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AN OSCILLATION THEOREM FOR A HILL'S TYPE EQUATION

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Let us consider the differential equation

$$u'' + (\lambda+\varepsilon Q(\lambda,\varepsilon,s))u = 0 \qquad (u'=du/ds) \qquad (1)$$

with $Q(\lambda,\varepsilon,s)$ a real periodic function of period π.

If u_1, u_2 is a pair of linearly independent solutions of (1) then every other solution can be written as a linear combination of them. In particular, those obtained from u_j, $j=1,2$ by shifting the variable s in one period, that is, there exist numbers a_{ij}, $i,j=1,2$ such that

$$u_j(s+\pi) = a_{j1}u_1(s) + a_{j2}u_2(s)$$

A solution $u(s)$ of (1) is called normal if, for some σ, it has the following property:

$$u(s+\pi) = \sigma\, u(s)$$

If (1) posseses a normal solution then the fact that there is no term containing u' in the equation implies that the associated σ must satisfy

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$$\sigma^2 - \Delta\sigma + 1 = 0 \tag{2}$$

where

$$\Delta = a_{11} + a_{22}$$

A necessary condition for a normal solution u(s) of (1) to be stable (bounded) is that $|\sigma|\leq 1$, and in view of (2) both solutions are stable only if the roots σ_1, σ_2 of the characteristic equation (2) satisfy the relation

$$|\sigma_1| = |\sigma_2| = 1$$

which in turn implies that:

i) if $|\Delta| > 2$ then the solutions are unstable

ii) if $|\Delta| < 2$ the solutions are stable.

Being Δ real the transition from stability to instability occurs when $|\Delta|=2$, for which $\sigma_{1,2}=\pm 1$. It then follows that if λ and ε in (1) are such that a normal solution exists and $|\Delta|=2$ then there must exist at least one periodic solution of the differential equation. In this latter case the corresponding pair (λ,ε) is called a transition value.

The next theorem shows that under suitable conditions there exist two sequences $\lambda_n^+(\varepsilon)$ and $\lambda_n^-(\varepsilon)$ of transition values for which (1) posseses periodic solutions. Haupt [2], [3] proved, imposing less conditions, a more general result than the one presented here, however, this seems to be a simpler and straightforward proof.

Theorem: if for each positive integer n the real valued function $Q(\lambda,\varepsilon,s)$ in (1) is continuously differentiable with respect to

(λ,ε) in open neighborhoods of $(n^2,0)$, continuous, even and periodic with period π with respect to s and if $Q(\lambda,\varepsilon,s)$ and $\partial Q(\lambda,\varepsilon,s)/\partial\lambda$ converge uniformily in s to $Q(n^2,0,s)$, $\partial Q(n^2,0,s)/\partial\lambda$ when $(\lambda,\varepsilon)\to(n^2,0)$, then for each n there exists an open interval $I=(-n,n)\subset\mathbb{R}$ and functions $u_n^+,u_n^-: I\times\mathbb{R}\to\mathbb{R}$, $\lambda_n^+,\lambda_n^-: I\to\mathbb{R}$, continuously differentiable with respect to ε and such that for each $\varepsilon\in I$, u_n^+, u_n^- are periodic solutions of period 2π of (1) when $\lambda=\lambda_n^+(\varepsilon)$ and $\lambda=\lambda_n^-(\varepsilon)$ respectively. Furthermore, $\lambda_n^+(0)=\lambda_n^-(0)$ and u_n^+, u_n^- are respectively even and odd functions of s satisfying

$$u_n^+(0,s) = \cos ns \quad , \quad u_n^-(0,s) = \sin ns$$

Proof: The proof will be carried out only for the pair u_n^+, λ_n^+. The other case follows doing obvious modifications.

Let $C[0,2\pi]$ and $C^2[0,2\pi]$ be the Banach spaces of continuous and twice continuously differentiable functions $u: [0,2\pi]\to\mathbb{R}$ equipped with the norms

$$\|u\|_0 = \max_{s\in[0,2\pi]} |u(s)| \tag{3a}$$

$$\|u\|_2 = \max_{s\in[0,2\pi]} \{|u(s)|+|u'(s)|+|u''(s)|\} \tag{3b}$$

and let

$$F: C^2[0,2\pi]\times\mathbb{R}\times(-a,a)\to C[0,2\pi] \tag{4}$$

be defined by

$$F[u,\lambda,\varepsilon] = u'' + (\lambda+\varepsilon Q(\lambda,\varepsilon,s))u \quad .$$

We assume that $(-a,a) \subset \mathbb{R}$ in (4) is some interval contained in the domain of definition of Q with respect to the variable ε.

Thus we seek solutions of the equation

$$F[u,\lambda,\varepsilon] = 0 \tag{5}$$

Clearly

$$F[\cos ns, n^2, 0] = 0$$

for each positive integer n. Our goal is now to conveniently modify (5) in order to use the implicit function theorem [1]. Let us define to that end

$$U = \{u \in C^2[0,2\pi]/u \text{ is even and } \frac{1}{\pi}\int_0^{2\pi} u(s)\cos ns\, ds = 0\}$$

$$V = \{v \in \dot{} [0,2\pi]/v \text{ is even}\}$$

Both sets can be turned into Banach spaces using the natural restrictions of the norms (3a,b).

Let

$$G_n : U \times \mathbb{R} \times (-a,a) \to V$$

be defined by

$$G_n[\dot{u},\alpha;\varepsilon] = F[\cos ns + \dot{u}, n^2+\alpha, \varepsilon] \tag{6}$$

G_n satisfies

$$G_n[0,0,0] = 0 \quad ,$$

the hypothesis made on Q guarantee that it is a continuously differentiable operator with respect to the pair $(\dot{u},\alpha)$ and differentiable with respect to ε.

Its Frēchet derivative at $\dot{u}=0$, $\alpha=0$, $\varepsilon=0$ is

$$DG[\bar{u},\bar{\alpha}] = \bar{u}'' + n^2\bar{u} + \bar{\alpha}\cos ns \tag{7}$$

We claim that DG is an homeomorphism from $U \times \mathbb{R}$ onto V. First of all, if

$$DG[\bar{u},\bar{\alpha}] = 0 \tag{8}$$

for some $\bar{u} \in U$ and $\bar{\alpha} \in \mathbb{R}$, then, multiplying (8) by cos ns, using (7) and integrating by parts from zero to 2π results $\bar{\alpha}=0$, hence $\bar{u}$ satisfies

$$\bar{u}'' + n^2\bar{u} = 0 \quad .$$

Since $\bar{u}$ must belong to U, it can only be zero. Thus DG is injective.

Let us see that it is also surjective. Using the technique above and variation of parameters one obtains that the solution of

$$DG[\bar{u},\bar{\alpha}] = v \quad , \quad v \in V$$

is

$$\bar{\alpha} = \frac{1}{\pi}\int_0^{2\pi} v(s)\cos ns \, ds \tag{9}$$

$$\bar{u}(s) = A\cos ns + \int_0^s h(x)\,\frac{\sin n(s-x)}{n}\,dx$$

where

$$h(x) = v(x) - \hat{\alpha} \cos nx \quad , \quad \hat{\alpha} \text{ given in (9)}$$

$$A = \frac{1}{\pi} \int_0^{2\pi} \cos ns \int_0^s h(x) \frac{\sin n(s-x)}{n} dx \quad ,$$

consequently $\hat{u} \in U$.

Thus, the implicit function theorem guarantees the existence of an interval $I \subset \mathbb{R}$, containing zero, and continuously differentiable functions $u^+: I \times \mathbb{R} \to \mathbb{R}$, $\alpha^+: I \to \mathbb{R}$ such that

$$G_n[u^+(\varepsilon,s),\alpha^+(\varepsilon),\varepsilon] \quad 0 \qquad \varepsilon \in I$$

$$u^+(s,0) = 0 \qquad \alpha^+(0) = 0$$

The theorem's thesis follows using the definition of G_n in (6). □

REFERENCES

[1] Crandall, M.G., "An introduction to Constructive Aspects of Bifurcation and the Implicit Function Theorem", in Applications of Bifurcation Theory, Paul H. Rabinowitz ed., Academic Press, Inc., NY, (1977), pp. 1-35.

[2] Haupt, O., "Über eine Methode zum Berweis von Oscilations Theoremem, Math. Am., 76 (1914), 67-104.

[3] Magnus, W. and Winkler, S., Hill's Equations, Dover, NY (1979).