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DESTABILIZING EFFECT OF SMALL DAMPING

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In 1890, Lord Kelvin, [6], indicated that the stability or instability of the equilibrium state of a system sithout dissipative or gyroscopic forces is not changed by the introduction of damping. This need not be true when dissipative forces are applied to other than non-gyroscopic, conservative systems In fact, Ziegler, [7 , 8 , 9], proved that a critical load of a non-conservative system can be decreased by adding damping. Further studies on this subject have been done by Bolotin, [1], Herrmann, [4], Hagedorn, [3], and others.

Is our purpose to show this effect in the mechanical model of fig.1 It consists of two rigid wightless bars of equal length ℓ, carrying concentrated masses $m_1 = m_2 = m$ at the endpoints and connected by a linear torsional spring. The lower endpoint of the linkage is kept fixed and angular motion is resisted by another torsional spring. Friction is present in the system at the pivots.

After a suitable change of variables, the differential equations of motion, which can be derived using Lagrange equations, are:

$$(1)\quad 2x'' + y''\cos(x-y) + (y')^2\sin(x-y) + (b_1 + b_2)x' - b_2y' + 2x - y - \lambda\sin(x-y) = 0$$

$$(2)\quad x''\cos(x-y) + y'' - (x')^2\sin(x-y) - b_2x' + b_2y' - x + y = 0$$

$$(\ )' = \frac{d}{dt}$$

$x(t) = y(t) = 0$ is the only time independent solution of (1) and (2) since (2) implies $x = y$ and then (1) implies $x = 0$.

In order to study the stability of this equilibrium state, which we shall denote by X_o, we consider the variational problem of (1) and (2) with respect to X_o:

$$(3)\quad 2x'' + y'' + (b_1 + b_2)x' - b_2y' + (2-\lambda)x - (1-\lambda)y = 0$$

$$(4)\quad x'' + y'' - b_2x' + b_2y' - x + y = 0$$

We investigate solutions of the variational problem in the form $x = a_1e^{\sigma t}$, $y = a_2e^{\sigma t}$. This leads to the homogeneous system of algebraic equations:

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(5) $[\, 2\sigma^2 + (b_1 + b_2)\sigma + 2 - \lambda \,]a_1 + [\, \sigma^2 - b_2\sigma + \lambda - 1]a_2 = 0$

(6) $[\, \sigma^2 - b_2\sigma - 1 \,]a_1 + [\, \sigma^2 + b_2\sigma + 1 \,]a_2 = 0$

and to the characteristic equation:

(7) $\sigma^4 + (b_1 + 5b_2)\sigma^3 + (6 + b_1 b_2 - 2\lambda)\sigma^2 + (b_1 + b_2)\sigma + 1 = 0$

If $b_1 = b_2 = 0$,(no damping),the four roots of (7) are

(8) $\sigma_j = \pm[\, \lambda - 3 \pm \sqrt{(\lambda - 2)(\lambda - 4)}\,]^{\frac{1}{2}}$

These roots are pure imaginary for λ less than the critical loal $\lambda_c^u = 2$ taking on the values $\pm i$ at it. Thus, X_o is neutrally stable for λ on the interval [0,2]. For $\lambda > \lambda_c^u$ two of the roots have positive real part, then X_o is linearly unstable, it is then possible to prove that X_o is nonlinearly unstable for $\lambda > \lambda_c^u$. For $\lambda \leq \lambda_c^u$, the nonlinear stability or instability of X_o can not be decided by the nature of the solutions of the variational equations as was pointed out by Liapunov,[5]. It has not been determined if X_o is asymptotically stable or not in this case.

When the damping coefficients are not simultaneously zero, application of the Routh-Hurwitz criterion shows that there exists a critical load λ_c, defined by

(9) $\lambda_c = 2\,\dfrac{b_1^2 + 6\,b_1 b_2 + b_2^2}{b_1^2 + 6b_1 b_2 + 5b_2^2} + \dfrac{b_1 b_2}{2}$

such that the real parts of the four roots of (7) are negative if $\lambda < \lambda_c$. At $\lambda = \lambda_c$, the real parts of two of the roots vanish, becoming positive for $\lambda > \lambda_c$. Hence, X_o is asymptotically stable for $\lambda < \lambda_c$ and nonlinearly unstable for $\lambda > \lambda_c$. At $\lambda = \lambda_c$ the solution X_o is linearly neutrally stable.

Applying the argument theorem of analytic functions, it is possible to prove that two of the roots of (7), let us say σ_3 and σ_4 have negative real parts for every λ.

The behavior of the real and imaginary parts of the roots as functions of λ is displayed in figures 2a,b,c and 3a,b,c for some representative values of b_1 and b_2

we have then seen that the critical force is $\lambda_c^u = 2$ when $b_1 = b_2 = 0$

In the damped case, and when $b_2 = 0$, the critical load and $\sigma_1(\lambda_c)$, which are equal to 2 and i respectively, are independent of b_1, while for $b_2 \neq 0$, λ_c depends on the ratio as well as on the magnitudes of the damping coefficients The complex frequency

$$(10) \qquad \sigma_1(\lambda_c) = i\omega_o = i \quad \frac{b_1 + b_2}{b_1 + 5b_2}$$

depends only on the ratio of the damping coefficients

In the limiting case of vanishing damping, i.e when b_1 and b_2 tend simultaneously to zero, the results corresponding to the undamped model can not be obtained unless $\lim\ (b_1 / b_2) = \infty$ as $b_1, b_2 \to 0$.

Differentiating λ_c with respect to b_2 and equating to zero, we have:

$$(11) \qquad b_1 b_2^4 \left[\frac{b_1^4 + 4b_1(3b_1^2 - 8) b_2 + 2(23b_1^2 - 48) b_2^2 + 60 b_1 b_2^3 + 25 b_2^4}{(b_1^2 + 6\, b_1 b_2 + 5\, b_2^2)^2} \right] = 0$$

A root of this equation is $b_2 = 0$. In addition we have determined numerically that it also has two positive real roots, let us say $b_{21}(b_1)$ and $b_{22}(b_1)$, for fixed b_1 in the interval $0 < b_1 < \overline{b}_1$, $\overline{b}_1 = 1.334418$

For fixed b_1 in this interval, λ_c has a relative maximum at $b_{21}(b_1)$ and a relative minimum at $b_{22}(b_1)$, (Table I). This behavior is illustrated in fig.4 for which we have chosen an arbitrary value of b_1 in $0 < b_1 < \overline{b}_1$

At any point (b_1, b_2) with b_1 in the interval $(0\ \overline{b}_1)$ and $b_{21}(b_1) < b_2 < b_{22}(b_1)$, the damping has a destabilizing effect since λ_c is a monotone decreasing function of b_2 and therefore the critical force can be lowered by incrementing the damping in the upper joint of the mecanical model

As a consequence of the discontinuity of the critical force, in the limit case of vanishing damping, the analysis of the destabilizing effect of small damping requires a special treatment We have found numerically that as b_1 tends to zero, so does $b_{21}(b_1)$ Moreover, $db_{21}(b_1) / db_1$, which can be found by implicit differentiation of the left hand side of equation (11) and evaluated numerically, tends to zero too.

By equating λ_c to 2, one obtains:

$$0.5\, b_2\, [\, 5\, b_1 b_2^2 + 2(\, 3\, b_1^2 - 8\,) b_2 + b_1^3\,] = 0 \tag{12}$$

A solution of this equation is $b_2 = 0$, and the quadratic factor yields two other roots when solved for b_2 in terms of b_1. Those two roots, which we shall denote by $b_{23}(b_1)$ and $b_{24}(b_1)$, (fig. 5), are real and positive provided $0 < b_1 < b_1^o \equiv \sqrt{6 - 2\sqrt{5}} \simeq 1.236068$. and therefore (12) has three real roots in an interval which is contained in the interval where (11) has two positive real roots. On $0 < b_1 < b_1^o$, $b_{23}(b_1)$ is a monotone increasing function and its limit, as well as the limit of its derivative with respect to b_1, is zero as b_1 tends to zero. By Rolle's theorem, $0 < b_{21}(b_1) < b_{23}(b_1)$ on the interval where they are both defined, (Table I).

This indicates that if in the undamped case, X_o is proven to be nonlinearly stable for $0 < \lambda \leq 2$ then, by choosing the damping coefficients in the ranges $0 < b_1 < b_1^o$ and $b_{23}(b_1) < b_2 < b_{24}(b_1)$, the critical force decreases, i.e., $\lambda_c < 2$. Thus damping can destabilize the system. That is the critical force can be lowered in the presence of damping. On the other hand, if $0 < b_2 < b_{23}(b_1)$, then $\lambda_c > \lambda_c^u$ and damping has a stabilizing effect in that region.

Ziegler, [10], has shown that this phenomenon can also be found in a model consisting of a disk mounted in a shaft rotating with angular velocity ω, when only internal damping is considered. He points out that there exist differences between the critical velocity predicted by the theory and the experimental messures which do not reflect the destabilizing effect. He solves the discrepancy by considering external friction. Another possible explanation has been given by Dimentberg, [2], who conjectures that in a more realistic model, hysteresis effects should be considered. Other examples of destabilizing effect of damping can be found in Ziegler, [10].

Herrmann, [4], has studied the destabilizing phenomenon in a mechanical model like the one in fig. 1, with $m_1 = 2m$, $m_2 = m$, finding $\lambda_c^u = 3.5 - \sqrt{2}$ and

$$\lambda_c = \frac{4\,b_1^2 + 33\,b_1 b_2 + 4\,b_2^2}{2(\,b_1^2 + 7\,b_1 b_2 + 6\,b_2^2\,)} + \frac{b_1 b_2}{2} \tag{13}$$

Assuming $b_i \ll 1$, $i = 1,2$, and defining the critical force for small damping as

$$F_d = \frac{4\,b_1^2 + 33\,b_1 b_2 + 4\,b_2^2}{2(\,b_1^2 + 7\,b_1 b_2 + 6\,b_2^2\,)} \tag{14}$$

he neglected $b_1 b_2/2$ and considered the ratio F_d/λ_c^u as a function of $\beta = b_1/b_2$. The forces ratio is strictly less than one, except when $\beta = \beta^* = 4 + 5\sqrt{2}$ where it is equal to 1. In this way he concluded that the presence of damping has a destabilizing effect which is eliminated only at that particular value β^*. This is in contradiction with our result that damping has a stabilizing effect on an open region of the b_1-b_2 plane instead of only on a straight line. To see that Herrmann's conclusion is erroneous for (13) we consider the ratio λ_c/λ_c^u, (Herrmann's functions λ_c and λ_c^u), instead of F_d/λ_c^u and do not neglect $b_1 b_2$ compared to F_d. We numerically determine that λ_c/λ_c^u has the value $4/(7 - 2\sqrt{2}) \simeq .959$ at $b_2 = 0$, it increases to a relative maximum, decreases with increasing b_2 up to a point where it reaches a relative minimum and finally, increases, all this provided $0 < b_1 < 1.126$, i.e., the same type of behavior our function λ_c has, (fig. 6).

Within this interval, there are three different values of b_2, let us say b_{21}, b_{22}, b_{23} , all of them depending on b_1, where the ratio λ_c / λ_c^u takes on the value 1. Furthermore, b_{21} and b_{22} are differentiable functions of b_1 and $\lim b_{21} = \lim b_{22} = 0$ as b_1 tends to zero. In Table II are given some numerical values of b_{2i} , $i=1,2,3$ for small b_1, as well as the inverses of the derivatives of b_{21} and b_{22} with respect to b_1; it follows from those numerical results that both inverses approach the common value β^*.

These results can also be found analytically. First, it must be noted that $b_1 b_2 / 2$ can not be neglected when λ_c is compared with λ_c^u since their difference is equal to

(15) $$\lambda_c - \lambda_c^u = \tfrac{1}{2} b_1 b_2 - (3 - 2\sqrt{2}) \frac{[b_1 - \beta^* b_2]^2}{2(b_1^2 + 7b_1 b_2 + 6b_2^2)}$$

$$= \tfrac{1}{2} b_1 b_2 + (F_d - \lambda_c^u)$$

and both terms can have the same order of magnitude when b_1 and b_2 are small and $b_1 \simeq \beta^* b_2$. Moreover, if it neglected, one is missled to the result $b_1 = \beta^* b_2$, while this is only the first order approximation to two different positive real roots of

(16) $$\lambda_c - \lambda_c^u = 0$$

In order to find a better approximation, let us set

$$b_1 = \epsilon \cos\alpha(\epsilon) \quad ; \quad b_2 = \epsilon \sin\alpha(\epsilon) \quad (\epsilon \geq 0)$$

in (16) using its expression (15) . Assuming α is at least twice continuously differentiable with respect to ϵ, we apply perturbation techniques to solve the equation

(17) $$\tfrac{1}{2}\epsilon^2 \sin\alpha \cos\alpha - (1.5 - \sqrt{2}) \frac{(\cos\alpha - \beta^* \sin\alpha)^2}{(\cos^2\alpha + 7\cos\alpha \sin\alpha + 6\sin^2\alpha)} = 0$$

When $\epsilon = 0$, we have

$$\cot \alpha(0) = \beta^*$$

The derivative of the left hand side of (17) with respect to ϵ at $\epsilon = 0$ is zero. Differentiating twice with respect to ϵ and setting $\epsilon = 0$, we obtain

$$(2\sqrt{2} - 3)\frac{[\sin\alpha(0) + \beta^* \cos\alpha(0)]^2 \alpha_\epsilon^2(0)}{[\cos^2\alpha(0) + 7\cos\alpha(0)\cdot\sin\alpha(0) + 6\sin^2\alpha(0)]} + \sin\alpha(0)\cdot\cos\alpha(0) = 0, \tag{18}$$

or

$$\frac{\beta^*}{1 + (\beta^*)^2} - \frac{(3 - 2\sqrt{2})[1 + (\beta^*)^2]^2}{[(\beta^*)^2 + 7\beta^* + 6]}\alpha_\epsilon^2(0) = 0 \tag{19}$$

so

$$\alpha_\epsilon(0) = \pm\left\{\frac{(2\sqrt{2} + 3)[(\beta^*)^2 + 7\beta^* + 6]\beta^*}{[1 + (\beta^*)^2]^3}\right\}^{\frac{1}{2}} \tag{20}$$

$$\simeq \pm .083945$$

and therefore

$$b_1 = \epsilon \cos[\cot^{-1}\beta^* \pm .083945\epsilon + O(\epsilon^2)] \tag{21}$$

$$b_2 = \epsilon \sin[\cot^{-1}\beta^* \pm .083945\epsilon + O(\epsilon^2)] \tag{22}$$

The angle defined by the two curves given parametrically in **(21,22)** contains the line $b_1 = \beta^* b_2$, showing that the double root β^* splits in two, different ones, thus, the destabilizing effect desappears not only at $b_1 = \beta^* b_2$, but whenever b_1 and b_2 are in that angle, or more precisely, in the cuspidal region defined by b_{21} and b_{22}.

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Captions for figures

Figure 1 Mechanical model.

Figure 2a Real part of the solution $\sigma(\lambda)$ of (7), vs the force λ, $b_1 = .1, b_2 = 0$

Figure 2b Imaginary part of the solution $\sigma(\lambda)$ of (7), vs the force λ, $b_1 = .1, b_2 = 0$.

Figure 2c Real and imaginary parts of the solution $\sigma(\lambda)$ of (7) corresponding to figures 2a,b . $b_1 = .1, b_2 = 0$

Figure 3a Same as Figure 2a with $b_1 = .1, b_2 = .2$

Figure 3b Same as Figure 2b with $b_1 = .1, b_2 = .2$

Figure 3c Same as Figure 2c with $b_1 = .1, b_2 = .2$

Figure 4 Cirtical force λ_c vs. b_2. $b_1 = 1.126$

Figure 5 Locus in the b_1-b_2 plane where $\lambda_c = \lambda_c^u = 2$

Figure 6 The ratio λ_c/λ_c^u , λ_c and λ_c^u from reference 4, $b_1 = 1.125$

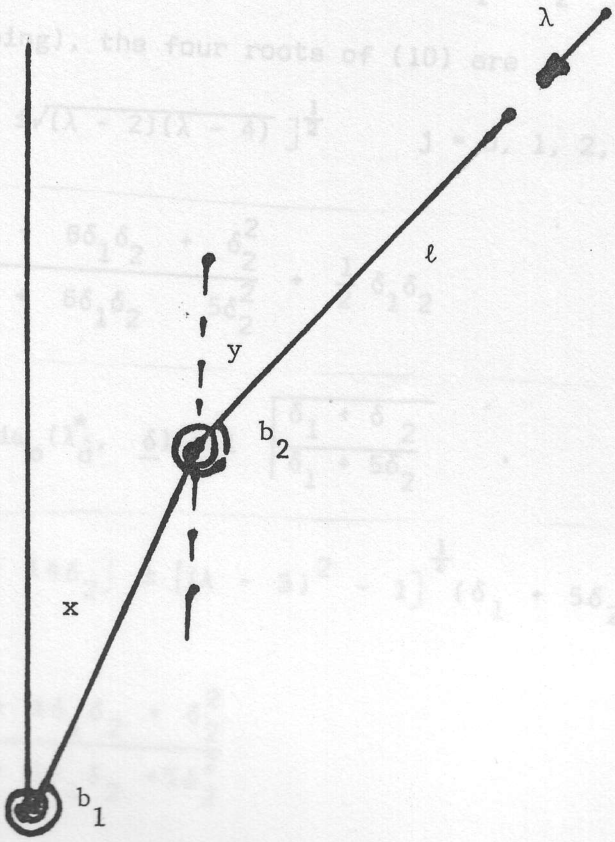


Figure 1

$$\lambda_d \sim \lambda_u + \frac{1}{D} \left(\sum_{k=1}^{s} \left[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial \delta_k} \omega_o^{2j-2} \right] \delta_k \right)^2$$

with

$$D = \left(\sum_{k=1}^{s} \left[\sum_{j=1}^{n} (j-1)(-1)^j \frac{\partial a_{2j-1}}{\partial \delta_k} \omega_o^{2j-4} \right] \delta_k \right)^2 B \quad -$$

$$- \left(\sum_{k=1}^{s} \left[\sum_{j=1}^{n} (-1)^j \frac{\partial a_{2j-1}}{\partial \delta_k} \omega_o^{2j-2} \right] \delta_k \right).$$

$$\left(\sum_{k=1}^{s} \left[\sum_{j=1}^{n} (-1)^j \left\{ 2 \frac{\partial^2 a_{2j-1}}{\partial \lambda \partial \delta_k} \omega_o^{2j-2} + 2(j-1) \frac{\partial a_{2j-1}}{\partial \delta_k} \omega_o^{2j-4} A + (j-1)(j-2) \frac{\partial a_{2j-1}}{\partial \delta_k} \omega_o^{2j-6} B \right\} \right] \delta_k \right)$$

if $\omega_o \neq 0$ and

$$\lambda \sim \lambda_u + \frac{\sum_k \frac{\partial a_1}{\partial \delta_k} \delta_u}{\sum_k \left[\frac{1}{a_2} \frac{\partial a_3}{\partial \delta_k} \frac{\partial a_o}{\partial \delta} - \frac{\partial^2 a_1}{\partial \delta_k \partial \lambda} \right] \delta_k} \quad \text{if } \omega_o = 0.$$

(8) $$2x'' + y'' \cos(x-y) + (y')^2 \sin(x-y) + (\delta_1 + \delta_2)x' - \delta_2 y' + 2x - y - \lambda \sin(x-y) = 0$$

(9) $$x'' \cos(x-y) + y'' - (x')^2 \sin(x-y) - \delta_2 x' + \delta_2 y' - x + y = 0$$

(10) $$z^4 + (\delta_1 + 5\delta_2) z^3 + (6 + \delta_1 \delta_2 - 2\lambda) z^2 + (\delta_1 + \delta_2) z + 1 = 0$$

If $\underline{\delta} = \underline{0}$, (no damping), the four roots of (10) are

$$z_j(\lambda) = \pm \left[\lambda - 3 \pm \sqrt{(\lambda - 2)(\lambda - 4)} \right]^{\frac{1}{2}} \qquad j = 0, 1, 2, 3$$

(11) $$\lambda_d^* = 2 \frac{\delta_1^2 + 6\delta_1\delta_2 + \delta_2^2}{\delta_1^2 + 6\delta_1\delta_2 \quad 5\delta_2^2} + \frac{1}{2} \delta_1 \delta_2$$

$$\sigma_o(\lambda, \underline{\delta}) = i\omega_o(\lambda_d^*, \underline{\delta}) = i \sqrt{\frac{\delta_1 + \delta_2}{\delta_1 + 5\delta_2}} .$$

$$-[\delta_1 + 5\delta_2]\lambda + [2\delta_1 + 14\delta_2] \pm [(\lambda - 3)^2 - 1]^{\frac{1}{2}} (\delta_1 + 5\delta_2) = 0$$

$$\lambda_d = 2 \frac{\delta_1^2 + 6\delta_1\delta_2 + \delta_2^2}{\delta_1^2 + 6\delta_1\delta_2 + 5\delta_2^2}$$

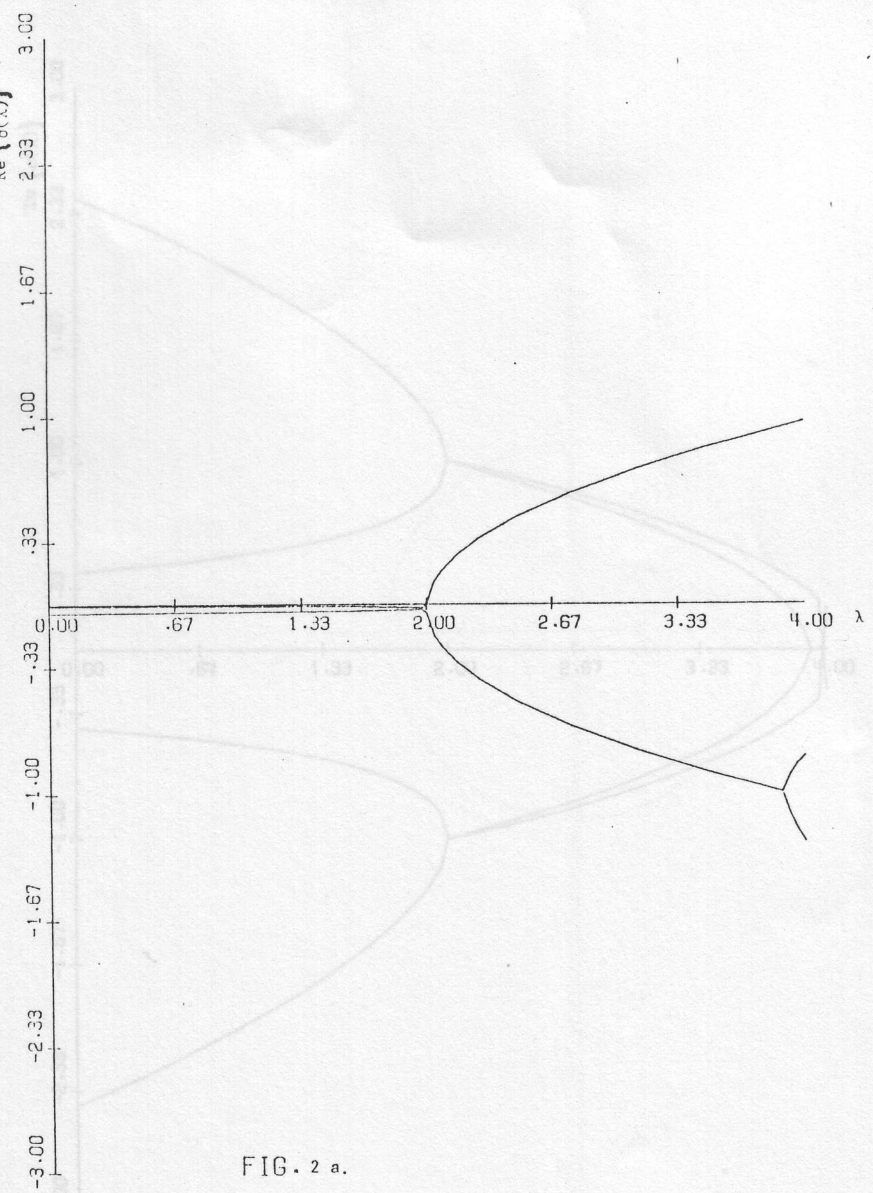


FIG. 2 a.

B1= .100

B2= 0.000

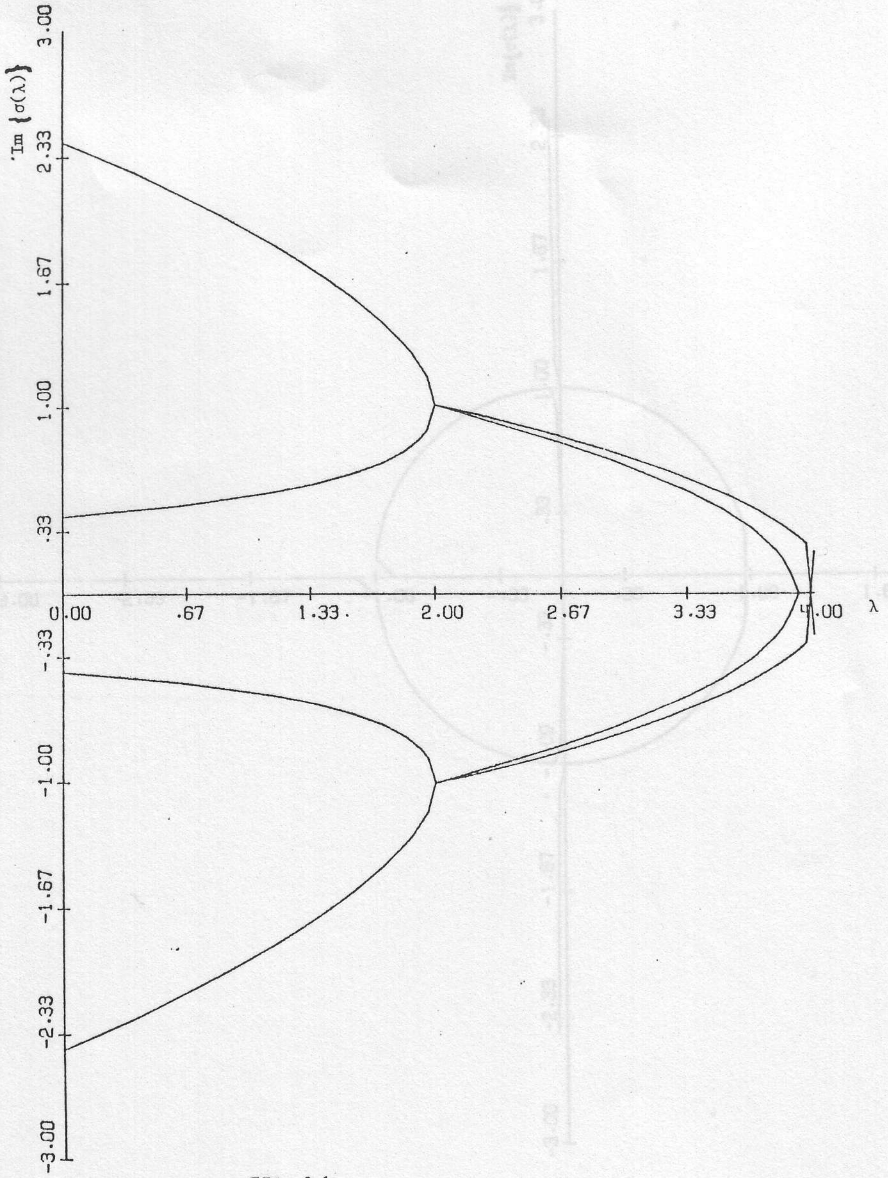
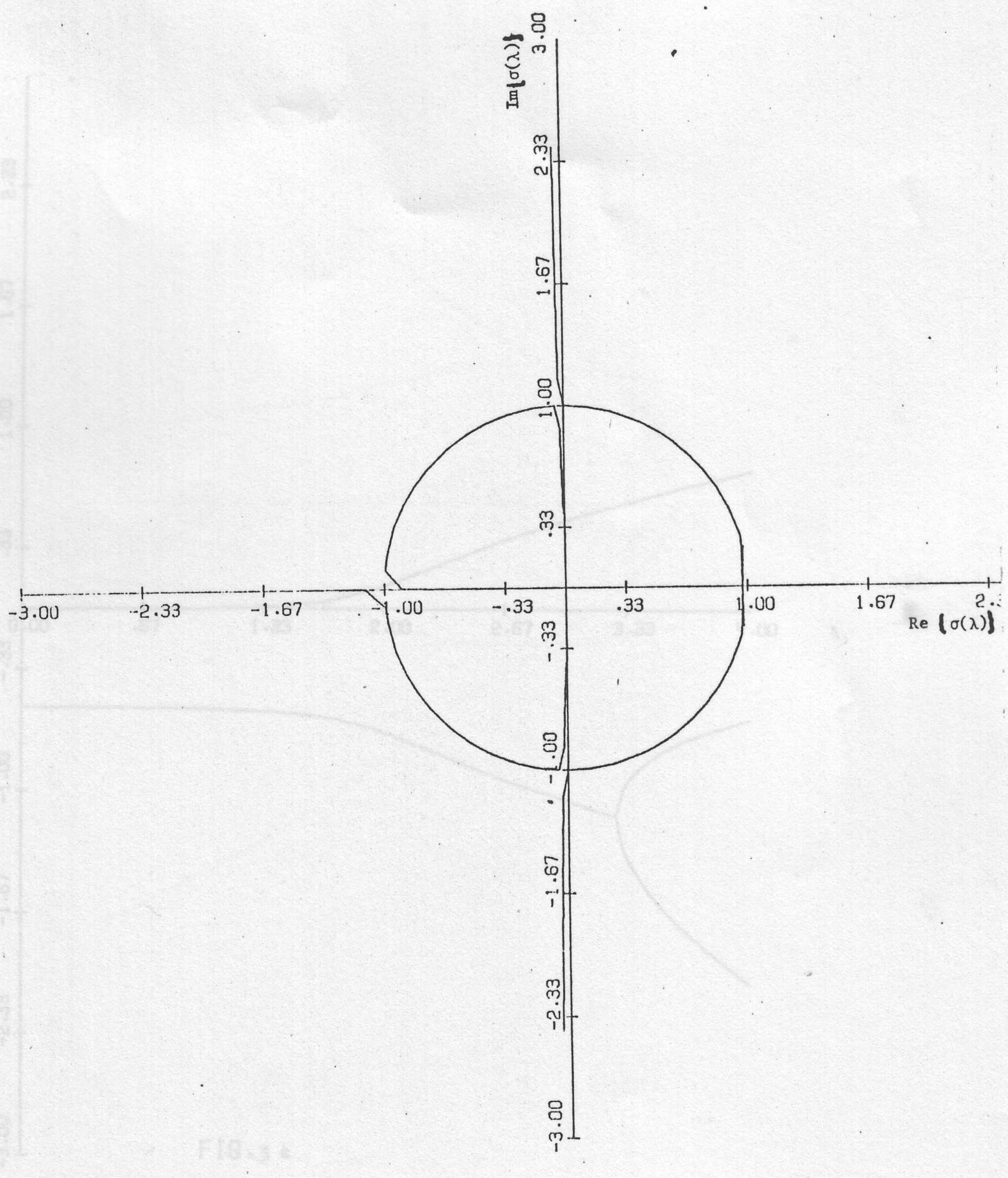


FIG. 2 b.

B1= .100

B2= 0.000



Im{σ(λ)}
3.00
2.33
1.67
1.00
.33
-.33
-1.00
-1.67
-2.33
-3.00
-3.00
-2.33
-1.67
-1.00
-.33
.33
1.00
1.67
Re {σ(λ)}

FIG. 2 c.

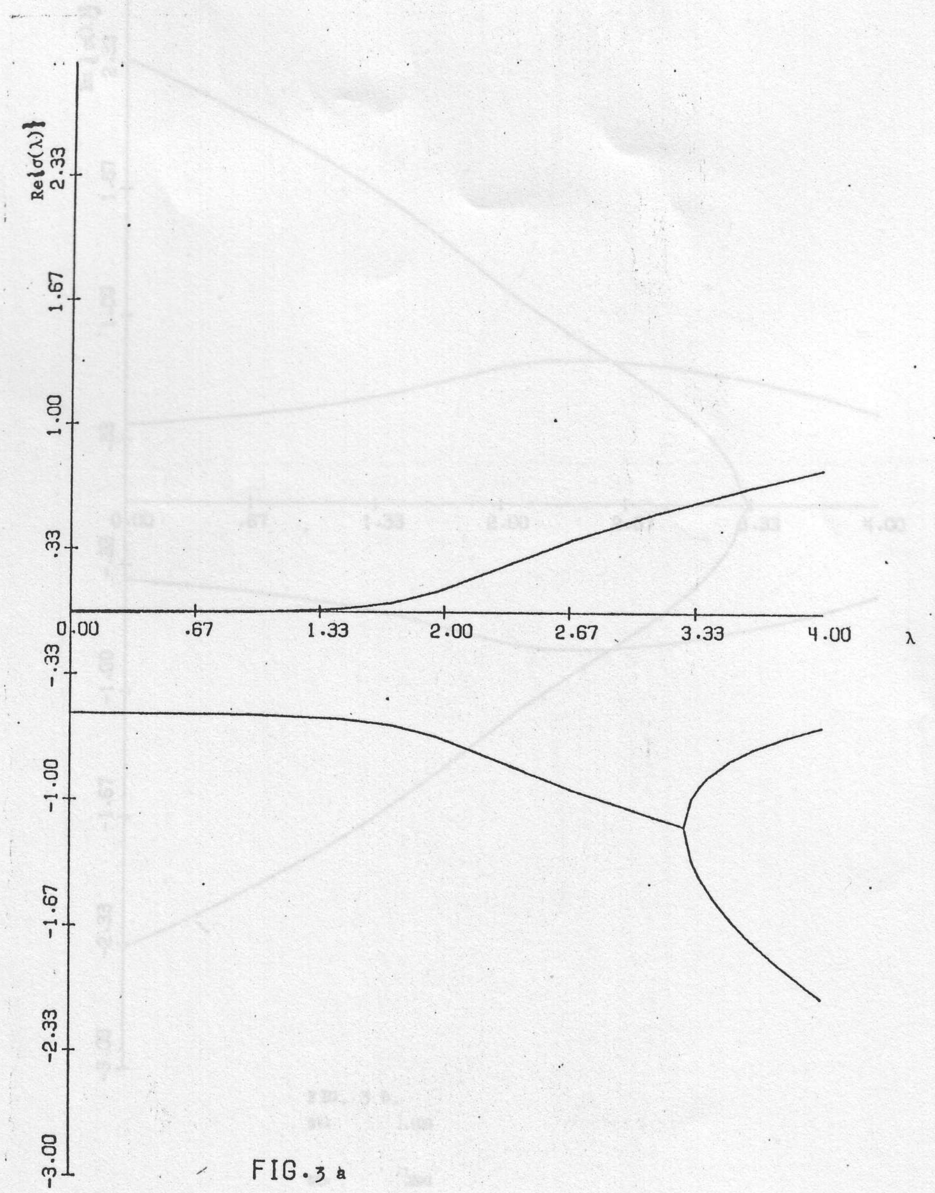


FIG.3 a

B1= .100

B2= .200 .

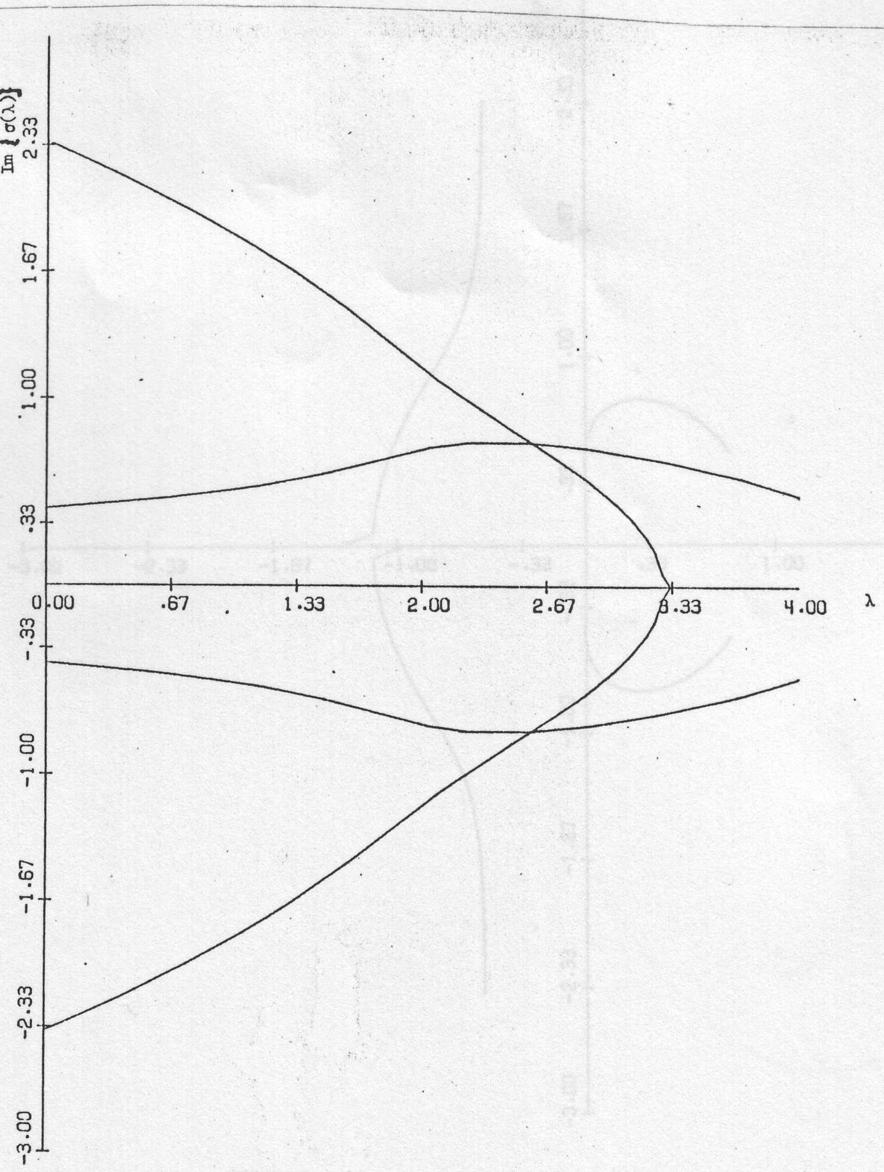
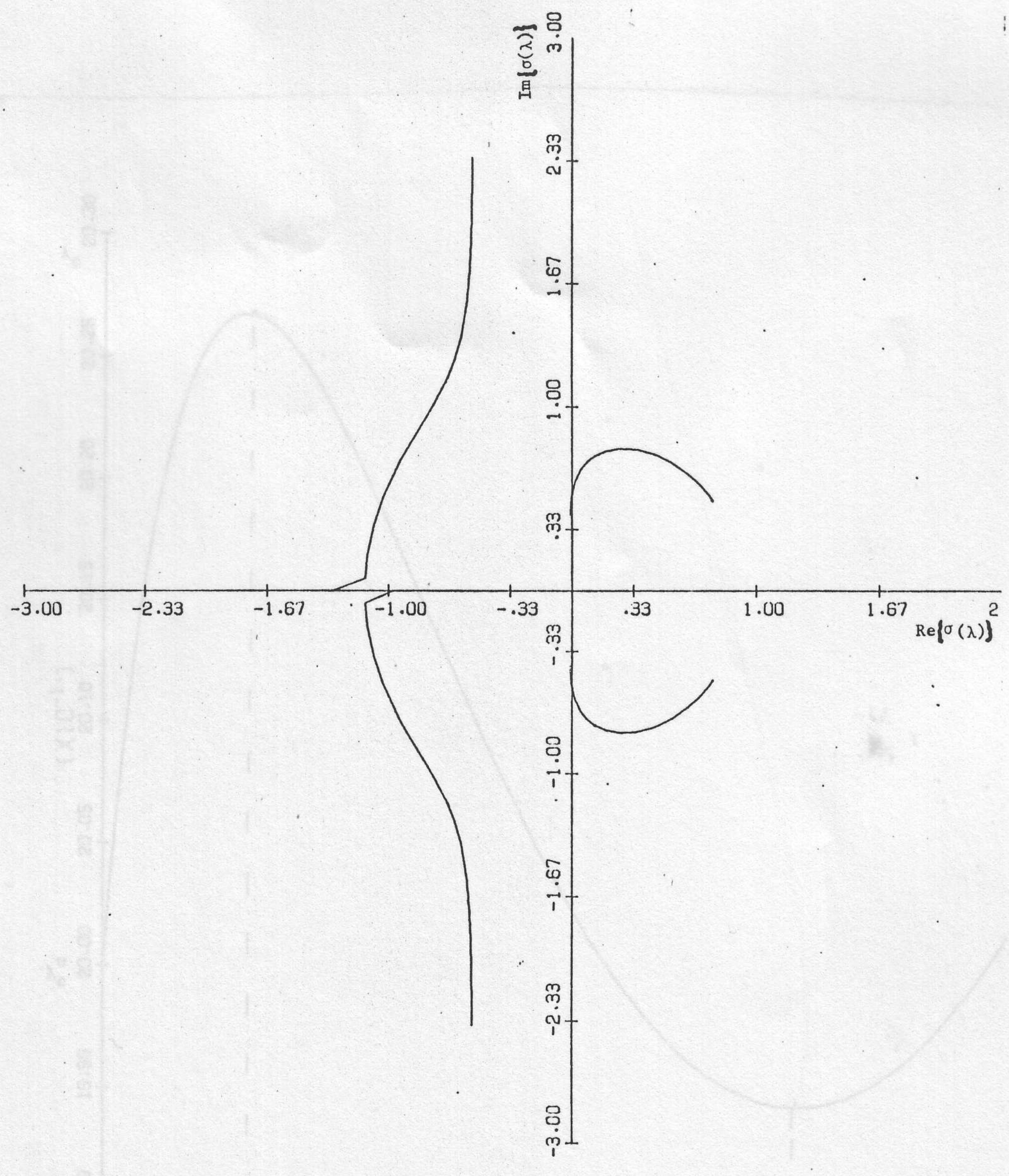


FIG. 3 b.

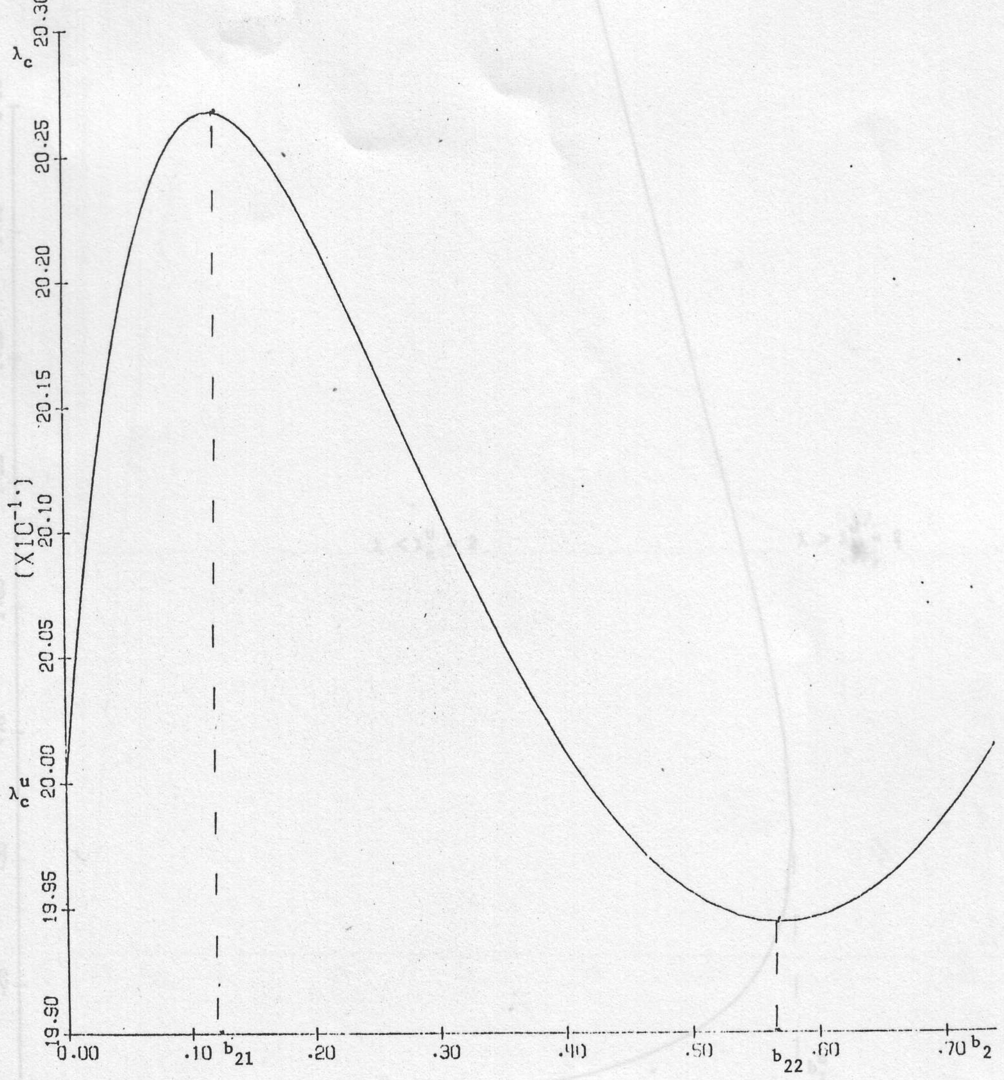
B1= .100

B2= .200



Im{σ(λ)}
3.00
2.33
1.67
1.00
.33
-.33
-1.00
-1.67
-2.33
-3.00
-3.00
-2.33
-1.67
-1.00
-.33
.33
1.00
1.67
2
Re{σ(λ)}

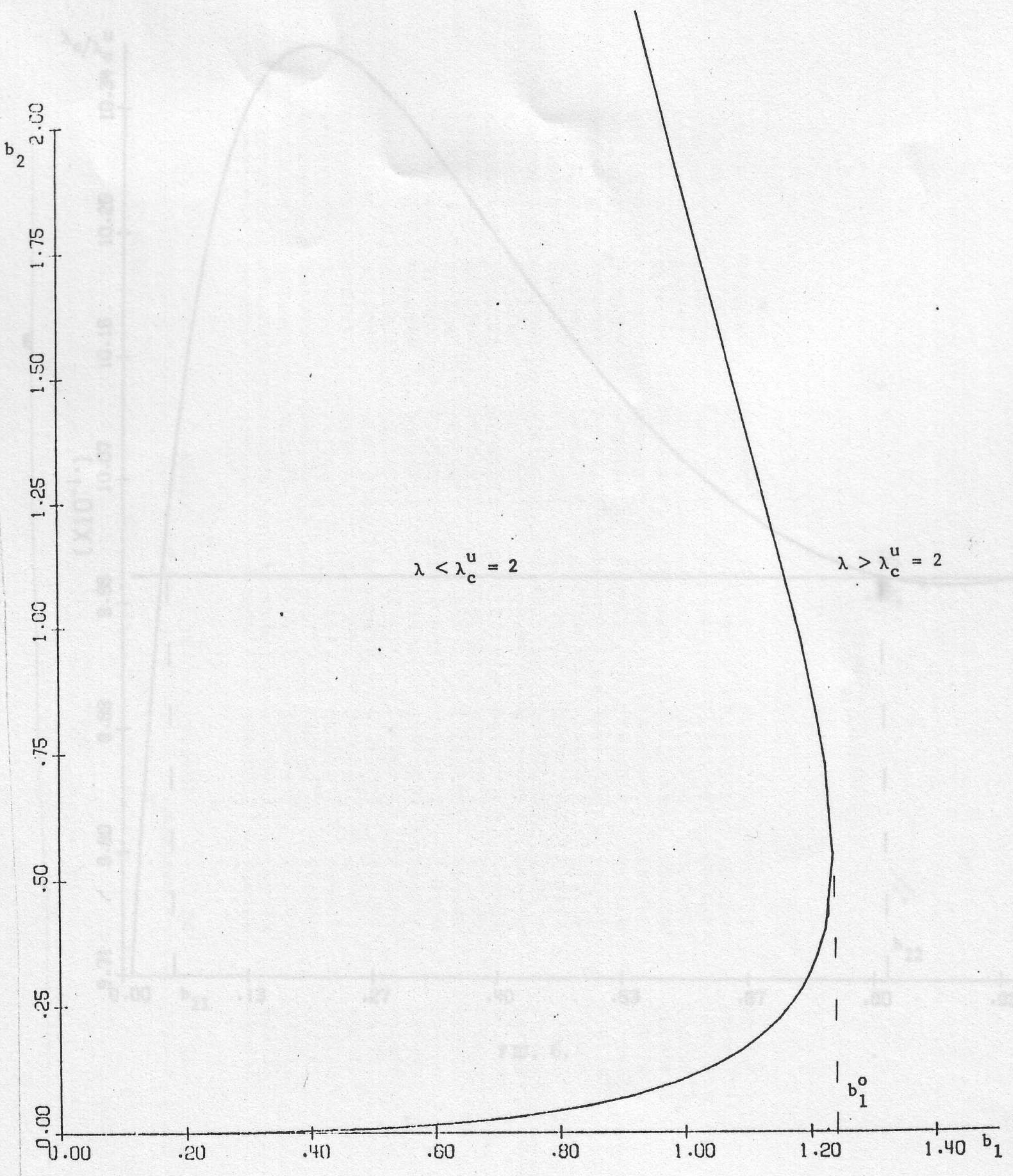
FIG. 3 c.



λc
20.30
20.25
20.20
20.15
20.10
20.05
20.00
19.95
19.90
(X10^-1.)
λc^u
0.00
.10
b21
.20
.30
.40
.50
b22
.60
.70
b2

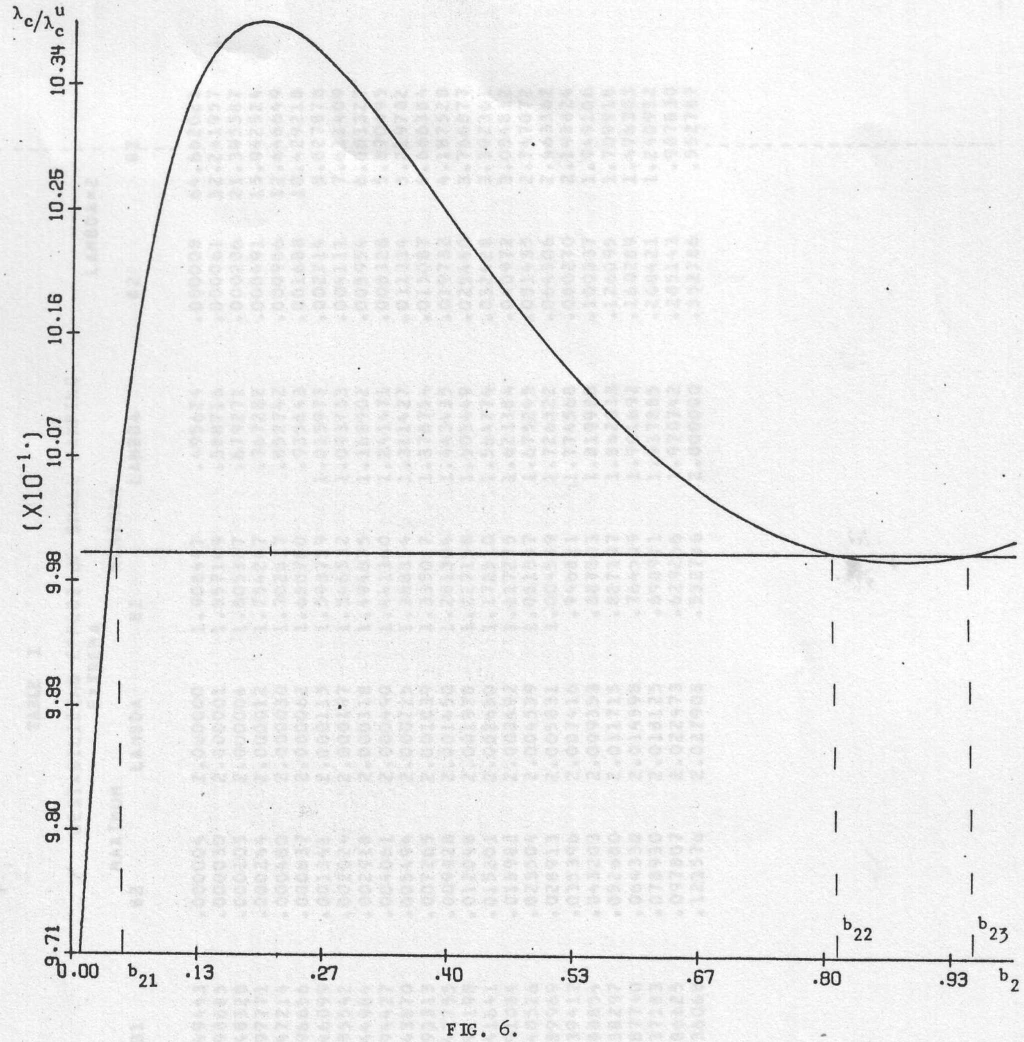
FIG. 4

B1= 1.226



b_2
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0.00
$\lambda < \lambda_c^u = 2$
$\lambda > \lambda_c^u = 2$
b_1^o
0.00
.20
.40
.60
.80
1.00
1.20
1.40
b_1

FIG.5



λc/λc u
10.34
10.25
10.16
10.07
10.00
(X10-1.)
9.98
9.89
9.80
9.71
0.00
b21
.13
.27
.40
.53
.67
.80
.93
b2
b22
b23

FIG. 6.

TABLE I

DESTABILIZING EFFECT OF SMALL DAMPING

EXTREMA

	MAXIMUM		MINIMUM		LAMBDA=2	
B1	B2	LAMBDA	B2	LAMBDA	B2	B2
.049443	.000004	2.000000	1.908447	.495624	.000008	64.662021
.098885	.000030	2.000001	1.857189	.588716	.000061	32.241957
.148328	.000103	2.000004	1.805307	.679271	.000206	21.395587
.197771	.000244	2.000012	1.754287	.767282	.000491	15.942524
.247214	.000480	2.000030	1.702617	.852742	.000966	12.646649
.296656	.000837	2.000062	1.650780	.935643	.001688	10.429218
.346099	.001341	2.000115	1.598759	1.015977	.002714	8.827876
.395542	.002024	2.000197	1.546532	1.093733	.004111	7.611409
.444984	.002918	2.000318	1.494075	1.168902	.005954	6.651327
.494427	.004061	2.000490	1.441360	1.241471	.008328	5.870495
.543870	.005494	2.000725	1.388354	1.311427	.011334	5.219782
.593313	.007265	2.001039	1.335017	1.378754	.015087	4.666384
.642755	.009428	2.001450	1.281304	1.443435	.019732	4.187528
.692198	.012048	2.001978	1.227158	1.505449	.025440	3.766877
.741641	.015201	2.002650	1.172510	1.564774	.032428	3.392361
.791084	.018982	2.003492	1.117275	1.621384	.040972	3.054812
.840526	.023504	2.004539	1.061347	1.675245	.051435	2.747072
.889969	.028913	2.005831	1.004589	1.726322	.064306	2.463362
.939412	.035396	2.007416	.946821	1.774568	.080270	2.198824
.988854	.043203	2.009353	.887803	1.819928	.100337	1.949106
1.038297	.052680	2.011715	.827197	1.862333	.126095	1.709918
1.087740	.064330	2.014598	.764509	1.901692	.160289	1.476303
1.137183	.078930	2.018125	.698971	1.937885	.208421	1.240932
1.186625	.097807	2.022473	.629266	1.970742	.285143	.987630
1.236068	.123576	2.027908	.552786	2.000000	.552786	.552787

TABLE II

DESTABILIZING EFFECT OF SMALL DAMPING

B1	B21	B22	B23	1/B21↓	1/B22↓
.00010000	.00000903	.00000903	35049.06195114	11.07207160	11.07007396
.00020000	.00001806	.00001807	17524.53077347	11.07315104	11.06907541
.00030000	.00002709	.00002711	11683.02029110	11.07419274	11.06798197
.00040000	.00003612	.00003614	8762.26498254	11.07523449	11.06691844
.00050000	.00004514	.00004518	7009.81174352	11.07627630	11.06586701
.00060000	.00005416	.00005423	5841.50953925	11.07731817	11.06482170
.00070000	.00006319	.00006327	5007.00792629	11.07836009	11.06378508
.00080000	.00007221	.00007231	4381.13168288	11.07940206	11.06274060
.00090000	.00008122	.00008136	3894.33901918	11.08044409	11.06169688
.00100000	.00009024	.00009041	3504.90486127	11.08148618	11.06065553
.00110000	.00009926	.00009946	3186.27688939	11.08252832	11.05961596
.00120000	.00010827	.00010851	2920.75355704	11.08357052	11.05857480
.00130000	.00011728	.00011757	2696.07994740	11.08461277	11.05753384
.00140000	.00012629	.00012662	2503.50254846	11.08565508	11.05649308
.00150000	.00013530	.00013568	2336.60211808	11.08669744	11.05545133
.00160000	.00014430	.00014474	2190.56422466	11.08773986	11.05441103
.00170000	.00015331	.00015380	2061.70724402	11.08878234	11.05336988
.00180000	.00016231	.00016286	1947.16769071	11.08982487	11.05232903
.00190000	.00017131	.00017193	1844.68491830	11.09086745	11.05128846
.00200000	.00018031	.00018099	1752.45040966	11.09191009	11.05024818
.00210000	.00018931	.00019006	1669.00012710	11.09295279	11.04920773
.00220000	.00019831	.00019913	1593.13622162	11.09399554	11.04816717
.00230000	.00020730	.00020820	1523.86916577	11.09503635	11.04712692
.00240000	.00021629	.00021727	1460.37435335	11.09608121	11.04608660
.00250000	.00022528	.00022635	1401.95911514	11.09712413	11.04504660

↓=D( )/DB1

Let $Q(\lambda, \underline{\delta}, z)$ be the 2n-th degree polynomial

$$Q(\lambda, \underline{\delta}, z) = \sum_{r=0}^{2n} a_r(\lambda, \underline{\delta}) z^r, \quad n \geq 2$$

$$P(\lambda, z) = Q(\lambda, \underline{0}, z) = \sum_{j=0}^{n} a_{2j}(\lambda, \underline{0}) z^{2j} .$$

(2)
$$\sigma_o(\lambda, \underline{\delta}) = -\frac{1}{2} \sum_{k=1}^{s} \left[\frac{\sum_{j=1}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j-1\ \omega(\lambda)^{2j-1}}{\sum_{j=1}^{n} j(-1)^j a_{2j} \quad \omega(\lambda)^{2j-1}} \right] \delta_k +$$

$$+ i \left\{ \omega(\lambda) - \frac{1}{2} \sum_{k=1}^{s} \left[\frac{\sum_{j=o}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j\ \omega(\lambda)^{2j}}{\sum_{j=1}^{n} j(-1)^j\ a_{2j}\ \omega(\lambda)^{2j-1}} \right] \delta_k \right\} + O(|\underline{\delta}|^2)$$

[3]
$$\sum_{k=1}^{s} \left[\sum_{k=1}^{n} (-1)^j \frac{\partial a}{\partial \delta_k} 2j-1 \quad \omega(\lambda)^{2j-1} \right] \delta_k = 0$$

(4)
$$\omega(\lambda) \sim \sqrt{\omega_o^2 + A(\lambda - \lambda_u) \pm \sqrt{\Delta}} \qquad \text{when} \quad \omega_o \neq 0 ;$$

where
$$\Delta = B(\lambda - \lambda_u) + A^2(\lambda - \lambda_u)^2 + o(|\lambda - \lambda_u|)$$

$$A = -\frac{\sum_{j=1}^{n} j(-1)^1 \frac{\partial a}{\partial \lambda} 2j\ \omega_o^{2j-2}}{\sum_{j=2}^{n} j(j-1)(-1)^j a_{2j} \omega_o^{2j-4}}$$

$$B = -2 \frac{\sum_{j=o}^{n} (-1)^j \frac{\partial a}{\partial \lambda} 2j \quad \omega_o^{2j}}{\sum_{j=2}^{n} j(j-1)(-1)^j a_{2j} \omega_o^{2j-4}} ,$$

with $P(\lambda, z_o(\lambda)) = o(|\lambda - \lambda_u|)$, and

$$\omega(\lambda) \sim \sqrt{\frac{a_2}{2a_4} - \frac{\operatorname{sgn} a_2}{2a_4} \sqrt{a_2^2 - 4a_4 \frac{\partial a_o}{\partial \lambda} (\lambda - \lambda_u)}} ,$$

with $p(\lambda, z_o(\lambda)) = o(|\lambda - \lambda_u|^2)$ if $\omega_o = 0$.