta|$ corresponds to the radial distance to the symmetry axis in cylindrical coordinates. Substituting the expression for $\overrightarrow{K}$ in Faraday's law it follows that

$$\nabla\zeta \times \nabla\left(h_\zeta K_T\right) = \sigma\delta\frac{\partial \overrightarrow{B}}{\partial t}.$$

Now, the magnetic induction is calculated in terms of the poloidal flux by

$$\overrightarrow{B} = \nabla \times \overrightarrow{A} = -\frac{1}{2\pi}\nabla\zeta \times \nabla\Phi_P.$$

This equation, combined with the previous one and the assumption of an uniform distribution over the small thickness δ, leads to a relation between the toroidal surface current density and the local value of the poloidal flux:

$$K_T = -\frac{\sigma\delta}{2\pi h_\zeta}\frac{\partial\Phi_P}{\partial t}.$$

Moreover the flux function must satisfy the boundary condition

$$\widehat{n} \cdot \nabla\Phi_P = -2\pi\mu_0 h_\zeta K_T,$$

which corresponds to the discontinuity of the magnetic induction across the surface layer of current ($\widehat{n}$ is the unit normal)

$$\widehat{n} \times \left[\overrightarrow{B}\right]_S = \mu_0\overrightarrow{K}.$$

In general, the vector potential at any point $\overrightarrow{r}$ not on the surface S' is given by the extension of the Biot-Savart law

$$\overrightarrow{A}\left(\overrightarrow{r}\right) = \frac{\mu_0}{4\pi}\iint_{S'}\frac{\overrightarrow{K}\left(\overrightarrow{r}'\right)}{\left|\overrightarrow{r} - \overrightarrow{r}'\right|}d^2r' + \overrightarrow{A}_{ext}\left(\overrightarrow{r}\right),$$

where $\overrightarrow{A}_{ext}$ stands for the external sources. Using the property $|\nabla\zeta|^2 = h_\zeta^{-2}$ the equivalent integral relation for the flux function is

$$\Phi_P\left(\overrightarrow{r}\right) = \frac{\mu_0}{2}h_\zeta^2\nabla\zeta \cdot \iint_{S'}\frac{h_{\zeta'}K_T\left(\overrightarrow{r}'\right)\nabla\zeta'}{\left|\overrightarrow{r} - \overrightarrow{r}'\right|}d^2r' + \Phi_{ext}\left(\overrightarrow{r}\right).$$

The differential element of area in the coordinate surface ρ that coincides with the surface layer of current is $d^2r\left(\rho\right) = h_\zeta d\ell\left(\theta\right)d\zeta$. Using the property $\nabla\zeta\cdot\nabla\zeta' = \cos\left(\zeta-\zeta'\right)/\left(h_\zeta h_{\zeta'}\right)$ the above equation can be written as

$$\Phi_P\left(\overrightarrow{r}\right) = \mu_0\oint K_T\left(\overrightarrow{r}'\right)\left\langle\frac{\pi h_\zeta h_{\zeta'}\cos\left(\zeta-\zeta'\right)}{\left|\overrightarrow{r}-\overrightarrow{r}'\right|}\right\rangle_{\zeta'} d\ell\left(\theta'\right) + \Phi_{ext}\left(\overrightarrow{r}\right),$$

where $\langle\ldots\rangle_\zeta = \left(2\pi\right)^{-1}\int\left(\ldots\right)d\zeta$. This defines the Green's function for the axisymmetric Ampère's law

$$G\left(\overrightarrow{r},\overrightarrow{r}'\right) = \left\langle\frac{\pi h_\zeta h_{\zeta'}\cos\left(\zeta-\zeta'\right)}{\left|\overrightarrow{r}-\overrightarrow{r}'\right|}\right\rangle_{\zeta'}.$$

The Green's function integral for Φ_P automatically satisfies the boundary condition $\widehat{n}\cdot\nabla\Phi_P = -2\pi\mu_0 h_\zeta K_T$.

Finally, taking the derivative with respect to time and using the relation between K_T and $\partial\Phi_P/\partial t$ provided by Faraday's law, the excitation of Foucault currents in a thin axisymmetric shell is governed by the equation

$$\frac{2\pi h_\zeta}{\sigma\delta}K_T\left(\overrightarrow{r}\right) = -\mu_0\oint\frac{\partial K_T\left(\overrightarrow{r}'\right)}{\partial t}G\left(\overrightarrow{r},\overrightarrow{r}'\right)d\ell\left(\theta'\right) - \frac{\partial\Phi_{ext}\left(\overrightarrow{r}\right)}{\partial t}.$$

This equation has local terms depending on the shell resistivity and non-local terms depending on mutual inductance effects between diverse regions of the current distribution. The total toroidal current induced in the shell is

$$I_T = \frac{1}{2\pi}\iint_{S(\rho)}\overrightarrow{K}\cdot\nabla\zeta\, d^2r\left(\rho\right) = \oint K_T\left(\theta\right)\, d\ell\left(\theta\right) = \int_0^{2\pi}K_T\left(\theta\right)\, h_\theta\, d\theta,$$

where the scale factor $h_\theta = \left|\partial\overrightarrow{r}/\partial\theta\right|$.

3. Spectral representation of the ETE vacuum vessel

In order to apply effectively the one-dimensional integral equation for the eddy currents obtained in the previous section, it is necessary to use a coordinate system coinciding with the contour of the axisymmetric shell. The centerline of the ETE vacuum vessel has an exact sectionally (piecewise) continuous representation given in the appendix and shown in Fig.1 as a continuous line.

The sectional continuous representation specifies the cylindrical coordinates $R\left(\omega\right)$, $Z\left(\omega\right)$ as functions of the poloidal angle ω in a pseudo-toroidal coordinate system centered in the cross-section of the vacuum vessel, as shown in Fig.1. Now, the centerline of the vacuum vessel can be represented approximately by a truncated spectral expansion in Chebyshev polynomials:

$$\begin{cases} R\left(\theta\right) &= C_0 + C_1\cos\theta - a\sum_{n=1}^{N}C_n\left[1-T_n\left(\cos\theta\right)\right] \\ Z\left(\theta\right) &= E_V\sin\theta\left[C_1 - a\sum_{n=1}^{N}C_nU_{n-1}\left(\cos\theta\right)\right]\end{cases}$$

The coefficients C_0 and C_1 are determined by the constraints

$$R\left(0\right) = R_0 + a, \qquad R\left(\pi\right) = R_0 - a,$$

where $R_0 = \left(D_V + d_V\right)/4$ and $a = \left(D_V - d_V\right)/4$ are the major and minor radii of the toroidal vessel, respectively (the geometrical parameters of the vacuum vessel are defined in the appendix). It follows that

$$C_0 = R_0 + a\sum_{n=1}^{[(N-1)/2]}C_{2n+1}, \qquad C_1 = a\left(1-\sum_{n=1}^{[(N-1)/2]}C_{2n+1}\right),$$

where $[N]$ denotes the greatest integer less than or equal to N. The elongation E_V and the remaining spectral coefficients $C_2, C_3, \ldots C_N$ can be determined by a least-squares fitting procedure. In the

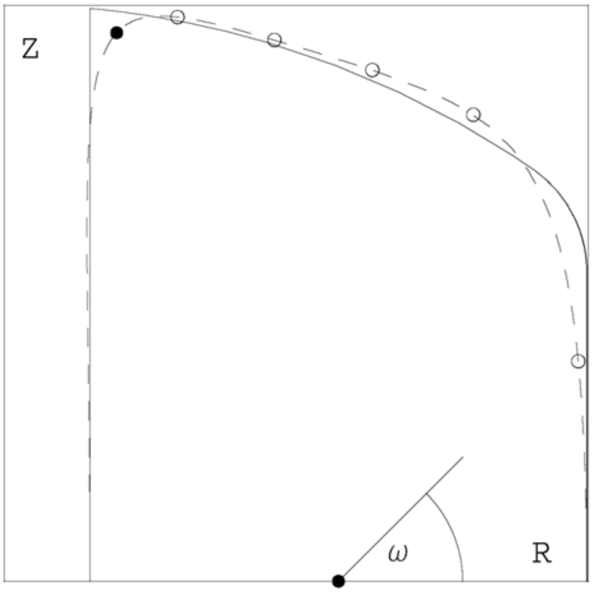

Figure 1: Centerline of the ETE vacuum vessel (continuous line) and spectral fit (dashed line).

case of the ETE vacuum vessel a reasonable spectral representation can be obtained including only elongation, triangularity and quadrangularity (squareness) corrections. The least-squares calculation gives $E_V = 2.164$, $C_2 = 0.0981$ and $C_3 = -0.110$, and the resulting spectral fit is show in Fig.1 as a dashed line. The least-squares fitting procedure includes also a determination of the best mapping between the pseudo-toroidal angle coordinate ω and the poloidal angle θ in the local curvilinear coordinate system. The adjusted $\theta - \omega$ mapping is shown in Fig.2. Finally, the spectral expansion $R(\theta)$, $Z(\theta)$ allows to determine the scale factors along the vacuum vessel centerline

$$h_\varsigma(\theta) = R(\theta), \qquad h_\theta(\theta) = \sqrt{\left(\frac{\partial R}{\partial\theta}\right)^2 + \left(\frac{\partial Z}{\partial\theta}\right)^2}.$$

4. Fourier components of the surface current

The integral equation for the Foucault currents in a thin shell, that was derived in Section 2, can be solved by expanding $K_T(\theta, t)$ in a Fourier series

$$K_T(\theta, t) = \frac{1}{2\pi h_\theta(\theta)}\left(I_T(t) + \sum_{n=1}^{\infty} I_n(t)\cos n\theta\right).$$

The total toroidal current flowing in the axisymmetric shell is $I_T(t)$ according with the definition in Section 2. Substitution of the Fourier series in the integral equation gives

$$\begin{aligned}\frac{h_\varsigma(\theta)}{\sigma\delta h_\theta(\theta)}\left(I_T(t) + \sum\nolimits_{n=1}^{\infty} I_n(t)\cos n\theta\right) &= -\mu_0\left(\frac{\partial I_T}{\partial t}\langle G(\theta,\theta')\rangle_{\theta'}\right. \\ &\quad \left. + \sum\nolimits_{n=1}^{\infty}\frac{\partial I_n}{\partial t}\langle G(\theta,\theta')\cos n\theta'\rangle_{\theta'}\right) - \frac{\partial\Phi_{ext}}{\partial t},\end{aligned}$$

where $\langle\ldots\rangle_\theta = (2\pi)^{-1}\int(\ldots)\,d\theta$. Limiting the Fourier coefficients to order ℓ, the $\cos m\theta$ harmonics of this equation result in a set of $\ell + 1$ linear equations for $I_T(t)$ and $I_n(t)$ that can be written in the

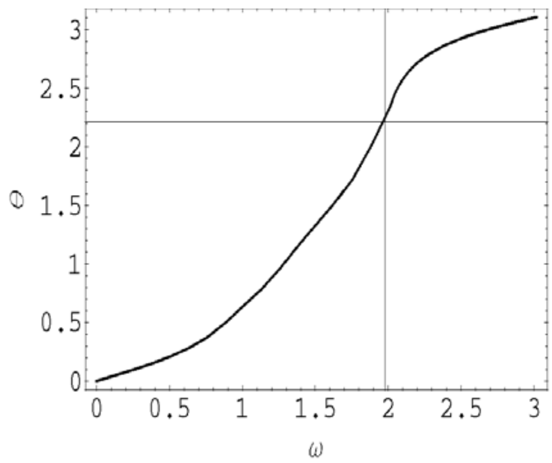

Figure 2: Adjusted $\theta - \omega$ mapping for the ETE vacuum vessel.

form

$$R_{0m} I_T(t) + L_{0m} \frac{\partial I_T}{\partial t} + \sum_{n=1}^{\ell} \left(R_{nm} I_n(t) + L_{nm} \frac{\partial I_n}{\partial t} \right) = -\frac{\partial}{\partial t} \left\langle \Phi_{ext}(\theta, t) \cos m\theta \right\rangle_\theta ,$$

where R_{nm} and L_{nm} are resistance and mutual inductance coefficients defined by:

$$\begin{cases} R_{nm} &= \dfrac{1}{\sigma\delta} \left\langle \dfrac{h_\zeta(\theta)}{h_\theta(\theta)} \cos n\theta \cos m\theta \right\rangle_\theta \\ L_{nm} &= \mu_0 \left\langle \left\langle G(\theta, \theta') \cos n\theta' \right\rangle_{\theta'} \cos m\theta \right\rangle_\theta \end{cases}$$

These definitions and the symmetry of the Green's function show that R_{nm} and L_{nm} are symmetric matrices.

In general, the external flux is the sum of the magnetizing flux $\Phi_M(t)$ produced by an ideal transformer and the fluxes $\Phi_k(\overrightarrow{r}, t)$ produced by sets of poloidal field coils:

$$\Phi_{ext}(\overrightarrow{r}, t) = \Phi_M(t) + \Phi_k(\overrightarrow{r}, t) = \Phi_M(t) + \mu_0 \sum_k I_k(t) G_k(\overrightarrow{r}, \overrightarrow{r}_k) .$$

Assuming that the external coils are formed by pairs of coils placed symmetrically with respect to the equatorial plane, and connected in series, the expression for the external flux becomes

$$\Phi_{ext}(\overrightarrow{r}, t) = \Phi_M(t) + \mu_0 \sum_k I_k(t) \left[G_k(\overrightarrow{r}, R_k, Z_k) + G_k(\overrightarrow{r}, R_k, -Z_k) \right] ,$$

where the Green's function is given in terms of the complete elliptic integrals K an E by

$$\begin{cases} G_k(\theta) &= \sqrt{R(\theta) R_k} \left(\dfrac{[2 - m_k(\theta)] K[m_k(\theta)] - 2E[m_k(\theta)]}{\sqrt{m_k(\theta)}} \right) \\ m_k(\theta) &= \dfrac{4R(\theta) R_k}{[R(\theta) + R_k]^2 + [Z(\theta) - Z_k]^2} \qquad (0 \le m_k \le 1) \end{cases}$$

Defining the mutual inductance coefficients

$$L_{km} = \mu_0 \left\langle [G_k(\theta) + G_k(-\theta)] \cos m\theta \right\rangle_\theta$$

the equations for the Fourier coefficients of the surface current density may be written (δ_{nm} is the

Kronecker delta)

$$R_{0m}I_T(t) + L_{0m}\frac{\partial I_T}{\partial t} + \sum_{n=1}^{\ell}\left(R_{nm}I_n(t) + L_{nm}\frac{\partial I_n}{\partial t}\right) = -\frac{\partial \Phi_M}{\partial t}\delta_{0m} - \sum_k L_{km}\frac{\partial I_k}{\partial t}.$$

In this way the problem of Foucault currents induced in a thin axisymmetric shell is reduced to the solution of a set of circuit-like coupled linear equations for the Fourier components of the surface current density.

The calculation of the mutual coefficients L_{nm} requires some attention because of the singular character of the Green's function

$$G(\theta,\theta') \underset{\theta'\to\theta}{\to} -h_\zeta(\theta)\left\{\frac{1}{2}\ln\left[\left(\frac{h_\theta(\theta)}{8h_\zeta(\theta)}\right)2\sin\left(\frac{\theta-\theta'}{2}\right)\right]^2 + 2\right\}.$$

Introducing the auxiliary function[2]

$$\mathcal{G}(\theta,\theta') = \frac{G(\theta,\theta')}{h_\zeta(\theta)} + \left\{\frac{1}{2}\ln\left[\left(\frac{h_\theta(\theta)}{8h_\zeta(\theta)}\right)2\sin\left(\frac{\theta-\theta'}{2}\right)\right]^2 + 2\right\},$$

which is nonsingular but nonsymmetric (not a true Green's function), the expression for the mutual coefficients becomes

$$\begin{aligned} L_{nm} &= \mu_0\left\langle h_\zeta(\theta)\left\langle \mathcal{G}(\theta,\theta')\cos n\theta'\right\rangle_{\theta'}\cos m\theta\right\rangle_\theta \\ &\quad -\mu_0\left\langle h_\zeta(\theta)\left\langle\left\{\frac{1}{2}\ln\left[\left(\frac{h_\theta(\theta)}{8h_\zeta(\theta)}\right)2\sin\left(\frac{\theta-\theta'}{2}\right)\right]^2+2\right\}\cos n\theta'\right\rangle_{\theta'}\cos m\theta\right\rangle_\theta. \end{aligned}$$

Using the integral

$$\frac{1}{4\pi}\int_0^{2\pi}\ln\left[2\sin\left(\frac{\theta-\theta'}{2}\right)\right]^2\cos n\theta' d\theta' = -\frac{\cos n\theta}{2n}(1-\delta_{0n})$$

it follows that

$$\begin{aligned} L_{nm} &= \mu_0\left\langle h_\zeta(\theta)G_n(\theta)\cos m\theta\right\rangle_\theta \\ &\quad +\mu_0\delta_{0n}\left\langle h_\zeta(\theta)\left[\ln\left(\frac{8h_\zeta(\theta)}{h_\theta(\theta)}\right)-2\right]\cos m\theta\right\rangle_\theta \\ &\quad +\mu_0\left(\frac{1-\delta_{0n}}{2n}\right)\left\langle h_\zeta(\theta)\cos n\theta\cos m\theta\right\rangle_\theta, \end{aligned}$$

where

$$G_n(\theta) = \left\langle\mathcal{G}(\theta,\theta')\cos n\theta'\right\rangle_{\theta'} = \frac{1}{2\pi}\mathcal{P}\int_0^{2\pi}\mathcal{G}(\theta,\theta')\cos n\theta' d\theta'$$

and $\mathcal{P}$ designates the principal value of the integral to make clear the absence of singularities. The logarithmic term in L_{00} corresponds to the self-field contribution to the inductance.

5. Solution of the circuit model and results

It is now an easy matter to solve the set of circuit equations for the Fourier components of the Foucault current. Introducing Laplace transformation in time and denoting the complex frequency by s, the equations for $I_T(s)$ and $I_n(s)$ ($n = 1, 2, \ldots, \ell$) can be written in matrix form

$$\underbrace{\begin{bmatrix} R_{00}+sL_{00} & R_{10}+sL_{10} & \ldots & R_{\ell 0}+sL_{\ell 0} \\ R_{01}+sL_{01} & R_{11}+sL_{11} & \ldots & R_{\ell 1}+sL_{\ell 1} \\ \vdots & \vdots & & \vdots \\ R_{0\ell}+sL_{0\ell} & R_{1\ell}+sL_{1\ell} & \ldots & R_{\ell\ell}+sL_{\ell\ell} \end{bmatrix}}_{R+sL}\underbrace{\begin{bmatrix} I_T(s) \\ I_1(s) \\ \vdots \\ I_\ell(s) \end{bmatrix}}_{I(s)} = -s\underbrace{\begin{bmatrix} \Phi_M(s)+\sum_k L_{k0}I_k(s) \\ \sum_k L_{k1}I_k(s) \\ \vdots \\ \sum_k L_{k\ell}I_k(s) \end{bmatrix}}_{\Phi(s)},$$

where $R + sL$ is a symmetric matrix. The initial values of the magnetizing sources are taken equal to zero at startup. The solution of the circuit model is obtained simply by multiplying the flux excitation vector $\Phi(s)$ by the inverse matrix $(R + sL)^{-1}$ and then calculating the inverse Laplace transform. One advantage of the method is that the inverse matrix depends only on the geometry of the problem, which is independent of the detailed excitation.

The resistance and inductance components scale as $A/(E_V \sigma \delta)$ and $\mu_0 R_0 [\ln(8A) - 2]$, respectively , where $E_V = 2.164$ is the elongation, $A = 1.346$ is the aspect ratio and $R_0 = 0.348\,\mathrm{m}$ is the major radius of the vacuum vessel. The conductivity of Inconel at room temperature is $\sigma \cong 7.8 \times 10^5\,(\Omega \cdot \mathrm{m})^{-1}$. The average surface current scales as $\overline{K_T} \sim AI_T/(2\pi R_0)$, where I_T is the total current induced in the vacuum vessel. Now, the thickness of the vacuum vessel is $\Delta_V = 6.35\,\mathrm{mm}$ for both the torispherical head and the external cylindrical wall, and $\delta_V = 1.00\,\mathrm{mm}$ for the internal cylindrical wall cf. the appendix. In the calculation of R_{nm} it is possible to split the θ integration in two sections to account for the change in the wall thickness. However, to get preliminary results in the case when only the ohmic transformer is excited, an effective thickness $\delta \cong 1.2\,\mathrm{mm}$ was assumed for the vacuum vessel taking into account that the surface current is concentrated in the inner wall.

Figure 3 shows the results of calculations performed for the eddy current behavior in space and time, which compares satisfactorily with measurements taken in the ETE vacuum vessel (only three harmonics, $\ell = 3$, were included in this computation). The figure shows: a model for the profile of the magnetizing flux applied by the ohmic heating system during the eddy current measurements; the calculated profile of the total current induced in the vacuum vessel; and the distribution of the surface current at four instants $\tau_0/2$, $2\tau_0$, $4\tau_0$ and $16\tau_0$ ($\tau_0 = \mu_0 \sigma \delta R_0 / A = 0.304\,\mathrm{ms}$ sets the time scale). The instant $\tau_0/2$ corresponds approximately to the maximum negative value of the induced current and $16\tau_0$ to the maximum positive value (of course, the induced current opposes the excitation according to Lenz's law). From the plots in Fig.3 and the mapping $\theta - \omega$ shown in Fig.2 one verifies that the eddy current distribution has two peaks at $\omega \sim 68^\circ$ and $\omega \sim 112^\circ$, near the two corners of the vacuum vessel contour and in accordance with rough measurements of the distribution excited by the ohmic heating system in ETE.

Based on the calculated and experimental results a pair of compensation coils is being designed to apply a vertical field bias before plasma breakdown in ETE. In addition, the eddy current distribution is being used to define an equivalent set of filaments that model the vacuum vessel effects in plasma discharge simulations during the early phase. In the zero-dimensional simulations the external inductance of the low aspect ratio ETE plasma and the mutual inductance coefficients between the plasma, the vacuum vessel filaments and the external poloidal field coils are calculated in accordance with a previous work [3] .

6. Appendix. Sectionally continuous representation of the ETE vacuum vessel

The centerline of the ETE vacuum vessel is described exactly by the following sectionally continuous representation:

$$\begin{cases} R(\omega) &= \dfrac{D_V}{2} \\ Z(\omega) &= \left(\dfrac{D_V - d_V}{4}\right)\tan\omega \end{cases}$$

$$\text{when } 0 \leqslant \omega \leqslant \arctan\left(\frac{H_V/2}{(D_V - d_V)/4}\right);$$

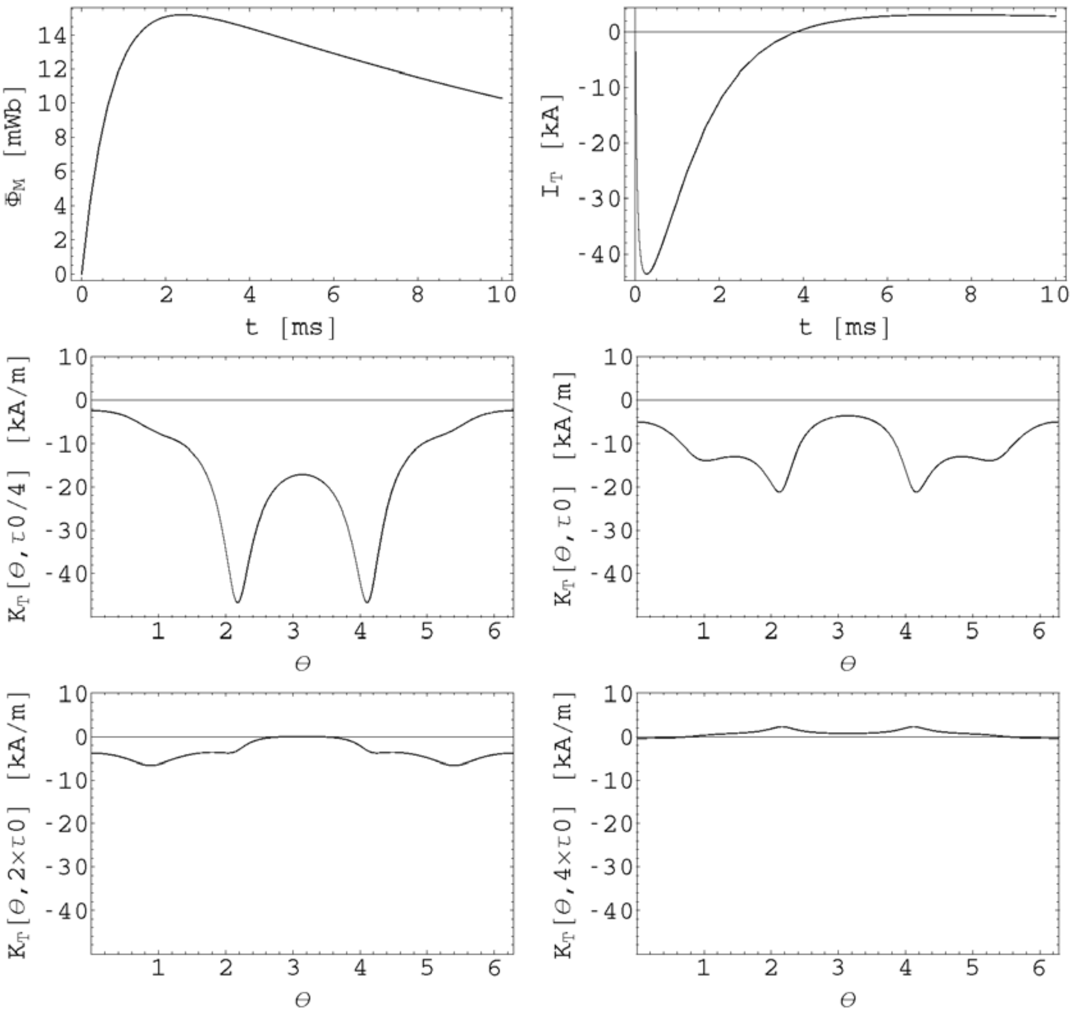

Figure 3: Magnetizing flux applied by the ohmic heating system, total eddy current in the vacuum vessel, and eddy current distribution at four instants of time.

$$
\begin{cases}
R(\omega) = & \left(\dfrac{D_V + d_V}{4}\right)\sin^2\omega + \dfrac{H_V}{2}\sin\omega\cos\omega + \left(\dfrac{D_V}{2} - r_V\right)\cos^2\omega \\
& + \cos\omega\left[H_V\left(\dfrac{D_V - d_V}{4} - r_V\right)\sin\omega\cos\omega\right. \\
& \left. - \left(\dfrac{D_V - d_V}{4}\right)\left(\dfrac{D_V - d_V}{4} - 2r_V\right)\sin^2\omega - \left(\left(\dfrac{H_V}{2}\right)^2 - r_V^2\right)\cos^2\omega\right]^{1/2} \\
Z(\omega) = & \left(\dfrac{D_V - d_V}{4} - r_V\right)\sin\omega\cos\omega + \dfrac{H_V}{2}\sin^2\omega \\
& + \sin\omega\left[H_V\left(\dfrac{D_V - d_V}{4} - r_V\right)\sin\omega\cos\omega\right. \\
& \left. - \left(\dfrac{D_V - d_V}{4}\right)\left(\dfrac{D_V - d_V}{4} - 2r_V\right)\sin^2\omega - \left(\left(\dfrac{H_V}{2}\right)^2 - r_V^2\right)\cos^2\omega\right]^{1/2}
\end{cases}
$$

when $\arctan\left(\dfrac{H_V/2}{(D_V - d_V)/4}\right) < \omega \leqslant \arctan\left(\dfrac{H_V/2 + r_V\sin\alpha_V}{(D_V - d_V)/4 - r_V(1 - \cos\alpha_V)}\right)$;

$$\begin{cases} R(\omega) &= \left(\dfrac{D_V + d_V}{4}\right)\sin^2\omega - Z_V \sin\omega\cos\omega \\ & \quad + \cos\omega \left\{\left[\left(\dfrac{D_V + d_V}{4}\right)\cos\omega + Z_V \sin\omega\right]^2 - \left(\dfrac{D_V + d_V}{4}\right)^2 + (R_V^2 - Z_V^2)\right\}^{1/2} \\ Z(\omega) &= -\left[\left(\dfrac{D_V + d_V}{4}\right)\cos\omega + Z_V \sin\omega\right]\sin\omega \\ & \quad + \sin\omega \left\{\left[\left(\dfrac{D_V + d_V}{4}\right)\cos\omega + Z_V \sin\omega\right]^2 - \left(\dfrac{D_V + d_V}{4}\right)^2 + (R_V^2 - Z_V^2)\right\}^{1/2} \end{cases}$$

$$\text{when } \arctan\left(\frac{H_V/2 + r_V \sin\alpha_V}{(D_V - d_V)/4 - r_V(1-\cos\alpha_V)}\right) < \omega \leqslant \arctan\left(\frac{R_V \sin\beta_V - Z_V}{-(D_V - d_V)/4}\right) \text{; and}$$

$$\begin{cases} R(\omega) &= \dfrac{d_V}{2} \\ Z(\omega) &= -\left(\dfrac{D_V - d_V}{4}\right)\tan\omega \end{cases}$$

$$\text{when } \arctan\left(\frac{R_V \sin\beta_V - Z_V}{-(D_V - d_V)/4}\right) < \omega \leqslant \pi.$$

The angle ω in this representation is the pseudo-toroidal angle centered on the midpoint of the vacuum vessel cross section cf. Fig.1. The geometrical parameters are: $D_V = 1.213\,\mathrm{m}$ (48"-1/4") is the average diameter, $R_V = 0.968\,\mathrm{m}$ (38"+1/8") is the average radius of dish, and $r_V = 0.130\,\mathrm{m}$ (5"+1/8") is the average knuckle radius of the torispherical head of thickness $\Delta_V = 0.00635\,\mathrm{m}$ (1/4"), respectively; $d_V = 0.179\,\mathrm{m}$ (0.180-0.001) is the average diameter and $h_V = 1.200\,\mathrm{m}$ is the total height of the internal cylindrical wall of thickness $\delta_V = 0.001\,\mathrm{m}$, respectively. The height of the external cylindrical wall, also of thickness $\Delta_{V,}$ is calculated by

$$H_V = 2\left[(R_V - r_V)\sin\alpha_V - Z_V\right],$$

where the center of the dish radius on the symmetry axis is specified by Z_V, which is calculated by

$$Z_V = \left(R_V + \frac{\Delta_V}{2}\right)\sqrt{1 - \left(\frac{d_V/2}{R_V + \Delta_V/2}\right)^2} - \frac{h_V}{2}.$$

The angular parameters α_V and β_V are calculated by

$$\alpha_V = \arccos\left(\frac{D_V/2 - r_V}{R_V - r_V}\right), \qquad \beta_V = \arccos\left(\frac{d_V/2}{R_V}\right).$$

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