KELVIN'N DESTABILIZING EFFECT OF SMALL DAMPING

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ABSTRACT:

In 1890, Lord Kelvin, [4], conjectured that the stability or instability of the equilibrium state of a system without dissipative forces is not changed by the introduction of damping. This need not be true when dissipative forces are applied to other than conservative systems. In this paper we study the corresponding phenomenon for first order systems of 2n ordinary differential equations, showing that, if the destabilizing effect occurs, then, under suitable conditions, one can find a formula giving the leading term of an assymptotic expansion of the critical force as a function of the damping coefficients.

SOME RESULTS ON POLYNOMIALS

Let $Q(\lambda, \underline{\delta}, z)$ be the 2n-th degree polynomial

$$Q(\lambda, \underline{\delta}, z) = \sum_{r=0}^{2n} a_r(\lambda, \underline{\delta}) z^r, \quad n \geq 2$$

with real coefficients $a_r(\lambda, \underline{\delta})$, $0 \leq r \leq 2n$, differentiable functions of the real parameter λ and the s-dimensional real vector

$$\underline{\delta} = (\delta_1, \ldots, \delta_s) \quad \delta_k \geq 0, \quad 1 \leq k \leq s.$$

Let us suppose that

$$\lim_{\underline{\delta} \to \underline{0}} a_{2j-1}(\lambda, \underline{\delta}) = 0, \quad 1 \leq j \leq n$$

and let

$$P(\lambda, z) = Q(\lambda, \underline{0}, z) = \sum_{j=0}^{n} a_{2j}(\lambda, \underline{0}) z^{2j}.$$

We observe the following facts about the roots of $P(\lambda, z)$:

R_1: If $z_o(\lambda)$ is any solution branch of the equation

$$P(\lambda, z) = 0, \tag{1}$$

then, so are $-z_o(\lambda)$, $\overline{z}_o(\lambda)$ and $-\overline{z}_o(\lambda)$. Therefore, if zero is not a solution of (1), then, either all its roots are pure imaginary or there is at least one with positive real part.

R_2: If there exists a value λ_u of λ such that all the solutions branches of (1) are pure imaginary for $\lambda \leq \lambda_u$ and one of them, let us say $z_o(\lambda)$, has positive real part for $\lambda > \lambda_u$, then $z_o(\lambda_u)$ is a root of at least order two. We shall denote it by

$$z_o(\lambda) = i\omega(\lambda) \text{ for } \lambda < \lambda \text{ and } z_o(\lambda_u) = i\omega_o$$

(*) Partial results of this paper were pre-published in [3].

Let us assume, the above mentioned λ_u and $z_o(\lambda)$ exist, and that in addition, $z_o(\lambda)$ is a simple root of (1) for $\lambda < \lambda_u$. Then there exists a solution branch of

$$Q(\lambda, \underline{\delta}, z) = 0$$

in a neighborhood of $\underline{\delta} = \underline{0}$ and fixed $\lambda < \lambda_u$, given by

$$\sigma_o(\lambda, \underline{\delta}) = -\frac{1}{2} \sum_{k=1}^{s} \left[\frac{\sum_{j=1}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j-1\ \omega(\lambda)^{2j-1}}{\sum_{j=1}^{n} j(-1)^j a_{2j}\ \omega(\lambda)^{2j-1}} \right] \delta_k + \tag{2}$$

$$+ i \left\{ \omega(\lambda) - \frac{1}{2} \left[\frac{\sum_{j=o}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j\ \omega(\lambda)^{2j}}{\sum_{j=1}^{n} j(-1)^j a_{2j}\ \omega(\lambda)^{2j-1}} \right] \delta_k \right\} + O(|\underline{\delta}|^2)$$

Such a root will be pure imaginary for some $\lambda_d < \lambda_u$ if, keeping only first order terms, λ_d is a solution of the equation

$$\sum_{k=1}^{s} \left[\sum_{k=1}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j-1\ \omega(\lambda)^{2j-1} \right] \delta_k = 0$$

We now assume that such λ_d exists and seek a formula giving an approximate value of it. In order to do this, we observe that neglecting terms of order higher than four in equation (1), one obtains

$$\omega(\lambda) \sim \sqrt{\omega_o^2 + A(\lambda - \lambda_u) \pm \sqrt{\Delta}} \quad \text{when } \omega_o \neq 0\,; \tag{4}$$

where

$$\Delta = B(\lambda - \lambda_u) + A^2(\lambda - \lambda_u)^2 + o(|\lambda - \lambda_u|)$$

$$A = \frac{\sum_{j=1}^{n} j(-1)^1 \frac{\partial a}{\partial \lambda} 2j\ \omega_o^{2j-2}}{\sum_{j=2}^{n} j(j-1)(-1)^j a_{2j} \omega_o^{2j-4}}$$

$$B = -2\, \frac{\sum_{j=o}^{n} (-1)^j \frac{\partial a}{\partial \lambda} 2j\ \omega_o^{2j}}{\sum_{j=2}^{n} j(j-1)(-1)^j a_{2j} \omega_o^{2j-4}}$$

with $P(\lambda, z_o(\lambda)) = o(|\lambda - \lambda_u|)$, and

$$\omega(\lambda) \sim \sqrt{\frac{a_2}{2a_4} - \frac{\operatorname{sgn} a_2}{2a_4} \sqrt{a_2^2 - 4a_4 \frac{\partial a_o}{\partial \lambda} (\lambda - \lambda_u)}},$$

with $p(\lambda, z_o(\lambda)) = o(|\lambda-\lambda_u|^2)$ if $\omega_o = 0$. The derivatives $\frac{\partial a_o}{\partial\lambda}$ and $\frac{\partial a_{2j}}{\partial\lambda}$ are evoluated at $\lambda = \lambda_u$.

Substituting ω in (3) by (4), expanding in powers of $\lambda - \lambda_u$ and neglecting those higher than one, one obtains the desired formula

$$\lambda_d \sim \lambda_u + \frac{1}{D}\Big(\sum_{k=1}^{s} \Big[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial\delta_k} \omega_o^{2j-2} \Big]\delta_k\Big)^2$$

with

$$D = \Big(\sum_{k=1}^{s} \Big[\sum_{j=1}^{n} (j-1)(-1)^j \frac{\partial a_{2j-1}}{\partial\delta_k} \omega_o^{2j-4}\Big] \delta_k\Big)^2 B \; -$$

$$- \Big(\sum_{k=1}^{s} \Big[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial\delta_k} \omega_o^{2j-2} \Big] \delta_k \Big).$$

$$\Big(\sum_{k=1}^{s} \Big[\sum_{j=1}^{n} (-1)^j \Big\{ 2\frac{\partial^2 a_{2j-1}}{\partial\lambda\partial\delta_k} \omega_o^{2j-2} + 2(j-1)\frac{\partial a_{2j-1}}{\partial\delta_k} \omega_o^{2j-4} A + (j-1)(j-2)\frac{\partial a_{2j-1}}{\partial\delta_k}\omega_o^{2j-6} B\Big\}\Big]\delta_k\Big)$$

if $\omega_o \neq 0$ and

$$\lambda_d \sim \lambda_u + \frac{\sum_k \frac{\partial a_1}{\partial\delta_k} \delta_u}{\sum_k \Big[\frac{1}{a_2} \frac{\partial a_3}{\partial\delta_k} \frac{\partial a_o}{\partial\delta} - \frac{\partial^2 a_1}{\partial\delta_k\partial\lambda} \Big] \delta_k} \quad \text{if } \omega_o = 0.$$

(Negative powers of ω_o should be considered with zero coefficient).

APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

Lemma: Let $A(\lambda, \underline{\delta})$ be a $2n \times 2n$ real matrix, $n = 2$, differenciable function of the real parameter λ and the s-dimensional real vector $\underline{\delta} = (\delta_1, \ldots, \delta_s)$ $\delta_k \geq 0$, $1 \leq k \leq s$, such that its characteristic polynomial

$$Q(\lambda, \underline{\delta}, z) = \sum_{\ell=1}^{2n} a_\ell(\lambda, \underline{\delta}) z^\ell$$

satisfies

H_1: $\lim\limits_{\underline{\delta}\to 0} a_{2j-1} = 0 \quad 1 \leq j \leq n$.

H_2: There exists λ_u such that every solution branch $z(\lambda)$ of the equation

$$P(\lambda, z) = Q(\lambda, \underline{0}, z) = 0$$

is pure imaginary for $\lambda \leq \lambda_u$ and there exists $z_o(\lambda)$ such that $\mathrm{Re}\{z_o(\lambda)\} > 0$ for $\lambda > \lambda_u$.

H_3: There exists $\lambda_d < \underline{\lambda_u}$ satisfying equation (3)

H_4: The root $\sigma_o(\lambda_d, \underline{\delta}) = i\omega(\lambda)$ of $Q(\lambda_d, \underline{\delta}, z) = 0$ such that $\sigma_o(\lambda_d, \underline{0}) = z_o(\lambda_d)$

is simple and

$$\left[2 \sum_{j=1}^{n} j(-1)^j a_{2j}\, \omega^{2j-1} \right] \left[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial \delta_\lambda}\, \omega^{2j-1} \right] -$$

$$\left[\sum_{j=1}^{n} (2j-1)(-1)^j a_{2j-1}\, \omega^{2j-2} \right] \left[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j}}{\partial \lambda}\, \omega^{2j} \right] > 0$$

Let addition $N: R^{2n+1+s} \to R^{2n}$ be a differentiable function such that

$N(\lambda, \underline{\delta}, \underline{X}) = o(|\underline{X}|)$,

then:

T_1 The solution $\underline{X}_o = \underline{X}(t, \lambda, \underline{0})$ of

(7) $\underline{X}'(t) = A(\lambda, \underline{\delta})\underline{X} + N(\lambda, \underline{\delta}, \underline{X})$

when $\underline{\delta} = 0$, is neutrally stable for $\lambda \leq \lambda_u$ and unstable for $\lambda > \lambda_u$.

T_2: The solution $\underline{X}_1 = X(t, \lambda, \underline{\delta})$ of (7) when $\underline{\delta} \neq \underline{0}$, is linearly unstable for $\lambda > \lambda_d$, and therefore, $\underline{X}_o$ is destabilized by the introduction of $\underline{\delta}$.

T_3: λ_d is given by formula (5)

T_4: There is Hopf bifurcation of $\underline{X}_1$ at $\lambda = \lambda_d$.

Proof:

Being the variational equation of (7),

$\underline{Y}' = A(\lambda, \underline{\delta})\, \underline{Y}$,

the properties of $\underline{X}_o$ follow by definition. According to the preceeding theory, λ_d is also given by (5).

In order to show the linear instability of $\underline{X}_1$ for $\lambda > \lambda_d$, we set $z=u+iv$ in (6) and separate real and imaginary parts, differentiate with respect to λ at $\lambda = \lambda_d$, that is, $u = 0$, $v = \omega$ and equate to zero, i.e.,

$$M \frac{d \operatorname{Re} z(\lambda)}{d\lambda} + 2K \frac{d \operatorname{Im} z(\lambda)}{d\lambda} + \sum_{j=0}^{n} (-1)^j \frac{\partial a_{2j}}{\partial \lambda}\, \omega^{2j} = 0$$

$$-2k \frac{d \operatorname{Re} z(\lambda)}{d\lambda} + M \frac{d \operatorname{Im} z(\lambda)}{d\lambda} + \sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial \lambda}\, \omega^{2j-1} = 0$$

where

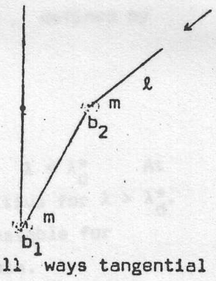
$$M = \sum_{j=1}^{n} (2j-1)(-1)^j a_{2j-1}\, \omega^{2j-2}$$

$$K = \sum_{j=1}^{n} j(-1)^j a_{2j}\, \omega^{2j-1}$$

Since the determinant of the system is $M^2 + 4K^2 > 0$, (otherwise ω would be a double root), T_2 and T_4 follow.

EXAMPLE:

Let us consider the double peddulum shown in the figure below



It consists of two rigid weightless bars of equal lenght ℓ, carrying concentrated masses $m_1 = m_2 = m$ at lower and upper hing is and connected by a linear torsional spring. The lower endpoint of the linkage is kept fixed and angular motion is resisted by another torsional spring. Friction is present at the pivots, and the whole system is subjected to a followr force applied at the free endpoint and, all ways tangential to the upper rod.

After a suitable change of variables, the differential equations of motion, which can be derived using Lagrange equations, are:

$$2x'' + y''\cos(x-y) + (y')^2\sin(x-y) + (\delta_1 + \delta_2)x' - \delta_2 y' + 2x - y - \lambda\sin(x-y) = 0 \tag{8}$$

$$x''\cos(x-y) + y'' - (x')^2\sin(x-y) - \delta_2 x' + \delta_2 y' - x + y = 0 \tag{9}$$

$(\,)' = \frac{d}{dt}$

$\underline{X}_o = (x(t, \lambda, \underline{0}),\ y(t, \lambda, \underline{0})) = \underline{0}$ is the only time independent solution of (8), (9) since (9) implies $x = y$ and then (8) implies $x = 0$.

The corresponding variational problem is

$$2x'' + y'' + (\delta_1 + \delta_2)x' - \delta_2 y' + (2 - \lambda)x - (1 - \lambda)y = 0$$

$$x'' + y'' - \delta_2 x' + \delta_2 y' - x + y = 0 .$$

Investigating solutions of the variational problem in the form $x = a_1 e^{zt}$, $y = a_2 e^{zt}$, one obtains the homogeneous system of algebraic equations:

$$\left[2z^2 + (\delta_1 + \delta_2)z + 2 - \lambda \right] a_1 + \left[z^2 - \delta_2 z + \lambda - 1 \right] a_2 = 0$$

$$\left[z^2 - \delta_2 z - 1 \right] a_1 + \left[z^2 + \delta_2 z + 1 \right] a_2 = 0$$

With characteristic equation

$$z^4 + (\delta_1 + 5\delta_2)z^3 + (6 + \delta_1\delta_2 - 2\lambda)z^2 + (\delta_1 + \delta_2)z + 1 = 0 \tag{10}$$

If $\underline{\delta} = \underline{0}$, (no damping), the four roots of (10) are

$$z_j(\lambda) = \pm \left[\lambda - 3 \pm \sqrt{(\lambda - 2)(\lambda - 4)} \right]^{\frac{1}{2}} \qquad j = 0, 1, 2, 3$$

These roots are pure imaginary for λ less than the critical load $\lambda_u = 2$, taking on the values $\pm i$ at it. Thus, $\underline{X}_o$ is neutrally stable for λ on the interval $[0, 2]$. For $\lambda > \lambda_u$, two of the roots have positive real part, then $\underline{X}_o$ os linearly unstable. It is then possible to prove that $\underline{X}_o$ id nonlinearly unstable for $\lambda > \lambda_u$.

When the damping coefficients are not simultaneously zero, application of the Routh-Hurwitz criterion shows that there exists a critical load λ_d^*, defined by

$$\lambda_d^* = 2\,\frac{\delta_1^2 + 6\delta_1\delta_2 + \delta_2^2}{\delta_1^2 + 6\delta_1\delta_2 \quad 5\delta_2^2} + \frac{1}{2}\delta_1\delta_2 \tag{11}$$

such that the real parts of the four roots of (10) are negative if $\lambda < \lambda_d^*$ At $\lambda = \lambda_d^*$, the real parts of two of the roots vanish, becoming positive for $\lambda > \lambda_d^*$. Hence $\underline{X}_o$ is asymptotically stable for $\lambda < \lambda_d^*$ and nonlinearly unstable for $\lambda > \lambda_d^*$. At $\lambda = \lambda_d^*$ the solution $\underline{X}_o$ is linearly neutrally stable.

Applying the argument theorem of analytic functions, it is possible to prove that two of the roots have negative real part for every λ. Straightforwand calculations also show that the complex frequency is, at $\lambda = \lambda_d^*$,

$$\sigma_o(\lambda, \underline{\delta}) = i\omega_o(\lambda_d^*, \underline{\delta}) = i\sqrt{\frac{\delta_1 + \delta_2}{\delta_1 + 5\delta_2}}$$

Applying the theory developed in this paper, one finds that equation (3) is in this case

$$-[\delta_1 + 5\delta_2]\lambda + [2\delta_1 + 14\delta_2] \pm [(\lambda - 3)^2 - 1]^{\frac{1}{2}}(\delta_1 + 5\delta_2) = 0$$

and using (5), we obtain

$$\lambda_d = 2\,\frac{\delta_1^2 + 6\delta_1\delta_2 + \delta_2^2}{\delta_1^2 + 6\delta_1\delta_2 + 5\delta_2^2}$$

which differs from λ_d^* in (11) only in terms of second order in δ_1, δ_2. Furthermore, applying the preceeding technique one has

$$\operatorname{Re}\left.\frac{dz(\lambda)}{d\lambda}\right|_{\lambda=\lambda_d^*} = \frac{(\delta_1 + \delta_2)[\delta_1^2 + 6\delta_1\delta_2 + 5\delta_2^2]}{[(\delta_1+5\delta_2)(\delta_1^2+6\delta_1\delta_2+5\delta_2^2)]^2 + 4[\delta_2(\delta_1+3\delta_2)]^2} > 0$$

and therefore, $\underline{X}_1$ bifurcates into a periodic solution of (8), (9) at $\lambda = \lambda_d^*$. An extense study on this solution can be found in [2].

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