DYNAMICS DAYS SOUTH AMERICA 2010
International Conference on Chaos and Nonlinear Dynamics

INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – July 26-30, 2010

The Role of Periodic Boundary Conditions on the Synchronization of Coupled Oscillators in a Ring

Hassan F. El-Nashar[1,2], *Hilda A. Cerdeira*[3]

[1] Department of Physics, Faculty of Science, Ain Shams University, 11566 Cairo, Egypt, hfelnashar@gmail.com
[2] Department of Physics, Faculty of Eductation, Alkharj University, P.O.Box 21034, 11942 Alkharj, K.S.A
[3] Instituto de Fìsica Téorica, UNESP-Universidade Estadual Paulista, Caixa Postal 70532-2, 01156-970, São Paulo, SP, Brazil, cerdeira@ift.unesp.br

Abstract

We study the synchronization of N nearest neighbors coupled oscillators in a ring. At synchronization, we always find two distinct quantities which characterize four of the oscillators, two pairs of nearest neighbors, only two of them have a phase difference of $\pm\pi/2$. We use N-1 equations of the time evolution of the phase differences between neighboring oscillators to derive an analytic form for the phase difference among neighboring oscillators, which shows the dependency on the periodic boundary conditions. Therefore, we build a technique based on geometric properties and numerical observations to arrive to an exact analytic expression for the coupling strength, where synchronization occurs, as well as to directly point to the two oscillators that have a phase difference equal to $|\pi/2|$.

keywords: Coupled Oscillators, Synchronization.

Introduction: In recent years we have seen coupled oscillators to be used to understand the behavior of systems in physics, chemistry, biology, neurology as well as other disciplines, to model several phenomena such as: Josephson junction arrays, multimode Lasers, vortex dynamics in fluids, biological information processes, neurodynamics [1,2]. The oscillators in systems have been observed to synchronize themselves to a common frequency, when the coupling strength between oscillators is increasing [3]. The synchronization features of many of the above mentioned systems might be described using a simple model of weakly coupled phase oscillators such as the Kuramoto model [4,5] as well as its variations to adapt it for finite range interactions which are more realistic to mimic many physical systems. In this context, we present a simplified version of the Kuramoto model with nearest neighbors coupling in a ring topology, which is a good candidate to describe the dynamics of coupled systems with local interactions. Several reports exist where the Kuramoto model with nearest neighbor coupled oscillators in a ring has been used to represent the dynamics of a variety of systems, such as Josephson junctions, coupled lasers, neurons, chains with disorders, multi-cellular systems in biology and in communication systems [6].

MODEL: The local model of nearest neighbor interactions is expressed as

$$\dot{\theta}_i = \omega_i + K\left(\sin\phi_i - \sin\phi_{i-1}\right), \qquad (1)$$

with periodic boundary conditions $\theta_{i+N} = \theta_i$ and phase difference $\phi_i = \theta_{i+1} - \theta_i$ for $i = 1,2,...,N$. The set of the initial values of frequencies $\{\omega_i\}$ are the natural frequencies which are taken from a Gaussian distribution and K is the coupling strength. These nonidentical oscillators (1) cluster in time averaged frequency until they completely synchronize to a common value given by the average frequency $\omega_0 = \sum_{i=1}^{N} \omega_i / N$ at a critical coupling K_c as shown in Fig.(1). At the vicinity of K_c, major clusters of successive oscillators have sets of nearest neighbors at the borders. An interesting fact emerges: the phase-locked solution of $|\pi/2|$ is always valid for one and only one phase difference, and this is the difference between the phases of the two oscillators at the border of the clusters [7].

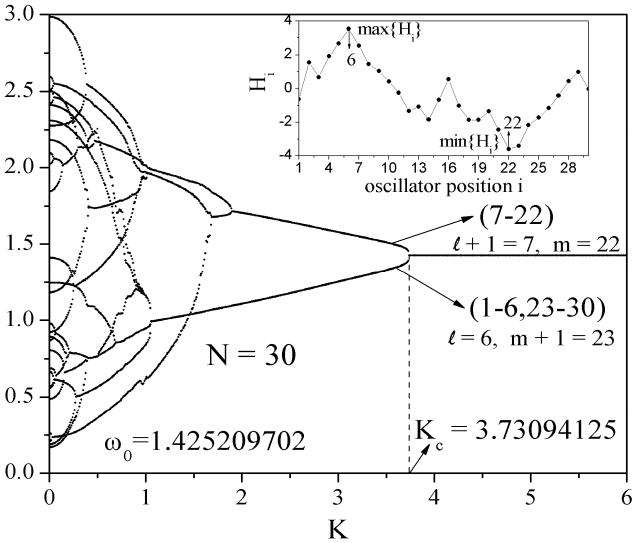

Figure 1 – Synchronization tree for 30 oscillators. The inset shows H_i vs i, where max{Z_i} corresponds to Z_ℓ and min{Z_i} to Z_m.

Figure (1) shows that the four oscillators, now labeled ℓ, ℓ+1, m, and m+1, at the borders of the major clusters in the vicinity of K_c, while Fig. (2) shows that $\sin(\phi_\ell)$ and $\sin(\phi_m)$ are always the maximum and the minimum of the $\sin(\phi_i)$ for all phase differences and only one of them satisfies the phase-lock condition. For any two neighboring oscillators, the equation for the time evolution of the phase difference is

$$\dot{\phi}_i = \Delta_i - 2K\sin(\phi_i) + K\sin(\phi_{i-1}) + K\sin(\phi_{i+1}), \qquad (2)$$

Where $\Delta_i = \omega_{i+1} - \omega_i$. At K_c, using equation (2), we arrive to

$$K_c \sin(\phi_i) = H_i + K_c \sin(\phi_N) \qquad (3),$$

where, $H_i = \dfrac{(N-i+1)}{N}\left[\sum_{i=1}^{N-1} i\Delta_i + \left(\sum_{j=1}^{i-1} j\Delta_j\right)\delta_{ij}\right]$.

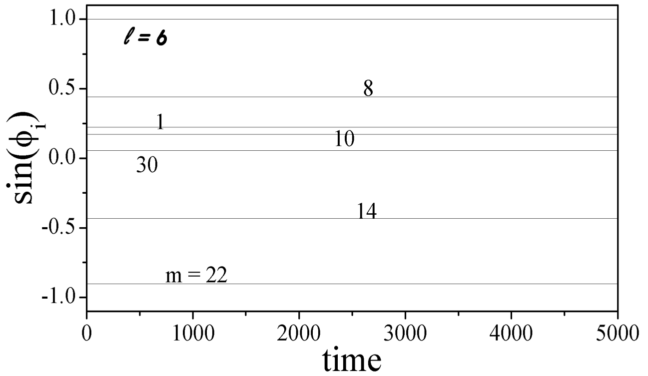

Figure 1 – Selected values of $\sin(\phi_i)$ at K_c for the system of 30 oscillators of Fig. 1.

Equation (3) shows that for the two oscillators which have $|\sin(\phi_j)| = 1$, the critical coupling K_c depends on H_j in addition to the periodic boundary conditions multiplied by K_c. However, it is not possible to determine directly the values of K_c and $\sin(\phi_N)$, or either ϕ_ℓ or ϕ_m, where one of them satisfies the phase-lock condition $|\pi/2|$. It should be noted that for the case without periodic boundary condition (chain of free ends), the value of $K_c = \max\{H_i\}$. Thus, according to (3), for the phase difference ϕ_j satisfies the phase-lock condition, the periodic boundary conditions decides finally the K_c, with $\max\{|H_j|\} \neq K_c$. Therefore, we decide to study the characteristics of the quantity H_i. We find that always the maximum and the minimum values of such quantity refer to the four oscillators which have phase differences ϕ_ℓ and ϕ_m. As shown in the inset of Fig. (1), two distinct quantities are labeled H_ℓ and H_m (a study for different numbers of oscillators and different sets of $\{\omega_i\}$ shows the same behaviors for the quantity H_i). Two major cases can be identified depending on the maximum and the minimum of H_i. When $H_\ell > 0$ and $H_m < 0$, we find two sub-cases and only one of them decides the exact value of the critical coupling. These two sub-cases are $\sin(\phi_l) = 1$ with $\sin(\phi_m) > -1$, or $\sin(\phi_l) < 1$ with $\sin(\phi_m) = -1$. For the first sub-case, we use three equations of system (3) for *l*, *l-1* and *m*, to get the sine law $\dfrac{\sin(\alpha/2)}{a} = \dfrac{\sin(\beta/2)}{b} = \dfrac{\sin(\gamma/2)}{c}$, of the triangle shown in Fig. (3), where the angles are $\alpha = -\phi_N + \pi/2$, $\beta = -\phi_m + \pi/2$ and $\gamma = -\phi_\ell + \pi/2$, and the sides are $a = \sqrt{H_\ell}$, $b = \sqrt{H_\ell - H_m}$ and $c = \sqrt{H_\ell - H_{\ell-1}}$.

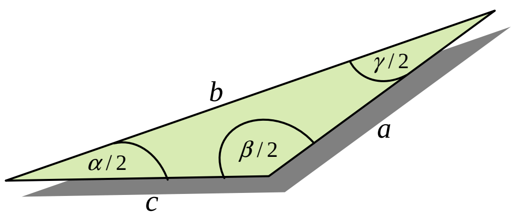

Figure 3- Triangle of known sides.

After some trigonometric manipulation we get $\beta = 2\cos^{-1}\left\{\dfrac{a^2 + c^2 - b^2}{2ac}\right\}$, from which we can calculate the value of the angle ϕ_m. Thus the value of the critical coupling becomes: $K_c^\ell = \dfrac{H_\ell - H_m}{1 - \sin(\phi_m)}$. Applying the same method to the second sub-case ($\sin(\phi_l) < 1$ with $\sin(\phi_m) = -1$), we get $K_c^m = \dfrac{-(H_m - H_l)}{1 + \sin(\phi_l)}$. The value of the critical coupling in the case $H_\ell > 0$ and $H_m < 0$ is

$$K_c = \max\{K_c^\ell, K_c^m\} \quad . \qquad (4)$$

Following the same method for the case $H_\ell < 0$ and $H_m > 0$, we find: $K_c^\ell = \dfrac{-(H_\ell - H_m)}{1 + \sin(\phi_m)}$ and $K_c^m = \dfrac{H_m - H_l}{1 - \sin(\phi_l)}$. The value of the critical coupling is given by equation (4). Table (1) shows good agreement between the results from numerical simulations of system (1) when compared to the values obtained from equation (4) for the same sets of initial frequencies.

N	K_c: simulation	K_c: equation (4)
30	3.73094125	3.72862539
50	4.48415639	4.47935214
100	5.86827841	5.86639415
200	7.96892973	7.95857428

Table 1 – Values of K_c from simulation of (1) and from (4).

Equation (4) allows us to determine whether ϕ_ℓ or ϕ_m has the phase-lock condition $|\pi/2|$. Determining such phase difference in addition to the value of K_c, we can determine the value of ϕ_N. Also, the following phase relations are satisfied at the stage of synchronization:

a) $|\sin(\phi_l)| = 1$, then

$$|\phi_N + \phi_m + \phi_{\ell-1}| = \frac{\pi}{2} \qquad (5)$$

b) $|\sin(\phi_m)| = 1$, then,

$$|\phi_N + \phi_{m-1} + \phi_\ell| = \frac{\pi}{2} \qquad (6)$$

References

[1] A. T. Winfree, "Geometry of Biological Time", (Springer, New York, 1990).

[2] S. Manrubbia, A. Mikhailov and D. Zanette, "Emergence of dynamical Order: Synchronization Phenomena in Complex Systems", (World Scientific, Singapore, 2004).

[3] H. Haken, "Brain Dynamics: Synchronization and activity Patterns in Pulse-Coupled Neural Nets with Delays and Noise", (Springer, Berlin, 2007).

[4] Y. Kuramoto, "Chemical Oscillations, Waves and Turbulences", (Springer, Berlin, 1984).

[5] J. A. Acebron, L. L. Bonilla, C. J. P. Vicente, F. Ritort and R. Spigler, Rev. Mod. Phys. 77, 137 (2005).

[6] Hassan F. El-Nashar and Hilda A. Cerdeira, CHAOS, 19, 033127 (2009).

[7] H. F. El-Nashar, P. Muruganandam, F. F. Ferreira, and H. A. Cerdeira, Chaos 19, 013103 (2009).