AN APPLICATION OF THE THIN AIRFOIL THEORY TO POTENTIAL FLOW ANALYSIS IN CENTRIFUGAL IMPELLERS

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The classical thin airfoil theory is extended to the calculation of the potential, incompressible and steady flow in centrifugal impellers with infinitely thin and constant width blades. The impeller physical plane is mapped on a linear cascade plane by means of the König conformal transformation, and the cascade effect is reproduced with vortex distributions on the blades chord in terms of a Glauert series, by satisfying the flow tangency condition. Satisfactory results are obtained with only three terms in the series for the case of logarithmic spiral blades.

INTRODUCTION

The analysis of the potential flow around aerodynamic airfoils, isolated or arranged in rectilinear cascades, can be made by applying the thin airfoil theory described in detail by Sholz (1965). In the case of centrifugal impellers, it is common to employ preliminarly the König conformal transformation by mapping the impeller radial cascade (physical plane) in a rectilinear cascade (transformed plane). If the transformed blades result relatively thin and slightly cambered, it is possible to simulate their displacement flow effect by means of a vortex distribution on the chord line (logarithmic line in the physical plane). One of the first works in this area is due to Hoffmeister (1960) who applied the rectilinear cascade theory by Schlichting (1955) to radial cascades. In this theory, the flow tangency condition is imposed only in some discret points on the chord, but this restriction can be removed by an integration process, as described by Mellor (1959). He also developed a more general and flexible calculation technic for the aerodynamic characteristics of airfoil families. For centrifugal impellers, a similar formulation was presented by Murata and Ogawa (1978) however, without the generality of the Mellor procedure.

In the present work, an extension of the classical thin airfoil theory by Glauert (1948) to the potential, incompressible flow calculation in centrifugal impellers with infinitely thin and constant width blades is presented. The analysis has a semianalytical content and is similar to the rectilinear cascade analysis effectuated by Mellor (1959) and Castro (1981). Some results obtained for impellers with logarithmic blades are presented in comparison with corresponding results obtained by means of the panel method.

GENERAL FORMULATION

In Fig. 1, a scheme is shown of a centrifugal impeller with internal and external radii r_1 and r_2, respectively, composed of N infinitely thin blades rotating clockwise with angular velocity ω in the physical complex plane $z = r\ exp(i\theta)$. The logarithmic spiral line joining the leading and the trailing edges of the blade forms an angle β with the radial direction. The

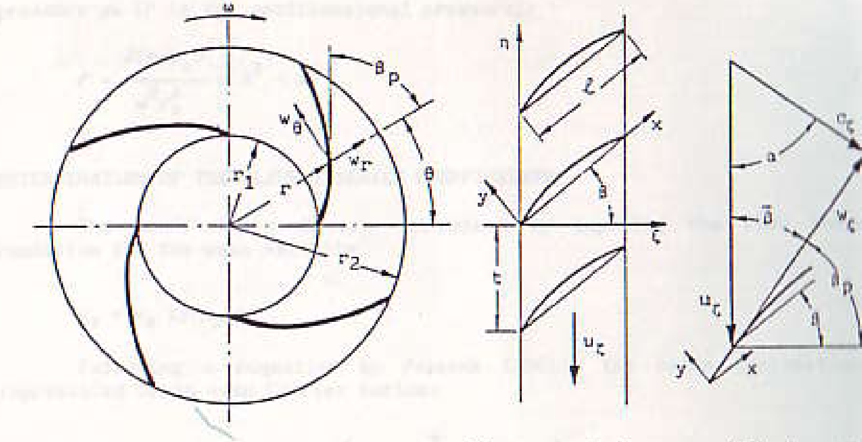

Figure 1 Centrifugal impeller scheme: (a) physical plane z; (b) transfo
plane ζ; (c) flow tangency condition.

radial cascade is mapped in a rectilinear cascade with blade spacing t, ℓ and setting angle β on the transformed complex plane $\zeta = \xi + i\eta$, by mea the König (1922) conformal transformation:

$$\zeta(z) = \frac{Nt}{2\pi}\ log\left(\frac{z}{r_1}\right) \quad , \qquad \ell cos\beta = \zeta(r_2) = \frac{Nt}{2\pi}\ ln\left(\frac{r_2}{r_1}\right) \qquad (1$$

The relative velocity components of the transformed flow in the $\zeta^* = \zeta\ exp(-i\beta) = x+iy$, with a reference blade chord on the x axis, are

$$w_x = \frac{Q_0}{Nt}\ cos\beta - \left(\frac{\Gamma_0}{Nt} + \frac{\Gamma_p}{2t} - u_\zeta\right) sin\beta + c_{sx}$$

$$w_y = -\frac{Q_0}{Nt}\ sin\beta - \left(\frac{\Gamma_0}{Nt} + \frac{\Gamma_p}{2t} - u_\zeta\right) cos\beta + c_{sy}\ ;$$

Q_0 and Γ_0 represent the intensities of a source and a clockwise vortex z origin and correspond to the flow rate and the internal circulation; the displacement circulation of one blade; the transformed tangential ve u_ζ is

$$u_\zeta = \frac{2\pi\omega r^2}{Nt} = \frac{2\pi\omega r_2^2}{Nt}\left(\frac{r_1}{r_2}\right)^{2(1-x/\ell)}\ ;$$

c_{sx} and c_{sy} are the displacement absolute velocities induced by the ca They will be simulated by continuous vortex distributions of density γ the chords, in the form of a trigonometric Glauert (1948) series:

$$\gamma = \frac{Q_0}{Nt} cos\beta \sum_{n=0}^{\infty} A_n\gamma_n\ ,$$

where $\gamma_0 = 2(1+cos\phi)/sin\phi$, $\gamma_n = 4sin\phi$ for $n \geq 1$, and $cos\phi = 1-2x/\ell$. The determi of the coefficients A_n will be discussed in the next section.

Following the formulation presented by Castro (1981) and Fernandes (1985), valid for cascades with thin and slightly cambered airfoils, the induced velocities calculated on the chord are:

$$c_{sx}^{\pm} = -\frac{Q_0}{Nt}\cos\beta \sum_{n=0}^{\infty} h_n A_n \pm \frac{\gamma}{2}, \tag{5.a}$$

$$c_{sy} = -\frac{Q_0}{Nt}\cos\beta \sum_{n=0}^{\infty} g_n A_n , \tag{5.b}$$

where the signal (+) indicates the suction surface and (-) the pressure surface of the blade and

$$g_0 = 1 + \frac{\lambda}{2}\int_0^{\pi} R(1+\cos\phi')d\phi' , \tag{6.a}$$

$$g_n = \frac{\lambda}{2}\int_0^{\pi} 2R\sin\phi'\sin n\phi'\,d\phi' - 2\cos n\phi , \quad n\geq 1 , \tag{6.b}$$

$$h_0 = \frac{\lambda}{2}\int_0^{\pi} I(1+\cos\phi')d\phi', \tag{6.c}$$

$$h_n = \frac{\lambda}{2}\int_0^{\pi} 2I\sin\phi'\sin n\phi'\,d\phi' , \quad n\geq 1 , \tag{6.d}$$

Here, $\lambda = \ell/t$ is the cascade solidity ratio; R and I represent the real and imaginary parts, respectively, of the cascade interference function, Z:

$$Z = e^{i\beta}\coth\left[\pi\frac{(x-x')}{t}e^{i\beta}\right] - \frac{t}{\pi(x-x')} . \tag{7}$$

The relative velocity on the blade surface in the physical plane is denoted by w and is obtained in the nondimensional form $W=w/(\omega r_2)$, with $R=r/r_2$:

$$W^{\pm} = \frac{\sec(\beta_p-\beta)}{R}\left\{\Phi\cos\beta\left[1+\sum_{n=0}^{\infty}\left(\pm\frac{\gamma_n}{2}-h_n\right)A_n\right]-\left(\frac{\Gamma_0}{Q_0}\Phi+\frac{\psi}{4}-R^2\right)\sin\beta\right\} , \tag{8}$$

where ϕ and ψ represent the flow and pressure coefficients, respectively:

$$\Phi = \frac{Q_0}{2\pi\omega r_2^2} , \qquad \psi = \frac{N\Gamma_p}{\pi\omega r_2^2} . \tag{9.a,b}$$

The displacement blade circulation is related to the impeller specific work Y_p by the formula $Y_p=\omega N\Gamma_p/2\pi$, and is invariant in the conformal transformation. The determination of Y_p is made by the integration of the circulation elements γdx on the chord. By Eq. (9.b), the result is

$$\psi = N\,\Phi\,(A_0+A_1)\,\ln\left(\frac{r_2}{r_1}\right) . \tag{10}$$

The static pressure distribution p is calculated by applying the Bernoulli equation with centrifugal effect and by adopting a reference

pressure p_0 (P is the nondimensional pressure):

$$P = \frac{2(p-p_0)}{\omega^2 r_2^2} = R^2 - W^2 . \tag{11}$$

DETERMINATION OF THE GLAUERT SERIES COEFFICIENTS

The coefficients A_n are determined by imposing the flow tangency condition for the mean velocity

$$w_y = w_x\,tg(\beta_p-\beta) . \tag{12}$$

Following a sugestion by Polasek (1961), the blade inclination is represented by an even Fourier series:

$$tg(\beta_p-\beta) = C_b F_c(\phi) = C_b\left(C_0 + 2\sum_{m=1}^{\infty} C_m\cos m\phi\right) , \tag{13}$$

where $F_c(\phi)$ represents the inclination of the basic blade of a given family of blades in the transformed plane and C_b is a camber factor, and

$$C_m = \frac{1}{\pi}\int_0^{\pi} F_c(\phi)\cos m\phi\,d\phi . \tag{14}$$

Eqs. (14), (2), (3), (5) and (6) are substituted in Eq. (12), without the term $\pm\gamma/2$ in (5.a). According to Mellor (1959), the partial result is multiplied by $(\cos k\phi)/\pi$ ($k = 0,1,2,\ldots$) and the new result is integrated in the interval $0 \leq \phi \leq \pi$. The final result can be writen as the following system of linear algebraic equations for the coefficients A_n

$$\sum_{n=0}^{\infty} a_{nk}A_n = -\left(tg\beta+\frac{\Gamma_0}{Q_0}\right)\delta_{ck} - \left(1-\frac{\Gamma_0}{Q_0}tg\beta\right)C_b C_k + \frac{\Omega_k}{\Phi} , \tag{15}$$

where

$$a_{nk} = g_{nk} - C_b\left[C_0 h_{nk} + \sum_{m=1}^{\infty} C_m\left(h_{n|m-k|}+h_{n|m+k|}\right)\right.$$
$$\left. + \left(\delta_{n0}+\delta_{n1}\right)\left(\delta_{nk}-C_b C_k\,tg\beta\right)\frac{N}{4}\ln\left(\frac{r_2}{r_1}\right)\right] , \tag{16}$$

$$\Omega_k = f_k - C_b\left[C_0 f_k + \sum_{m=1}^{\infty} C_m\left(f_{|m-k|}+f_{|m+k|}\right)\right] , \tag{17}$$

and

$$g_{0k} = \delta_{0k} + \frac{\lambda}{2\pi}\int_0^{\pi}\int_0^{\pi} R(1+\cos\phi')\cos k\phi\,d\phi'd\phi , \tag{18.a}$$

$$g_{nk} = \delta_{nk} + \frac{\lambda}{2\pi}\int_0^{\pi}\int_0^{\pi} 2R\sin\phi'\sin n\phi'\cos k\phi\,d\phi'd\phi , \quad n\geq 1, \tag{18.b}$$

$$h_{0k} = \frac{\lambda}{2\pi} \int_0^\pi \int_0^\pi I(1 + \cos \phi') \cos k\phi \, d\phi' d\phi \ , \tag{18.c}$$

$$h_{nk} = \frac{\lambda}{2\pi} \int_0^\pi \int_0^\pi 2I \sin \phi' \sin n\phi' \cos k\phi \, d\phi' d\phi \ , \tag{18.d}$$

$$f_k = \frac{1}{\pi} \int_0^\pi \left(\frac{r_1}{r_2}\right)^{1+\cos\phi} \cos k\phi \, d\phi \ , \tag{18.e}$$

$$\delta_{nk} = \begin{cases} 1 \text{ for } n = k \\ 0 \text{ for } n \neq k \end{cases} \tag{18.f}$$

The f_k values represent the centrifugal effect of the impeller; they are easily determined by numerical quadrature. The other values, g_{nk} and h_{nk}, were tabulated by Castro (1981) for ranges $0 \le \beta \le \pi/3$, $0 \le \lambda \le 1, 5$, $n \le 4$, $k \le 4$.

It is possible to isolate the flow coefficient effect, by decomposing the coefficients A_n in the following manner:

$$A_n = A_{n\delta}\left(tg\beta + \frac{\Gamma_0}{Q_0}\right) + A_{nc}\left(1 - \frac{\Gamma_0}{Q_0} tg\beta\right) C_b + \frac{A_{n\phi}}{\phi} \ . \tag{19}$$

Thus, the system of equations (15) is decomposed in the following way:

$$\sum_{n=0}^{\infty} a_{nk} \, A_{n\delta} = -\delta_{0k} \ , \tag{20.a}$$

$$\sum_{n=0}^{\infty} a_{nk} \, A_{nc} = -C_k \ , \tag{20.b}$$

$$\sum_{n=0}^{\infty} a_{nk} \, A_{n\phi} = \Omega_k \ . \tag{20.c}$$

In this manner, the determination of the smooth flow entry condition is straightforward, by imposing $A_0 = 0$ in Eq. (19). In this condition one obtains a finite velocity on the blade leading edge; this would be impossible if $A_0 \neq 0$.

EXAMPLE

With logarithmic spiral blades, the calculations are substantially simplified because $\beta_p = \beta$ and thus $F_c(\phi) = 0$, $C_a = 0$; $A_{nc} = 0$. In this case, the blade in the transformed plane coincides with the chord and one would expect that the obtained results approach the exact potential flow solution as the number of terms in the Glauert series increases.

Some cases of potential flow in centrifugal impellers with logarithmic spiral blades, smooth flow entry and $\Gamma_0 = 0$ were computed with the method described in this work, by employing 3,4 and 5 terms in the Glauert series. The results are shown in Fig.2, Tables 1 and 2, in comparison with corresponding results obtained by a panel method developed by Manzanares (1982).

In Table 1, one observes that the results obtained with only 3 terms for the coefficients ϕ and ψ are at least as good as the results obtained with 40 panels. The results for the pressure distributions are also reasonable as one can see in Fig. 2. Numerical pressure values, shown in Table 2 for one of the cases, indicate that the solution quality with 3 terms is slightly

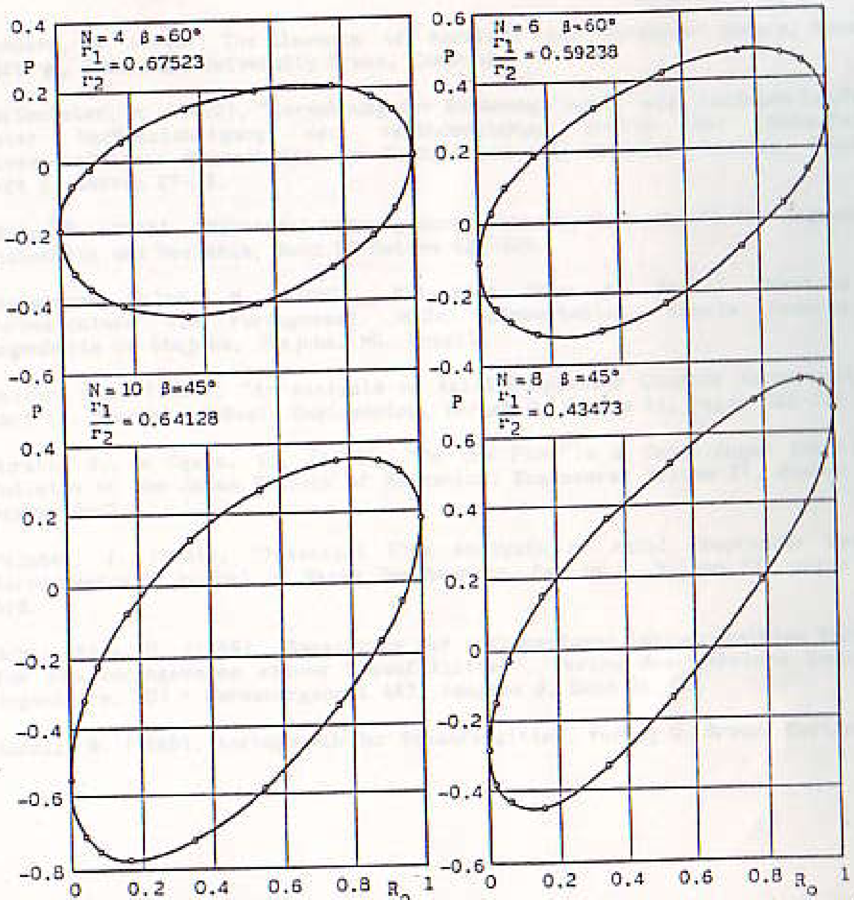

Figure 2 Pressure distribution $P \times R_0$ for centrifugal impellers with logarithmic blades: ∘ ∘ present method (3 terms); —— panel method (40 panels). $R_0 = (r - r_1)/(r_2 - r_1)$.

Table 1 Present method x panel method - results for flow (ϕ) and pressure (ψ) coefficients at smooth flow entry conditions.

Logarithmic Blades		Present Method			Panel Method	
		3 terms	4 terms	5 terms	20 panels	40 panels
N=4 β=60°	ϕ	0.3725	0.3725	0.3725	0.3731	0.3726
r1/r2=0.67523	ψ	0.2228	0.2228	0.2229	0.2188	0.2221
N=10 β=45°	ϕ	0.5250	0.5250	0.5250	0.5272	0.5254
r1/r2=0.64128	ψ	0.6002	0.6002	0.6002	0.5910	0.5984
N=6 β=60°	ϕ	0.2830	0.2830	0.2830	0.2844	0.2832
r1/r2=0.59238	ψ	0.5672	0.5676	0.5676	0.5575	0.5656
N=8 β=45°	ϕ	0.2652	0.2654	0.2654	0.2684	0.2660
r1/r2=0.43473	ψ	1.0444	1.0437	1.0437	1.0299	1.0409

inferior to that with 40 panels. With 4 terms, however, a better solution is obtained, up to 4 significative digits for the majority of the points, as one concludes in a comparison with the 5 term solution.

Table 2 Present method x panel method - pressure distribution results for the case N=8, β=45°, r_1/r_2=0.43473. (+) suction side; (-) presure side.

$R_0=\frac{r-r_1}{r_2-r_1}$	P	Present Method			40 Panels
		3 terms	4 terms	5 terms	
0.060048	(+)	-0.4191	-0.4384	-0.4352	-0.4366
	(-)	-0.0265	-0.0155	-0.0153	-0.0178
0.154513	(+)	-0.4391	-0.4417	-0.4413	-0.4439
	(-)	0.1476	0.1492	0.1490	0.1463
0.343567	(+)	-0.3313	-0.3182	-0.3180	-0.3253
	(-)	0.3574	0.3541	0.3541	0.3523
0.534269	(+)	-0.1377	0.1305	0.1308	0.1384
	(-)	0.5167	0.5148	0.5149	0.5130
0.784945	(+)	0.1829	0.1758	0.1757	0.1738
	(-)	0.7013	0.7028	0.7028	0.7009
0.909467	(+)	0.3908	0.3817	0.3820	0.3807
	(-)	0.7564	0.7597	0.7596	0.7580

CONCLUSIONS

An extension of the thin airfoil theory to the potential flow analysis in centrifugal impellers was presented.

With moderate computational effort, it was possible to obtain satisfactory results for impellers with logarithmic spiral blades. Aparently, the theory can also be applied in other situations without major difficulties. For this, however, it is necessary to proceed with more detailed analysis, because the thin aifoil theory constitutes an approximation for the exact potential flow model. One expects the developed technic will be satisfactory only for slightly cambered blades in the transformed plane. However, the blades can be highly cambered in the physical plane.

It is important to note that a major part of the calculation time is expended in the computation of the expressions h_{nk}, g_{nk} and f_k. It is possible to determine them in advance for a required range of geometric parameters of the impeller, in the case of selected families of blade geometry. This is very convenient when a rapid and systematic comparative evaluation is required for a great number of concurrent impeller geometries, specially when simple calculators or microcomputers are available.

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