

wzrost - rozwój FLOWS IN CONICAL DUCTS WITH INTERNAL OBSTACLES

JERZY TADEUSZ SIELAWA

INPE - Cachoeira Paulista, SP/EFEI, Itajubá, MG, Brazil

MARIA C. IANINO

Universidade Federal, Viçosa, MG, Brazil

NELSON MANZANARES FILHO

EFEI - Escola Federal de Engenharia, Itajubá, MG, Brazil

VALDIR DE OLIVEIRA

EFEI - Escola Federal de Engenharia, Itajubá, MG, Brazil

A potential incompressible flow generated by a source located at the apex of a conical duct is considered. A distribution of source and sinks along the axis of the cone is used to produce a streamline, which represents the surface of an axisymmetric body lying in the cone axis. It results in streamlines of flow past an internal obstacle inside the duct.

INTRODUCTION

Consider a conical duct with a source point located at its apex. Assuming a potential, incompressible flow, the streamlines are constituted by straight lines, leaving the apex.

If there is a body situated within the cone, the flow field can be obtained by determining a distribution of sources and sinks along the axis, producing a streamline along the surface of the body.

In the present paper this problem is discussed assuming axial symmetry and using the conical coordinate system.

1. CONICAL COORDINATE SYSTEM (R, μ, ν)

As it is known [1], the relationship between the conical coordinates R, μ, ν and cartesian ones, x, y, z , is:

$$\begin{cases} \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - \nu^2} - \frac{z^2}{\nu^2 - \mu^2} = 0 \\ \frac{x^2}{\nu^2} + \frac{y^2}{\mu^2 - \nu^2} - \frac{z^2}{\nu^2 - \mu^2} = 0 \end{cases} \quad (1)$$

where

$$0 \leq \nu \leq a \leq \mu \leq b, \quad 0 \leq R \leq \infty \quad (1')$$

If the configuration is axially symmetrical (about the z axis), then $a = 0$ (and, thus, $\nu = 0$, $0 \leq \mu \leq b$). Then one can write:

$$\begin{cases} \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2} - \frac{z^2}{\nu^2 - \mu^2} = 0 \\ x^2 + y^2 + z^2 = R^2 \end{cases}$$

The metric matrix of conical coordinates is:

$$\{g_{ij}\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{R^2(\mu^2 - \nu^2)}{(\mu^2 - a^2)(b^2 - \mu^2)} & 0 \\ 0 & 0 & \frac{R^2(\mu^2 - \nu^2)}{(a^2 - \nu^2)(b^2 - \nu^2)} \end{pmatrix}$$

The Jacobian of the transformation is

$$J = \frac{(\mu^2 - \nu^2) R^2}{\sqrt{(\mu^2 - a^2)(b^2 - \mu^2)(a^2 - \nu^2)(b^2 - \nu^2)}}$$

The Laplace equation in conical coordinates is:

$$\begin{aligned} (\mu^2 - \nu^2) \left\{ R^2 \frac{\partial^2 V}{\partial R^2} + 2R \frac{\partial V}{\partial R} \right\} + (\mu^2 - a^2) (b^2 - \mu^2) \frac{\partial^2 V}{\partial \mu^2} + \mu(b^2 + a^2 - 2\mu^2) \frac{\partial V}{\partial \mu} \\ + (\nu^2 - a^2) (\nu^2 - b^2) \frac{\partial^2 V}{\partial \nu^2} + \nu(2\nu^2 - a^2 - b^2) \frac{\partial V}{\partial \nu} = 0 \end{aligned}$$

In the axially symmetrical case ($a = 0$, $\nu = 0$), one gets

$$\mu^2 \left\{ R^2 \frac{\partial^2 V}{\partial R^2} + 2R \frac{\partial V}{\partial R} \right\} + \mu^2 (b^2 - \mu^2) \frac{\partial^2 V}{\partial \mu^2} + \mu(b^2 - 2\mu^2) \frac{\partial V}{\partial \mu} = 0$$

The solution can be obtained by separation of variables:

$$V(R, \mu) = S(R) C(\mu)$$

where

$$S(R) = R^{-1/2} e^{\pm k(\ln R)} \quad ; \quad k = -i(n + \frac{1}{2})$$

$$C_k(\cos \theta) = P_{-\frac{1}{2}}^{+k}(\cos \theta) = P_n(\cos \theta)$$

where P_n are the Legendre polynomials and C_k are conical functions. As it is known, [2], one can obtain:

$$C_k(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^{\theta} \frac{\cosh(kz)}{\sqrt{\cos^2 z - \cos^2 \theta}} dz =$$

$$= \frac{\sqrt{2}}{\pi} \cosh(k\pi) \int_0^\infty \frac{\cos(kz)}{\sqrt{\cosh^2 z + \cos^2 \theta}} dz \quad (5)$$

2. VELOCITY POTENTIAL

The distance d between two arbitrary points, P_1 and P_2 , can be obtained from Fig. 1.

It can be easily seen that:

$$d^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\psi \quad (6)$$

where

$$\cos\psi = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\omega \quad (7)$$

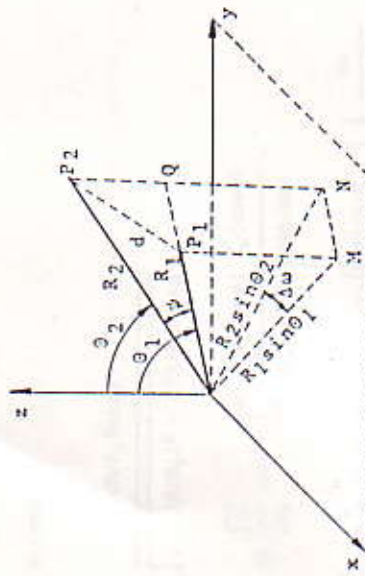


Fig. 1 - Distance d between points P_1 and P_2

Applying Fourier cosine transform, one can get, after some algebra:

$$d^{-1} = \frac{1}{\sqrt{R_1 R_2}} \int_0^\infty \frac{C_k(-\cos\psi)}{\cosh(k\pi)} \cos\left[k \ln \left(\frac{R_1}{R_2} \right)\right] dk \quad (8)$$

In order to get the velocity potential, one can note that the source located on the z axis, in a free space, produces a potential which satisfies the Laplace Equation:

$$\phi_0 = -\frac{I}{d} = \frac{-I}{\sqrt{R_2^2 - 2R_1R_2\cos\theta + R_1^2}} \quad (9)$$

where I is the source intensity.

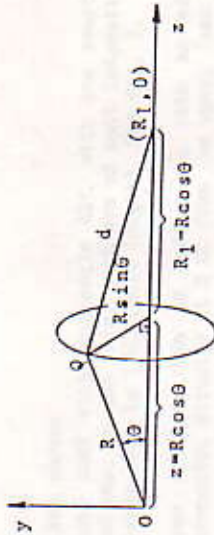


Fig. 2 - Distance between a point $z(R, 0)$ and an arbitrary point Q

To get the potential of a source situated on the axis within the circular cone (i.e. with $\theta = \alpha$), one has to consider that it should satisfy the Neumann condition:

$$\left. \frac{\partial \phi}{\partial n} \right|_{\text{wall}} = 0 \quad (10)$$

or

$$\left. \frac{\partial \phi}{\partial \theta} \right|_{\theta=\alpha} = 0, \quad (10')$$

as well as the Laplace equation.

Assuming that the new potential is composed of two parts

$$\phi = \phi_0 + \phi_c$$

after some calculations, one can get, [3]:

$$\phi_c = -\frac{I}{\sqrt{R R_1}} \int_0^\infty \frac{C'_k(-\cos\alpha) C_k(\cos\theta)}{C'_k(\cos\alpha) \cosh(\pi k)} \cos\left[k \ln \left(\frac{R}{R_1} \right)\right] dk \quad (11)$$

where it is understood that

$$C'_k(\cos\alpha) = \frac{dC_k(\cos\alpha)}{d(\cos\alpha)} \quad (12a)$$

$$C'_k(-\cos\alpha) = \frac{dC_k(-\cos\alpha)}{d(-\cos\alpha)} \quad (12b)$$

It can be easily shown that $\phi = \phi_0 + \phi_c$ satisfies both the Laplace equation and the Neumann condition and thus, that it constitutes the proper potential.

Finally, if one puts a source at the cone apex, its potential is

$$\phi_\infty = -\frac{I}{R} \quad (13)$$

The total potential of the flow field within the cone, due to the source at the apex and sources along the cone axis is, thus:

$$\phi(R, \theta) = \phi_{\infty} + \phi_0 + \phi_c \quad (14)$$

where ϕ_{∞} , ϕ_0 and ϕ_c are given by equations (13), (9) and (11), respectively.

3. STREAM FUNCTION

In order to be able to adapt the method to generate arbitrary shapes of obstacles, one will switch to the stream function ψ (instead of the potential ϕ).

Using well known relations between ϕ and ψ , one gets, after some algebra:

$$\psi(R, \theta) = \psi_{\infty} + \psi_0 + \psi_c \quad (15)$$

where

$$\psi_{\infty} = A \cos \theta \quad (16a)$$

$$\psi_0 = \frac{1}{\sqrt{1 + (R/R_1)^2 - 2(R/R_1)\cos\theta}} \quad (16b)$$

$$\psi_c = 4I \left[\frac{R}{R_1} \sin^2 \theta \int_0^{\infty} \frac{C_k'(-\cos \alpha) C_k'(\cos \theta)}{C_k'(\cos \alpha) \cosh(\pi k)} \cdot \left\{ \frac{1}{2} \cos \left[k \ln(R/R_1) \right] + k \sin \left[k \ln(R/R_1) \right] \right\} \frac{dk}{1 + 4k^2} \right] \quad (16c)$$

Expressing ψ in cylindrical coordinates ($R^2 = r^2 + z^2$, $\cos \theta = z/R$), one gets

$$\psi_{\infty} = \frac{Az}{\sqrt{r^2 + z^2}} \quad (17a)$$

$$\psi_0 = \frac{1}{\sqrt{r^2 + (z - z_1)^2}} \quad (17b)$$

$$\psi_c = \frac{4I}{\left[r^2 + z^2 \right]^{3/4} \sqrt{z_1}} \int_0^{\infty} \frac{C_k'(-\cos \alpha) C_k'[z(r^2 + z^2)^{-1/2}]}{C_k'(\cos \alpha) \cosh(\pi k)} \cdot \left\{ \frac{1}{2} \cos \left[k \ln \frac{\sqrt{r^2 + z^2}}{z_1} \right] + k \sin \left[k \ln \frac{\sqrt{r^2 + z^2}}{z_1} \right] \right\} \frac{dk}{1 + 4k^2} \quad (17c)$$

4. EXAMPLES

Two examples are shown:

(1) A conical duct with semi-angle 45° , with one source located at the apex and a second one, on the axis; both are of unit intensity ($A = I = 1$) and the distance between them is 1 UL (unit length). See Fig. 3;

(2) The same configuration as before but with an additional sink of negative unit intensity, situated at 2 UL from the apex, (see Fig. 4).

Observe that any streamline in both examples can be substituted by a solid surface situated within the cone. But, of course, the most interesting surface is the one which has a stagnation point in front of it, i.e. where

$$\frac{\partial \phi}{\partial z} = 0$$

In this way, the example (1) produces an axially symmetrical body which extends itself to infinity (on one side), while the example (2) produces a closed body (since the sink and the source have the same intensity). As an additional bonus one gets a flow within these bodies (where they hollow), due to the distribution of sources and sinks.

5. CONTINUOUS DISTRIBUTION OF SOURCES AND SINKS

If the sources and sinks are distributed in a continuous manner between, say, s_1 and s_2 , we get

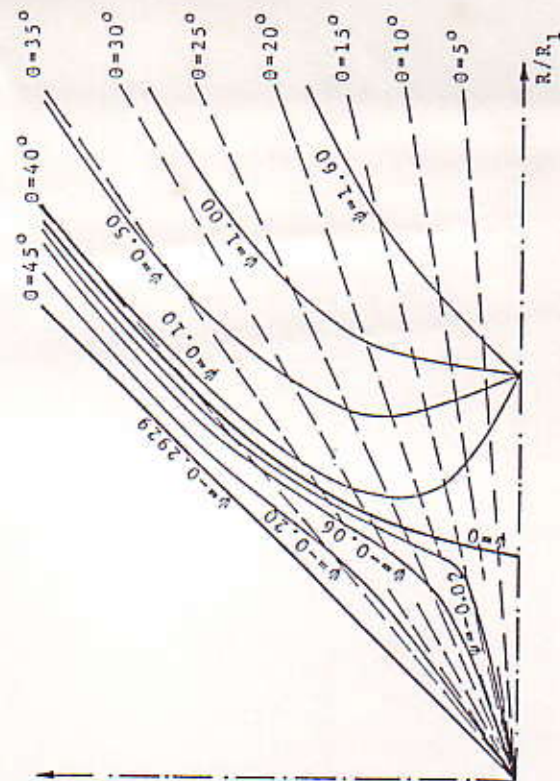


Fig. 3 The flow in the cone of example 1.

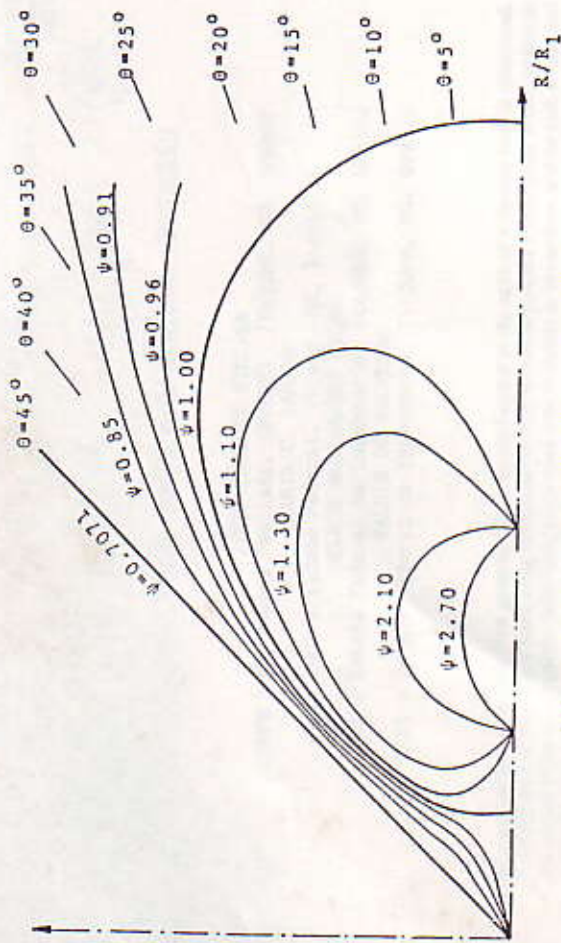


Fig. 4 The flow in the cone of example 2.

$$\psi_{\infty} = \frac{Az}{\sqrt{r^2+z^2}} \quad (18a)$$

$$\psi_0 = \int_{S_1}^{S_2} \frac{I(z_1) (z-z_1)}{\sqrt{r^2+(z-z_1)^2}} dz_1 \quad (18b)$$

$$\psi_c = \frac{4r^2}{(r^2+z^2)^{3/4}} \int_0^{\pi} \frac{dk}{1+4k^2} \frac{C'_k(-\cos\alpha) C'_k[z(r^2+z^2)^{-1/2}]}{C'_k(\cos\alpha) \cosh(\pi k)} \quad (18c)$$

$$\cdot \int_{S_1}^{S_2} \frac{I(z_1)}{\sqrt{z_1}} \left\{ \frac{1}{2} \cos[k \ln \frac{\sqrt{r^2+z^2}}{z_1}] + k \sin[k \ln \frac{\sqrt{r^2+z^2}}{z_1}] \right\} dz_1$$

where $\psi = \text{const}$ are the flow streamlines.

Assuming that within the cone duct we have a solid surface

$$r = f(z), \quad z_a \leq z \leq z_b$$

and that the streamline coinciding with this surface is

$$\psi = A,$$

one gets

$$F(z) + \int_{S_1}^{S_2} M(z, z_1) I(z_1) dz_1 = 0 \quad (19)$$

where

$$F(z) = \frac{Az}{\sqrt{r^2(z)+z^2}} - A \quad (20)$$

and where $M(z, z_1)$ can be easily obtained from the above expressions. Eq. (19) is the Fredholm integral equation of the first kind. Determining $I(z_1)$ we get the distribution of sources and sinks which produces the conical flow inside the conical duct, along the given obstacle.

REFERENCES

- [1] P.M. Morse & H. Feshbach - Methods of Theoretical Physics - Mc Graw-Hill & Kogakusha, 1953
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- [3] M.C. Ianino, M.Fortes & J.T. Sielawa - Potential of a source within a conical duct, in the presence of a radial flow (in Portuguese) - Revista Brasileira de Ciências Mecânicas vol.4, 1982, 3-10.