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A PHASE TRANSITION IN A TWO DIMENSIONAL HAMILTONIAN MAP

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Abstract: The transition from integrability to non-integrability for a two dimensional Hamiltonian mapping exhibiting mixed phase space is considered. The phase space of such mapping show a large chaotic sea surrounding KAM islands and limited by a set of invariant tori. The description of the phase transition is made by the use of scaling functions for average quantities of the mapping averaged along the chaotic sea. The critical exponents are obtained via extensive numerical simulations. Given the mapping the critical exponents that characterize the scaling functions are obtained. Therefore classes of universality are defined.

We present and discuss some dynamical properties for a set of two dimensional Hamiltonian mappings. We assume that there is a two-dimensional integrable system that is slightly perturbed. The Hamiltonian function that, in principle, describes the system is written as

$$H(I_1, I_2, \theta_1, \theta_2) = H_0(I_1, I_2) + \varepsilon H_1(I_1, I_2, \theta_1, \theta_2) \;, \quad (1)$$

where the variables I_i and θ_i with $i = 1, 2$ correspond respectively to the action and angle. One can see clearly that the control parameter ε controls a transition from integrability to non integrability. To use the characterization of the dynamics in terms of a mapping, we can now consider a Poincaré section defined by the plane $I_1 \times \theta_1$ and assume θ_2 as constant (mod 2π). A generic two dimensional mapping which qualitatively describes the behavior of (1) is

$$T : \left\{ \begin{array}{l} I_{n+1} = I_n + \varepsilon h(\theta_n, I_{n+1}) \\ \theta_{n+1} = [\theta_n + K(I_{n+1}) + \varepsilon p(\theta_n, I_{n+1})] \mod(2\pi) \end{array} \right. \quad (2)$$

where h, K and p are assumed to be nonlinear functions of their variables while the index n corresponds to the nth iteration of the mapping. The variables I and θ correspond indeed to I_1 and θ_1.

Since the mapping (2) should be area preserving, the expressions for $h(\theta_n, I_{n+1})$ and $p(\theta_n, I_{n+1})$ have to obey some properties, in particular some intrinsic relations. The relations are obtained considering that the determinant of the Jacobian matrix is the unity. After some straightforward algebra, it is easy to conclude that area preservation will be observed only if the condition

$$\frac{\partial p(\theta_n, I_{n+1})}{\partial \theta_n} + \frac{\partial h(\theta_n, I_{n+1})}{\partial I_{n+1}} = 0 \;, \quad (3)$$

is matched. For many mappings considered in the literature, the function $p(\theta_n, I_{n+1}) = 0$. Hence, if we keep h as $h(\theta_n) = \sin(\theta_n)$, and vary K, to illustrate applicability of the formalism, we nominate the following mappings that have already been studied:

- Considering $K(I_{n+1}) = I_{n+1} + \zeta I_{n+1}^2$, the logistic twist mapping is obtained;
- $K(I_{n+1}) = I_{n+1}$, then the Taylor-Chirikov's map is recovered;
- $K(I_{n+1}) = 2/I_{n+1}$, then the Fermi-Ulam accelerator model is obtained [1, 2];
- $K(I_{n+1}) = \zeta I_{n+1}$, with ζ constant, then the bouncer model is found;
- For the case of

$$K(I_{n+1}) = \left\{ \begin{array}{l} 4\zeta^2 (I_{n+1} - \sqrt{I_{n+1}^2 - \frac{1}{\zeta^2}}) \;\text{ if }\; I_{n+1} > \frac{1}{\zeta} \;, \\ 4\zeta^2 I_{n+1} \;\text{ if }\; I_{n+1} \leq \frac{1}{\zeta} \;. \end{array} \right. \quad (4)$$

  where ζ is a constant, then we recovered the so called Hybrid-Fermi-Ulam-bouncer model.

In this work, we consider the following expression for the two dimensional mapping [3]:

$$T : \left\{ \begin{array}{l} x_{n+1} = \left[x_n + \frac{a}{(y_{n+1})^{2/3}} \right] \mod 1 \\ y_{n+1} = |y_n - b \sin(2\pi x_n)| \end{array} \right. \;, \quad (5)$$

where a and b and γ are the control parameters. The determinant of the Jacobian matrix is $\mathrm{Det\ J} = \mathrm{sign}(y_n - b\sin(2\pi x_n))$ where $\mathrm{sign}(u) = 1$ if $u > 0$ and $\mathrm{sign}(u) = -1$ if $u < 0$.

It is important to emphasize that there are two control parameters in mapping (5) that control the transition from integrability to no integrability, namely $a = 0$ or $b = 0$. The phase space generated from iteration of the mapping (5) for $a = 2$ and $b = 10^{-3}$ is shown in Fig. 1.

Now we concentrate to discuss some scaling properties present in the chaotic sea. The average quantity to be explored is the deviation of the average $\bar{y}$ for chaotic orbits,

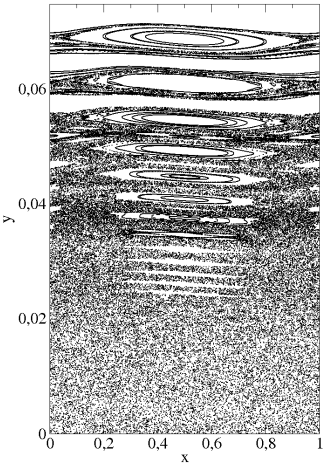

Figure 1 – Phase space generated by the mapping (5) for the control parameters, $a = 2$ and $b = 10^{-3}$.

denoted as ω. In fairness, the behavior of ω shows the same properties of the average $\bar{y}$. It is defined as

$$\omega(n, a, b) = \frac{1}{M} \sum_{i=1}^{M} \sqrt{\overline{y_i^2}(n, a, b) - \overline{y_i}^2(n, a, b)}\ , \qquad (6)$$

where M corresponds to an *ensemble* of different initial conditions $x_i \in (0, 1)$ randomly chosen for a fixed $y_0 = 10^{-3}b$ and $\bar{y}_i$ is given by

$$\overline{y_i}(n, a, b) = \frac{1}{n} \sum_{j=1}^{n} y_{j,i}\ . \qquad (7)$$

The behavior of $\omega \times n$ for different control parameters, as labeled in the figure, is shown in Fig.2. However, similar results would indeed be observed for other values of γ too.

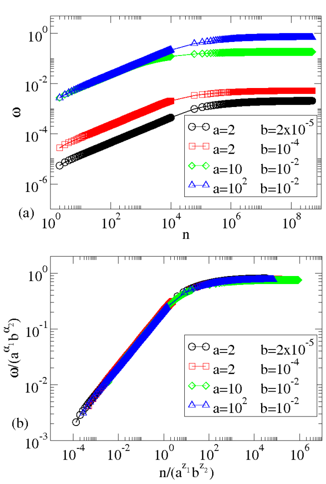

Figure 2 – (Color online) (a) Plot of different ω curves as function of n for different values of a and b for an ensemble of $M = 5000$ different initial conditions. (b) Their collapse onto a single and universal plot.

Let us now discuss the behavior observed in Fig 2. The curves start growing for small n and after reaching a critical crossover iteration number, n_x, they bend toward a regime of convergence. Based on the behavior seen in Fig. 2(a) we can suppose that:

- (i) For $n \ll n_x$, ω grows according to a power law of the type
$$\omega \propto \left(nb^2\right)^{\beta}, \qquad (8)$$
where β is a critical exponent;
- (ii) For large n, say $n \gg n_x$, the behavior of ω is
$$\omega \propto a^{\alpha_1} b^{\alpha_2}, \qquad (9)$$
where α_1 and α_2 are critical exponents;
- (iii) The crossover n_x, that characterizes the transition of the growing regime for the saturation is
$$n_x b^2 \propto a^{z_1} b^{z_2}, \qquad (10)$$
where z_1 and z_2 are called as dynamical exponents.

The critical exponents α_1, α_2 and the dynamical exponents z_1 and z_2 can be obtained from extensive numerical simulations. Firstly, fitting the initial regime of growth, we obtain that the critical $\beta \cong 0.5$. We have obtained $z_2 = -0.757(4)$, $\alpha_2 = 0.607(1)$, $z_1 = 1.162(5)$ and (d) $\alpha_1 = 0.587(1)$. Since we have now obtained the critical exponents, the scaling hypotheses can be verified. In this case, it is shown in Fig 2(b) a merger of four different curves of ω generated for different values of the control parameters a and b into a single and universal plot. Finally, the critical exponents could be used to define classes of universality and compared to other kinds of transition observed in dynamical systems.

To summarize our conclusions, we have studied in this work a phase transition from integrability for non-integrability for a two dimensional Hamiltonian map. The critical exponents were obtained via extensive simulations and scaling hypotheses were all supported by a perfect collapse of all the curves of the deviation around the average quantities for the chaotic sea.

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