Stability of Nonadiabatic Cellular Flames near Extinction

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Stability of Nonadiabatic Cellular Flames near Extinction

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Abstract

Combinations of power series and Fourier series are employed here to study the stability of the solutions of the Joulin-Sivashinsky evolution equations that describe the dynamics of cellular premixed flames near the condition of extinction by heat loss. Regions of stability to space-periodic perturbations are determined, as are the distances between the boundaries of these regions, when the perturbation has a wavelength that is a submultiple of the wavelength of the cellular flame.

I. Introduction

The boundary-value problem defined by the Joulin-Sivashinsky equations describes the dynamics of premixed flame fronts near the condition of extinction by heat loss. Joulin and Sivashinsky[1] derived this system under the restriction that the planar flame is unstable to cellular perturbations. Joulin[2] numerically integrated steady-state one-dimensional versions of the problem, showing that there are solutions, corresponding to cellular flames, that propagate steadily with rates of heat loss greater than the maximum value for extinction of the planar flame. Recently, Sinay and Williams[3] proved analytically the existence of such solutions and solved the one-space-dimension problem utilizing a combination of power series and Fourier series. We shall refer to this combination as the power Fourier series (PFS) algorithm.

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Let D be the thermal diffusivity and U the burning velocity of the planar flame with Lewis number $Le = 1$ having the maximum rate of heat loss. Use these variables to define length and time scales, and let ζ and τ be correspondingly scaled space and time variables, respectively. If the flame-front position (in units D/U) in a frame moving towards the fresh mixture with velocity U is denoted by ϕ and a nondimensional flame temperature decrement as ψ, then the Joulin-Sivashinsky boundary-value problem is

$$\phi_\tau + \tfrac{1}{2}|\nabla\phi|^2 = \Delta\phi + \psi \qquad \text{on } \Omega \times \mathbb{R}$$

$$\psi_\tau + \nabla\psi\cdot\nabla\phi = \Delta\psi - \Delta\phi + \tfrac{1}{3}(\psi^2 + \nu) \tag{1}$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial n} = 0 \qquad \text{on } \partial\,\Omega$$

where ν is the scaled heat-loss deviation parameter, Ω is a region in $\mathbb{R}$ or $\mathbb{R}^2$, $\partial\Omega$ is its boundary, and $\partial/\partial n$ stands for the derivative in the normal direction. A detailed description of the physical meaning of the variables can be found in Ref. 3. In that paper, solutions of the one-space-dimension version of Eq. (1) were obtained in the form

$$\phi = -\mu\,\tau + f(\zeta)$$
$$\psi = -\mu + g(\zeta) \tag{2}$$

where $f(\zeta)$ and $g(\zeta)$, which are periodic in ζ with wavelength $2\pi/k$, can be represented parametrically for each fixed k, in terms of a parameter δ, as power series in δ with coefficients which are Fourier series in ζ. The scaled velocity deviation μ and the loss parameter ν were also obtained as power series in δ.

The purpose of the present paper is to describe some of the linear-stability boundaries of Eq. (2) and their physical significance. To that end, we first study the linear stability of plane flames, corresponding to $\delta = 0$, and then, using the same kind of power series and Fourier series employed in Ref. 3, we determine transitions between regions of stability and instability.

II. Stability of Planar-Front Flames

Linearizing the one-dimensional version of Eq. (1) and using Eq. (2), we obtain

$$\dot\phi_\tau + f_\zeta(\zeta,k,\delta)\dot\phi_\zeta = \dot\phi_{\zeta\zeta} + \dot\psi$$

$$\dot\psi_\tau + g_\zeta(\zeta,k,\delta)\dot\phi_\zeta + f_\zeta(\zeta,k,\delta)\dot\psi_\zeta = \dot\psi_{\zeta\zeta} - \dot\phi_{\zeta\zeta} + \tfrac{2}{3}(-\mu(\delta) + g(\zeta,k,\delta))\dot\psi \tag{3}$$

where the subscripts ζ and τ indicate partial derivatives with respect to the corresponding variable and the functions $f(\zeta,k,\delta)$, $g(\zeta,k,\delta)$, as well as the parameter $\mu(\delta)$, are series in δ given previously in Ref. 3.

Setting

$$\begin{aligned}\dot{\phi} &= e^{b\tau}\,[\cos\omega\tau\; u^{+}(\zeta) + \sin\omega\tau\; u^{-}(\zeta)]\\ \dot{\psi} &= e^{b\tau}\,[\cos\omega\tau\; v^{+}(\zeta) + \sin\omega\tau\; v^{-}(\zeta)]\end{aligned} \tag{4}$$

and, further, transforming the resulting equations in the usual way, we obtain a first-order system of ordinary differential equations of the form

$$X' = F[\zeta, k, \delta, b, \omega]\, X \;; \qquad ()' = \frac{\partial}{\partial\zeta} \tag{5}$$

where F is an 8×8 real periodic matrix of period $2\pi/k$. It follows from Floquet theory[4] that corresponding to any real fundamental solution matrix $X(\zeta)$ of Eq. (5), there exist a real periodic matrix $P(\zeta)$ of period $4\pi/k$, and a real constant matrix C, such that

$$X(\zeta) = P(\zeta)\,\exp(\zeta C) \tag{6}$$

From the experimental viewpoint it is desirable to establish the stability of the cellular flames under perturbations by excitations that are periodic in space with wavelengths that are smaller than the dimensions of the burner, or rather, exact submultiples of the wavelength of the cellular flame. We shall thus concentrate our study on the determination of the conditions under which X in Eq. (6) is periodic; i.e., we shall seek values $b = b(\delta)$, $\omega = \omega(\delta)$ for which Eq. (5) and therefore Eq. (3) with $\dot{\phi}$, $\dot{\psi}$ given by Eq. (4) possesses a solution periodic in ζ, with period a multiple (or submultiple) of $4\pi/k$.

When $\delta = 0$, Eq. (3) reduces to

$$\begin{aligned}\dot{\phi}_{0,\tau} &= \dot{\phi}_{0,\zeta\zeta} + \dot{\psi}_0\\ \dot{\psi}_{0,\tau} &= \dot{\psi}_{0,\zeta\zeta} - \dot{\phi}_{0,\zeta\zeta} - (1-k^2)\dot{\psi}_0\end{aligned} \tag{7}$$

Here, the subscript 0 stands for $\delta = 0$, and one must recall that $f_0 = g_0 = 0$, whereas $\mu_0 = \frac{3}{2}(1 - k^2)$ (see Ref. 3).

Since Eq. (7) is a linear system of ordinary differential equations with constant coefficients, we can assume, using the fact that we seek periodic solutions with period $4\pi/k$,

$$\phi_0 = A \exp(\sigma\,\tau + i\,\frac{mk}{2}\zeta),\quad \psi_0 = B \exp(\sigma\,\tau + i\,\frac{mk}{2}\zeta) \tag{8}$$

where $\sigma = b + i\omega$, b, and ω are real numbers, and m is an integer. Substitution of Eq. (8) for ϕ_0, ψ_0 into Eq. (7) yields a homogeneous algebraic system, which has a nontrivial solution if and only if

$$b^2 + [1 + (\frac{m^2}{2} - 1)\,k^2\,]\,b + \frac{m^2 k^4}{4}(\frac{m^2}{4} - 1) - \omega^2 = 0 \tag{9a}$$

$$[\,1 + 2b + \left(\frac{m^2}{2} - 1\right)k^2\,]\,\omega = 0 \tag{9b}$$

It is easy to see that, for fixed m, the only possible solution of Eqs. (9a) and (9b) is

$$\omega = 0$$

$$b = b_m^{\pm} = \tfrac{1}{2}\{-[\,1 + (\frac{m^2}{2} - 1)\,k^2] \pm \sqrt{(k^2-1)^2 + (mk)^2}\,\},\qquad m \geq 0 \tag{10}$$

where the superscripts $\pm$ of b_m correspond to the signs of the square root.

Straightforward calculations show that $b_0^+ = 0$, $b_0^- = k^2 - 1$ if $k < 1$ and $b_0^+ = k^2 - 1$, $b_0^- = 0$ if $k > 1$; $b_1^+ > 0$ and $b_1^- < 0$ for all k, $b_2^+ = 0$, $b_2^- < 0$ for all k, and finally, $b_m^{\pm} < 0$ for all k and $m \geq 3$. Correspondingly, we have the real solutions

$$\begin{pmatrix}\phi_0\\ \psi_0\end{pmatrix} = e^{b\tau}\begin{pmatrix}u_0\\ v_0\end{pmatrix} \tag{11}$$

with

$$\begin{pmatrix}u_0\\ v_0\end{pmatrix} = \begin{pmatrix}1\\ b_m^{\pm} + (\frac{mk}{2})^2\end{pmatrix}\cos(\frac{mk\zeta}{2})\quad(\sin(\frac{mk\zeta}{2}))$$

We have thus established the linear stability behavior of plane flames subject to perturbations that are periodic in space, with period $4\pi/mk$. Since $b_1^+ > 0$, the known result is recovered, to the effect that the solutions for $\delta = 0$ are unstable to disturbances with period $4\pi/k$. The purpose of the next section is to study the solutions of Eq. (3) which are analytic continuations in δ of Eq. (11), and the functions $b = b(\delta)$, $\omega = \omega(\delta)$ for which these solutions exist and which reduce to Eq. (10) when $\delta = 0$.

III. Periodic Solutions

Introducing Eq. (4) for $\dot{\phi}$, $\dot{\psi}$ in Eq. (3), we obtain

$$b\,u^+ + \omega\,u^- + f\zeta\,u_\zeta^+ = u_{\zeta\zeta}^+ + v^+ \tag{12a}$$

$$b\,v^+ + \omega\,v^- + g\zeta\,u_\zeta^+ + f\zeta\,v_\zeta^+ = v_{\zeta\zeta}^+ - u_{\zeta\zeta}^+ + \frac{2}{3}(-\mu + g)\,v^+ \tag{12b}$$

$$b\,u^- - \omega\,u^+ + f\zeta\,u_\zeta^- = u_{\zeta\zeta}^- + v^- b\,v^- \tag{12c}$$

$$b\,v^- - \omega\,v^+ + g\zeta\,u_\zeta^- + f\zeta\,u_\zeta^-\,v_\zeta^- = v_{\zeta\zeta}^- - u_{\zeta\zeta}^- + \frac{2}{3}(-u + g)v^- \tag{12d}$$

It follows from the results of the previous section that the system in Eqs. (12a-12d) has a periodic solution of period $4\pi/(mk)$ when $\delta = 0$ if and only if $b = b_m^{\pm}$ and $\omega = 0$. Let us assume for a moment that, for some fixed k, $u^+ = \tilde{u}$, $v^+ = \tilde{v}$ is a solution of Eqs. (12a) and (12b) with $\omega = 0$ and δ in a neighborhood of zero. Then $u^- = \tilde{u}$, $v^- = \tilde{v}$ is also a solution of Eqs. (12c) and (12d) with $\omega = 0$ and δ in the same neighborhood of zero. Let

$$G = G[u^+, v^+, u^-, v^-, \omega, \delta]$$

be the homogeneous differential operator defined by Eqs. (12a-12d). Then $G[\tilde{u}, \tilde{v}, \tilde{u}, \tilde{v}, 0, \delta] = 0$ in a neighborhood of $\delta = 0$. If

$$h(\omega, \delta) = G[\tilde{u}, \tilde{v}, \tilde{u}, \tilde{v}, \omega, \delta]$$

then $h(0,\delta) = 0$ in a neighborhood of $\delta = 0$ and $h_\omega(0,\delta) = (\tilde{u}, \tilde{v}, -\tilde{u}, -\tilde{v})$. It follows from the implicit function theorem that the solution of the equation $h(\omega, \delta) = 0$ is unique and therefore $\omega \equiv 0$ in that neighborhood. We can thus try to reduce the problem of solving Eqs. (12a-12d) to the one of solving

$$b\,u + f\zeta\,u_\zeta = u_{\zeta\zeta} + v \tag{13a}$$

$$b\,v + g\zeta\,u_\zeta + f\zeta\,v_\zeta = v_{\zeta\zeta} - u_{\zeta\zeta} + \frac{2}{3}(-\mu + g)\,v \tag{13b}$$

Fixing $m = m_0$ in Eq. (10) we now seek values $b = b(\delta)$ for which Eq. (13) possesses a periodic solution with period $4\pi/(m_0 k)$. To obtain them, we set $z = k\zeta$ if m_0 is an even integer and $z = k\zeta/2$ if m_0 is odd. Two special cases must be observed. When $m_0 = 0$, $b = 0$, $u = 1$, $v = 0$ is a solution for all δ. Another solution is the analytic continuation of $u = \zeta$, v

$= 0$ which is not periodic. If $m_0 = 2$ then $b(\delta)$ is identically zero and the solutions of Eq. (13) are $u = f_\zeta$, $v = g_\zeta$ and $u = \tilde{f}_\zeta$, $v = \tilde{g}_\zeta$ where $\tilde{f}$, $\tilde{g}$ are odd solutions of Eq. (1) (see Ref. 5) that can be obtained in the same way that we calculated f and g by the PFS algorithm.

It follows from the structure of the Eq. (13) that one can have $u(z)$ and $v(z)$ either both even functions or both odd functions of z; hence, we employ the PFS algorithm for fixed m_0 and $\delta \neq 0$, assuming that u and v can be expanded in the power series

$$u(z) = \sum_{n=0}^{\infty} u_n(z)\,\delta^n; \qquad v(z) = \sum_{n=0}^{\infty} v_n(z)\,\delta^n \tag{14}$$

and define

$$\begin{aligned} u_n(z) &= \sum_{p=0}^{\infty} \hat{u}_n(p)\cos pz \quad (\text{or } \sin pz) \\ v_n(z) &= \sum_{p=0}^{\infty} \hat{v}_n(p)\cos pz \quad (\text{or } \sin pz) \end{aligned} \tag{15}$$

We must also expand b in powers of δ, and thus we set

$$b = b^e(\delta) = \sum_{n=0}^{\infty} b_n^e \delta^n$$

and

$$b = b^o(\delta) = \sum_{n=0}^{\infty} b_n^o \delta^n$$

for the cosine and sine expansions in Eq.(15), respectively. Obviously $b^e(0) = b^o(0) = b_0 = b_{m_0}$ where b_{m_0} can be either $b_{m_0}^+$ or $b_{m_0}^-$ as given in Eq. (10). In this way, one obtains the recursive system of algebraic equations

$$M[p]\widehat{U}_n(p) = b_n \widehat{U}_n^{\perp} + R_n(p) \tag{16}$$

where

$$M[p] = \begin{bmatrix} -(k^2p^2 + b_0) & 1 \\ -k^2p^2 & k^2p^2 + 1 - k^2 + b_0 \end{bmatrix}, \ \widehat{U}_0^{\perp} = (1, -((k\,m_0)^2/4 + b_0))$$

$$M[p] = \begin{bmatrix} -(k^2p^2 + 4\,b_0) & 4 \\ -k^2p^2 & k^2p^2 + 4\,(1 - k^2 + b_0) \end{bmatrix}, \ \widehat{U}_0^{\perp} = (4, -((k\,m_0)^2 + 4\,b_0))$$

if m_0 is even or odd, respectively. Here $\widehat{U}_n(p) = (\hat{u}_n(p), \hat{v}_n(p))$, $R_n(p)$, $p \geq 0$ depends on $\widehat{U}_j(p)$, $0 \leq j < n$, $0 \leq p \leq j$, and b_n stands for b_n^o or b_n^e.

Let $p_{\text{crit}} = m_0/2$ if m_0 is even and $p_{\text{crit}} = m_0$ if m_0 is odd. Then the matrix $M[p_{\text{crit}}]$ is singular. It can be proved by induction that if m_0 is even then $\hat{u}_n(p) = \hat{v}_n(p) = 0$ if $p > p_{\text{crit}} + n$ or $p < p_{\text{crit}} - n$. The Fourier coefficients of the even and odd solutions are the same if $n < p_{\text{crit}}$, and therefore

$$b^o(\delta) - b^e(\delta) - O(\delta^{p_0}) \tag{17}$$

When m_0 is odd the three preceeding inequalities become $p > p_{\text{crit}} + 2n$, $p < p_{\text{crit}} - 2n$ and $p < p_{\text{crit}}$ respectively, and the conclusion Eq. (17) remains the same. This is an important result since, as predicted by Floquet theory, the region between $b^o(\delta)$ and $b^e(\delta)$ in the b-δ plane is of instability in the Floquet sense.

Given the singular character of $M[p_{\text{crit}}]$, we must impose, in order to have a unique solution, an additional condition, which we choose as

$$\langle \widehat{U}_n(p_{\text{crit}}) ; \widehat{U}_0(p_{\text{crit}}) \rangle = 0 \qquad n > 0 \tag{18}$$

where

$$\widehat{U}_0(p_{\text{crit}}) = (1,\ k^2 m_0^2/4 + b_0)$$

and $\langle\ ;\ \rangle$ is the usual dot product in $\mathbb{R}^2$. Then, for each $n \geq 1$, we start with $p = p_{\text{crit}}$, choose b_n so that the right-hand side of Eq. (16) is orthogonal to the kernel of the adjoint matrix $M^*[p_{\text{crit}}]$, determine the solution that satisfies Eq. (18), and use this result to solve Eq. (16) for $1 \leq p \leq n$, $p \neq p_{\text{crit}}$. The results are shown in Figs. 1 and 2 and are discussed in Section IV.

IV. Discussion and Conclusions

In the present work we have analyzed the linear stability of the space-periodic solutions of the Joulin-Sivashinsky equations under small periodic perturbations. Two special cases were considered: namely, when the wavelength λ_p of the perturbation is an integer submultiple of the wavelength λ_f of the cellular flame, and when it is an integer submultiple of $2\lambda_f$.

In the former case, in which m_0 is even, we have seen that for fixed $k < 1$, all of the exponential factors $b_{m_0}(\delta)$ are negative for δ in intervals $0 \leq \delta < \delta_{m_0}$, where δ_{m_0} depends on k and m_0. Correspondingly, we have that the cellular flame is stable to these perturbations so long as the scaled heat-loss parameter ν remains in the range $-(9/4)(1-k^2)^2 \leq \nu < \nu_{m_0}$, where $-(9/4)(1-k^2)^2$ is the value of ν at the bifurcation point[3] and $\nu_{m_0} = \nu(\delta_{m_0})$. In particular, there exists a minimum value of δ, δ_4, determined by $b_4(\delta) = 0$, such that the cellular flame is stable to all perturbations with

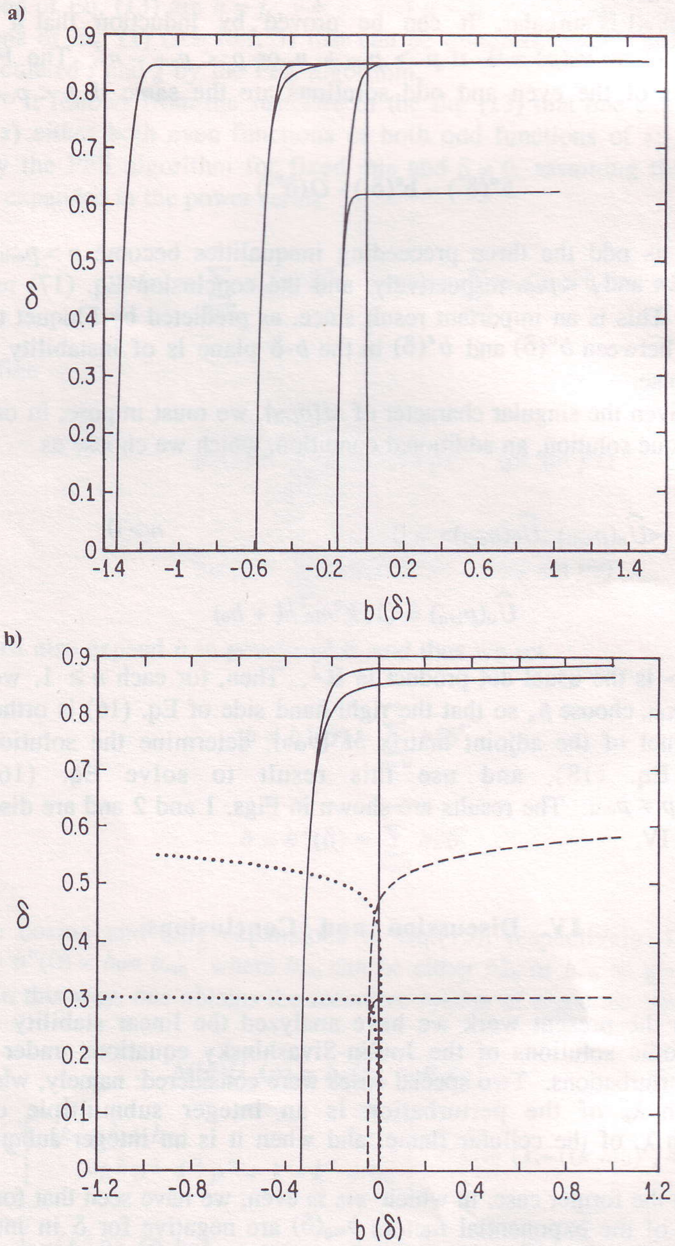

Fig. 1 Exponential factors $b_{m_0}(\delta)$ a) for $k = 0.4$ and $m_0 = 4, 6, 8$ and b) for $m_0 = 1, 3, 5$; the lower and upper branches for each m_0, correspond to cosine and sine expansions respectively.

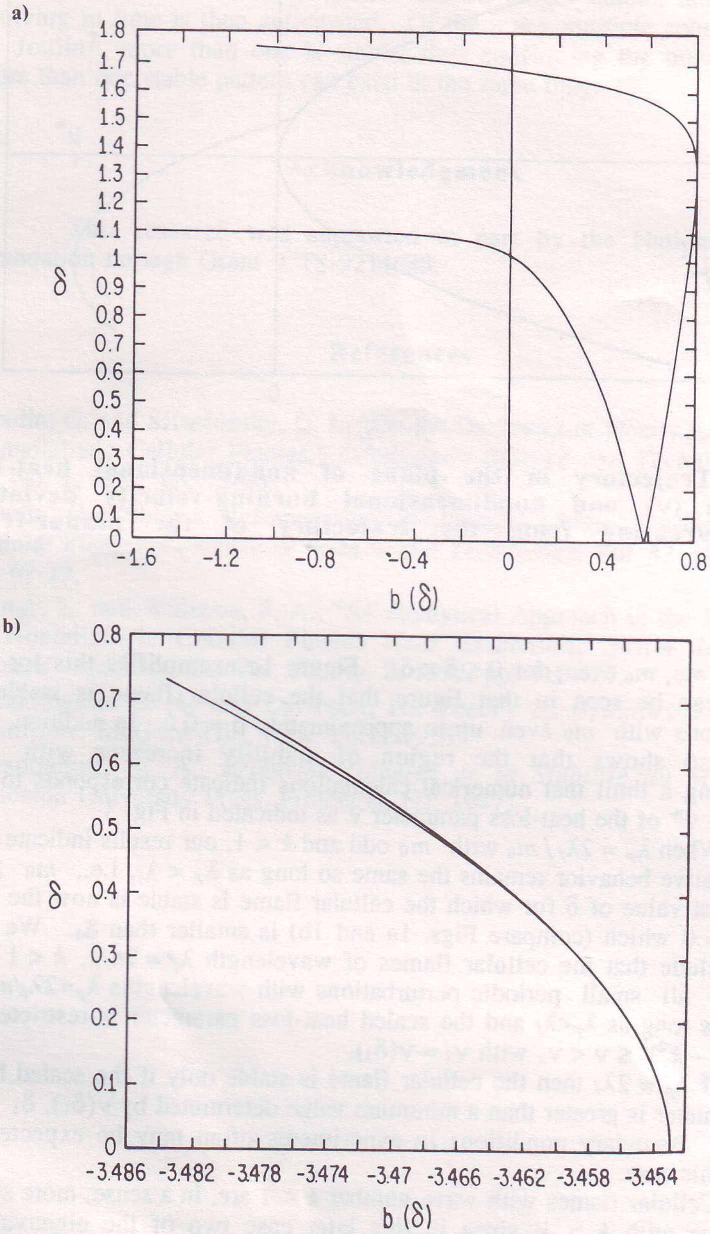

Fig. 2 Exponential factors for $k = 1.25$, a) $b_1(\delta)$, b) $b_4(\delta)$.

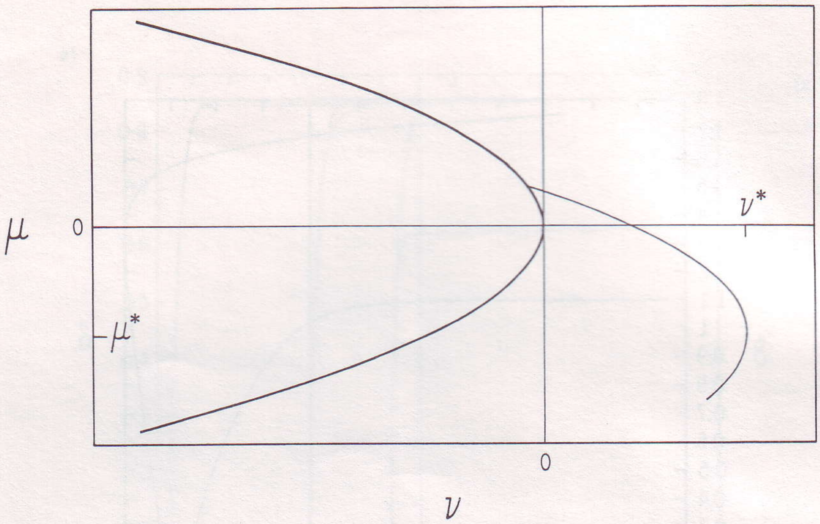

Fig. 3 Trajectory in the plane of nondimensional heat-loss deviation (ν) and nondimensional burning-velocity deviation (μ) bifurcating from the trajectory of the planar-front solutions.

$\lambda_p = 2\lambda_f / m_0$, m_0 even, for $0 \le \delta < \delta_4$. Figure 1a exemplifies this for $k = 0.4$. It can be seen in that figure that the cellular flame is stable to perturbations with m_0 even, up to approximately $\delta = 0.6$. In addition, that figure also shows that the region of stability increases with m_0, approaching a limit that numerical calculations indicate corresponds to the maximum ν^* of the heat-loss parameter ν as indicated in Fig. 3.

When $\lambda_p = 2\lambda_f / m_0$ with m_0 odd and $k < 1$, our results indicate that the qualitative behavior remains the same so long as $\lambda_p < \lambda_f$, i.e., $m_0 \ge 3$. The largest value of δ for which the cellular flame is stable is now the root of $b_3(\delta) = 0$ which (compare Figs. 1a and 1b) is smaller than δ_4. We can thus conclude that the cellular flames of wavelength $\lambda_f = 2\pi/k$, $k < 1$ are stable to all small periodic perturbations with wavelengths $\lambda_p = 2\lambda_f/m$, m integer, as long as $\lambda_p < \lambda_f$ and the scaled heat-loss parameter is restricted to $-(9/4)(1-k^2)^2 \le \nu < \nu_3$ with $\nu_3 = \nu(\delta_3)$.

If $\lambda_p = 2\lambda_f$ then the cellular flame is stable only if the scaled heat-loss parameter is greater than a minimum value determined by $\nu(\delta_1)$, δ_1 root of $b_1(\delta)$. Boundary conditions in experiments often may be expected to exclude this case.

Cellular flames with wave number $k < 1$ are, in a sense, more stable than those with $k > 1$, since in this later case two of the eigenvalues, b_0^+ and b_1^+, are positive for small δ, while all the other results remain the same as those for $k < 1$.

These results imply that the cellular patterns predicted for $\mu > 0$ and $\nu < \nu^*$ are indeed stable patterns and therefore should be observable experimentally for flames for which the assumptions of Joulin and

Sivashinsky[1] apply. As μ decreases along these bifurcated solutions, a point is reached at which these solutions are no longer stable, and a pattern evolving in time is then anticipated. Of the many multiple solutions found by Joulin[2], more than one is stable, thus confirming the possibility that more than one stable pattern can exist at the same time.

Acknowledgment

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References

[1]Joulin, G. and Sivashinsky, G. I., "On the Dynamics of Nearly Extinguished Nonadiabatic Cellular Flames," *Combustion Science and Technology*, Vol. 31, No. 1, 1983, pp. 75-90.

[2]Joulin, G., "Shapes, Velocities and Propagation Limits of a Nonadiabatic Cellular Flame," *Combustion Science and Technology*, Vol. 47, No. 1, 1986, pp. 69-79.

[3]Sinay, L. and Williams, F. A., "An Analytical Approach to the Description of Nonadiabatic Cellular Flames Near Extinction," *SIAM Journal on Applied Mathematics*, Vol. 52, No. 2, 1992, pp. 416-427.

[4]Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5]Liapounoff, M. A., *Problème Général de la Stabilité du Mouvement*, Princeton University Press, Princeton, NJ, 1947.