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An Asymptotic Formula for the Distance Between Some Transition Curves of The Quasi-Periodic Mathieu's Equation

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AN ASYMPTOTIC FORMULA FOR THE DISTANCE BETWEEN SOME TRANSITION CURVES OF THE QUASI-PERIODIC MATHIEU'S EQUATION

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Abstract. In this work we determine explicit formulas which give the asymptotic distance between transition curves $\lambda^{\pm}(\varepsilon)$, bifurcating from $\lambda = m^2, \lambda = m^2\omega^2$, m a positive integer, in the $\lambda - \varepsilon$ plane, along which there exist bounded solutions of the quasi-periodic Mathieu's equation $\ddot{x} + (\lambda + \varepsilon(\cos t + \cos\omega t))x = 0$.

Key words: quasi-periodic, Mathieu's equation, Floquet theory, perturbation methods.

AMS subject classification: 34, 34D, 34E, 34D08, 34D10, 34E10

1. Introduction. Mathieu's equation

$$\ddot{x} + (\lambda + \varepsilon\cos t)x = 0, \qquad (\dot{\ }) = \frac{d}{dt} \tag{1}$$

generally associated with the problem of stability of many physical phenomena, is a well known subject on which there exists a vast literature [1],[2],[3]. The properties of its solutions can easily be described applying Floquet's theory and, in particular, those of the transition curves $\lambda = \lambda(\varepsilon)$ which separate regions in the $\lambda - \varepsilon$ plane where all the solutions are bounded from those where there is at least an unbounded one. A rather simple modification of (1), in which $\cos\omega t$, ω irrational, is added to the driving term, leads to the quasi-periodic Mathieu's equation

$$\ddot{x} + (\lambda + \varepsilon(\cos t + \cos\omega t))x = 0. \tag{2}$$

In a recent article of this journal[4], Zounes and Rand discuss the transition curves of (2), giving some analytical as well as numerical results. The purpose of the present work is to give an explicit, asymptotic formula for the distance between the curves corresponding to the normalized solutions which bifurcate from the points $\lambda = m^2$ and $\lambda = m^2\omega^2, m \geq 1$ integer, when $\varepsilon = 0$. Here we employ a combination of power series and Fourier series, a method already utilized by Levy and Keller for the Mathieu's equation[5], reducing the problem of solving the differential equation to a sequence of linear algebraic ones, a devise which allows us to explicitly find the asymptotic distance between the curves.

2. Power series-Fourier series solution. When $\varepsilon = 0$, the solutions of (2) with period $2\pi/m$ and $2\pi/m\omega, m$ a positive integer, satisfying $x(0) = 1, \dot{x}(0) = 0$ and $x(0) = 0, \dot{x}(0) = 1$, are $\cos mt$ ($\sin mt$) and $\cos m\omega t$ ($\sin m\omega t$) provided λ takes on, respectively, the values $\lambda = \lambda_0 = m^2$ and $\lambda = \lambda_0 = m^2\omega^2$. The goal of our investigation

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is to determine curves $\lambda = \lambda(\varepsilon)$ on which (2) posseses bounded solutions and the distance between them. To that end, we shall consider first the case $\lambda_0 = m^2$. In order to find an asymptotic solution of (2), we use the regular perturbation method[6], setting

$$x(t) = \sum_{n=0}^{\infty} x_n^{\pm}(t)\varepsilon^n, \tag{3}$$
$$\lambda = \sum_{n=0}^{\infty} \lambda_n^{\pm}\varepsilon^n, \ \lambda_0^{\pm} = m^2.$$

Here, the superscript $\pm$ will differentiate the even from the odd solutions. Substituting $x(t)$ and λ in (2) by (3), collecting powers of ε and equating their coefficients to zero we obtain:

$$\ddot{x}_0^{\pm} + m^2 x_0^{\pm} = 0, \tag{4}$$
$$\ddot{x}_n^{\pm} + m^2 x_n^{\pm} = -\sum_{j=1}^{n} \lambda_j^{\pm} x_{n-j}^{\pm} - (\cos t + \cos \omega t)\, x_{n-1}^{\pm}. \tag{5}$$

Let

$$x_0^{+}(t) = \cos mt,$$
$$x_0^{-}(t) = \sin mt.$$

Several properties of the solutions of (5) can easily be deduced from the structure of the equation itself, namely:

1. Being $\lambda_j^{\pm}$ constant in the first term of the right-hand-side, the corresponding particular solution is of the same type the $x_{n-j}^{\pm}$ are. i.e., it is a sum of the same harmonic functions which define $x_{n-j}^{\pm}$ but with, possibly, different coefficients.
2. The product of $x_{n-1}^{\pm}$ by $\cos t + \cos \omega t$ adds and subtracts 1 and ω to the harmonics of $x_{n-1}^{\pm}$,but not both at the same time. For example, the product $(\cos t + \cos \omega t)\ x_0^{+}$ contains terms of the form $\cos(m \pm 1)t$ and $\cos(m \pm \omega)t$, and therefore so does the solution x_1^{+}. It is also illustrative to see what happens when $n = 2$, i.e., with the product of x_1^{+}. In this case, the additions and subtractions have two main effects, on one hand one has $\cos(m + 1 \pm \omega)t$ $(\cos(m - 1 \pm \omega)t)$, on the other hand, $\cos(m + \omega \pm \omega)t$ $(\cos(m - \omega \pm \omega)t)$. Applying this reasoning recursively, it follows that $x_n^{+}(t)$ is a sum of terms of two different forms: $\cos pt$ and $\cos(q\omega + l)$, $p, q, l \epsilon \mathbf{Z}$. Although the subtractions may generate negative coefficients of ω, one can always make them positive, changing l by $-l$. Evidently that proper care of the signum must be taken in the case of $x_n^{-}(t)$.
3. It follows from the discussion in the previous item, that the integers p, q and l satisfy $\max(0, m - n) \leq p \leq m + n$, $|l| \leq m + n$, $1 \leq q \leq n$.

The immediate consequence of the properties above is that we can write

$$
(6)\qquad
\begin{aligned}
x_n^+(t) &= \sum_{p=0}^{m+n} \hat{u}_n^+(p)\cos pt + \sum_{q=1}^{n}\sum_{|l|=0}^{m+n} \hat{v}_n^+(q,l)\cos(q\omega+l)t,\\
x_n^-(t) &= \sum_{p=1}^{m+n} \hat{u}_n^-(p)\sin pt + \sum_{q=1}^{n}\sum_{|l|=0}^{m+n} \hat{v}_n^-(q,l)\sin(q\omega+l)t.
\end{aligned}
$$

Since the operator on the left-hand-side of equation (4) has a non-trivial kernel, the solutions of (5) are not unique. In order to solve this indeterminacy, we shall impose the condition

$$
(7)\qquad \hat{u}_n^{\pm}(m) = \delta_{n,0}
$$

where $\delta_{n,0}$ is the Kronecker delta.

Substituting $x_n^{\pm}(t)$ in (5) by (6) and equating like harmonics, we obtain

$$
(8)\qquad m^2\hat{u}_n^+(0) = -\sum_{j=1}^{n}\lambda_j^+\hat{u}_{n-j}(0) - \frac{1}{2}\hat{u}_{n-1}^+(1) - \frac{1}{2}\hat{v}_{n-1}^+(1,0),
$$

$$
(9)\qquad
\begin{aligned}
\left[m^2-p^2\right]\hat{u}_n^{\pm}(p) &= -\sum_{j=1}^{n}\lambda_j^{\pm}\hat{x}_{n-j}^{\pm}(p) - \frac{1}{2}\left[\hat{u}_{n-1}^{\pm}(p-1)+\hat{u}_{n-1}^{\pm}(p+1)\right]\\
&\quad -\frac{1}{2}\hat{u}_{n-1}^{+}(0)\delta_{p,1} - \frac{1}{2}\left[\hat{v}_{n-1}^{\pm}(1,p)\pm\hat{v}_{n-1}^{\pm}(1,-p)\right],
\end{aligned}
$$

$$
(10)\qquad
\begin{aligned}
\left[m^2-(q\omega+l)^2\right]\hat{v}_n^{\pm}(q,l) &= -\sum_{j=1}^{n}\lambda_j^{\pm}\hat{v}_{n-j}^{\pm}(q,l)\\
&\quad -\frac{1}{4}\left[(1\pm 1)+(1\mp 1)sgn(l)\right]\hat{u}_{n-1}^{\pm}(|l|)\delta_{q1}\\
&\quad -\frac{1}{2}\left[\hat{v}_{n-1}^{\pm}(q,l-1)+\hat{v}_{n-1}^{\pm}(q,l+1)\right]\\
&\quad -\frac{1}{2}\left[\hat{v}_{n-1}^{\pm}(q-1,l)+\hat{v}_{n-1}^{\pm}(q+1,l)\right].
\end{aligned}
$$

Choosing $p=m$ in (9) and using (7) we have:

$$
(11)\qquad \lambda_n^{\pm} = -\frac{1}{2}\left[\hat{u}_{n-1}^{\pm}(m-1)+\hat{u}_{n-1}^{\pm}(m+1)\right] - \frac{1}{2}\left[\hat{v}_{n-1}^{\pm}(1,m)\pm\hat{v}_{n-1}^{\pm}(1,-m)\right].
$$

2.1. The distance between transition curves.

LEMMA 1. *If $n<m$ then:*

a) $\hat{u}_n^+(p) = \hat{u}_n^-(p)\forall p>0,$

b)
$$
\begin{aligned}
\hat{v}_n^+(q,l) &= \hat{v}_n^-(q,l) \quad \forall q \; if \; l\ge 0,\\
\hat{v}_n^+(q,l) &= -\hat{v}_n^-(q,l) \quad \forall q \; if \; l<0,
\end{aligned}
$$

c) $\lambda_n^+ = \lambda_n^-$.

Proof. The lemma is trivially true for $n = 0$ because $\hat{u}_0^+(p) = \hat{u}_0^-(p) = \delta_{p,m}, \hat{v}_0^+(q,l) = \hat{v}_{0n}^-(q,l) = 0 \forall q, l$ and $\lambda_0^+ = \lambda_0^- = m^2$.

Let us suppose it is also true up to some $n = k < m-1$ then for $n = k+1$, we have in (11): $\hat{x}_k^+(m \pm 1) = \hat{x}_k^-(m \pm 1)$ and from item 3 in page 2 $\hat{y}_k^+(1, \pm m) = \hat{y}_k^-(1, \pm m)$, therefore $\lambda_{k+1}^+ = \lambda_{k+1}^-$. This and the assumption, using (9), imply $\hat{u}_{k+1}^+(p) = \hat{u}_{k+1}^-(p)$. In order to prove the properties of $\hat{v}_{k+1}^\pm(q,l)$ we observe in (10) that, due to the assumption, they are automatically satisfied when $q \neq 1$ and $l \geq 1$ or $l < -1$. In the cases $q \neq 1$, $l = 0, -1$ we have that the first term as well as the last four terms in the right-hand-side of (10) are zero because $\max(0, m-k) = m-k > 1$ and the second term is zero because $\delta_{q1} = 0$. If $q = 1$ and $l \geq 1$ or $l \leq -1$ the results is again immediate. If $l = 0$ then $\hat{v}_k^\pm(1,-1) = 0$ resulting in $\hat{v}_{k+1}^+(1,0) = \hat{v}_{k+1}^-(1,0)$. If $l = -1$, all but the second term are automatically zero, the right-hand-side of (10) is thus reduced to

$$-\frac{1}{4}\left[(1 \pm 1) - (1 \mp 1)\right] \hat{u}_k^\pm(1),$$

which proves the lemma.□

A first difference in the coefficients appears, besides the sign when $l < 0$, when $n = m$. Namely, $\hat{u}_m^-(0) = 0$ while $\hat{u}_m^+(0)$ is not necessarily zero. It is not difficult to prove by induction that this difference propagates in the values of $\hat{u}$ and $\hat{v}$ from $n = m$ and $n = m+1$ respectively. They will only affect $\lambda_n^\pm$ when they reach $p = m-1$ (see equation (11)) that is, when $n = 2m$ because $\hat{u}_{2m-1}^+(m-1)$ might be, and we shall prove it is, different from $\hat{u}_{2m-1}^-(m-1)$.

If follows from the above reasoning and equation (11) that

$$\lambda_{2m}^- - \lambda_{2m}^+ = -\frac{1}{2}\left[\hat{u}_{2m-1}^-(m-1) - \hat{u}_{2m-1}^+(m-1)\right]. \tag{12}$$

Using equation (9) we obtain:

$$\begin{aligned}\left[\hat{u}_{2m-k}^-(m-k) - \hat{u}_{2m-k}^+(m-k)\right] &= \\ -\frac{1}{2\left[m^2-(m-k)^2\right]} &\left[\hat{u}_{2m-(k+1)}^-(m-(k+1)) - \hat{u}_{2m-(k+1)}^+(m-(k+1))\right],\end{aligned}$$

and therefore

$$\hat{u}_{2m-1}^-(m-1) - \hat{u}_{2m-1}^+(m-1) = \left(-\frac{1}{2}\right)^{m-2} \prod_{k=1}^{m-2}\left(\frac{1}{m^2-(m-k)^2}\right)\left[\hat{u}_{m+1}^-(1) - \hat{u}_{m+1}^+(1)\right]. \tag{13}$$

It also follows from (9) with $n = m+1$ that

$$\hat{u}_{m+1}^-(1) - \hat{u}_{m+1}^+(1) = \frac{1}{(m^2-1)}\hat{u}_m^+(0), \tag{14}$$

and from (8):

(15) $$\hat{u}_m^+(0) = -\frac{1}{2m^2}\hat{u}_{m-1}^+(1).$$

Once again, using (9)

$$\hat{u}_{m-k}^+(k) = -\frac{1}{2\,(m^2-k^2)}\hat{u}_{m-(k+1)}^+(k+1),$$

thus

(16) $$\hat{u}_{m-1}^+(1) = \left(-\frac{1}{2}\right)^{m-1}\prod_{k=1}^{m-1}\frac{1}{(m^2-k^2)}.$$

Putting together equations (12)-(16):

(17) $$\lambda_{2m}^- - \lambda_{2m}^+ = -\left(\frac{1}{2}\right)^{2m-1}\frac{1}{(2m-1)!^2},$$

which is the desired formula.

The case $\lambda_0 = m^2\omega^2$ can be treated in a form completely analogous. We shall not repeat here the procedure, limiting ourselves to merely stating the final result

(18) $$\lambda_{2m}^- - \lambda_{2m}^+ = -\left(\frac{1}{2\omega^2}\right)^{2m-1}\frac{1}{(2m-1)!^2}.$$

It is interesting to observe that when $\lambda_0 = m^2$, the zero-th order approximation corresponds to a non-zero $\hat{u}$ coefficients and the $\hat{v}$ ones appear at the first order approximation. In contraposition, in this second case, the zero-th order approximation is of $\hat{v}$ type, but the $\hat{u}'s$ are zero up to order $m-1$.

3. Discussion of results and conclusions. We have determined formulas for two types of solutions which bifurcate from $\lambda_0 = m^2$ and $\lambda_0 = m^2\omega^2$, proving that in both cases the distance from the transition curves is of order $2m$ in ε. The method employed, at least for these special solutions is, from the computational view-point, fast and accurate, making unnecessary to deal with small divisors or determinants of infinite matrices. In fact, the recursive system of equations (8-11) can be used not only for the detemination of the width of the instability regions, but for actually finding asymptotic solutions of (2) to any desired order of approximation, giving also insight on different properties of the Fourier coefficients.

Much work remains to be done, for example, it would be desirable to extend the formulas to the more general form $\lambda_0 = (m_1 + m_2\omega)^2/4,\ m_1, m_2 \epsilon \mathbf{Z}$

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