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DIRECT HAMILTONIZATION - THE GENERALIZATION OF THE ALTERNATIVE HAMILTONIZATION

Maria Lewtchuk Espindola[1]

[1]Universidade Federal da Paraíba, DM/CCEN, João Pessoa, PB, Brazil, mariia@mat.ufpb.br

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1. INTRODUCTION

The direct Hamiltonization procedure for mechanical systems generalize the alternative one. Alternative or two fold Hamiltonization procedure is a generalized approach for the foundations of Analytical Mechanics, as it can be applied to any mechanical system described by a Lagrangian function. It also reproduces the usual Hamiltonization as a particular case in the non singular systems and furnishes the same Hamiltonian in the singular one as the Dirac's theory, but without any additional constraints [1–6].

Direct Hamiltonization procedure can be applied to non Lagrangian, Nambu, non holonomic and dynamical systems [5, 7, 8].

2. THE TWO FOLD HAMILTONIZATION PROCEDURE

A generalized approach to the foundations of Analytical Mechanics is furnished by alternative Hamiltonization procedure, the so called two fold Hamiltonization Procedure [1, 2].

Suppose a mechanical system with N degrees of freedom, described by the Lagrangian function $L(q, \dot{q}, t)$, where $q = q^{1}, q^{2}, \ldots, q^{N}$ and $\dot{q} = dq/dt$.

This procedure is developed defining the Hamiltonian function as one of the solutions of the partial differential equation obtained replacing the first set of the Hamilton equations of motion[1]

$$\dot{q}^{\,i} = \partial H/\partial p_{\,i} \tag{1}$$

in the usual definition of the Hamiltonian function from a Lagrangian one[2]

$$H(q, p, t) = p_{\,i}\, \dot{q}^{\,i} + L(q, \dot{q}, t), \tag{2}$$

where $p = p_1, p_2, ..., p_N$. Therefore the partial differential equation that defines the Hamiltonian function is

$$H = p_{\,i}\, \partial H/\partial p_{\,i} + L\,(\partial H/\partial p, q, t)\,, \tag{3}$$

whose solutions are the possible Hamiltonian functions. The choice of a solution is arbitrary, and can be the most simple.

This partial differential equation has a complete solution linear at the momenta given by

$$H = p_{\,i}\, A^{\,i} - L(q, A, t), \tag{4}$$

where the $A^{\,i\,\prime}$s are arbitrary functions of $q^{\,k}$ and t. This arbitrary functions are determined by the first set of canonical equations of motion (1) and the Euler-Lagrange equation of motion for the mechanical system

$$\Phi_i(\ddot{q},\ \dot{q}, q, t) = 0, \tag{5}$$

i.e.,

$$\Phi_i(\dot{A},\ A, q, t) = 0. \tag{6}$$

If the Lagrangian function is not singular then the equation (3) also has a envelope solution, determined by the conditions

$$\partial H/\partial A^{\,i} = 0. \tag{7}$$

Therefore if the Lagrangian function L is a regular (non singular) function then exists at least two different manner of Hamiltonizate the mechanical system: (a) the Hamiltonian is a particular solution given by (4); (b) the Hamiltonian is the envelope solution obtained from (7).

The momenta is obtained from the second set of the Hamilton equations of motion

$$\dot{p}_{\,i} = -\partial H/\partial q^{\,i}. \tag{8}$$

The usual Hamiltonization and the usual definition of the momentum, i.e., $p_{\,i} = \partial L/\partial \dot{q}^{\,i}$, when they exist, is recovered by the envelope solution of the above equation.

Any mechanical systems, singular or not, can be Hamiltonizated by this procedure as can be seen in several previous papers [1–5].

The singular Mechanics is one of the most important application of the alternative Hamiltonization, since in this case the partial differential equation which defines the Hamiltonian function is a linear one, then no envelope solution exists.

[1]Latin letters run from 1 to N.
[2]Repeated indices denotes sum (Einstein convention).

Therefore the usual Hamiltonization cannot be reproduced as they do not exist. Although the alternative Hamiltonization procedure gives the same Hamiltonian function as in Dirac theory for singular systems, there is no additional constraint to the Hamiltonian function. Also there is no need of new definition, as that made by Dirac, of weak equalities, or superphase space, as all the results is obtained in the usual phase space. As well as there is no necessity of new variational techniques to develop the procedure.

The main change in the foundations of Analytical Mechanics produced by the alternative Hamiltonization is that there is no definition a priori of the new variable, the so called generalized momenta to obtain the Hamiltonian function, as soon as it is a natural consequence of the imposition of a canonical description.

In the similar manner the procedure is developed for field theory [6], where the Hamiltonian density is the solution of a partial differential equation with variational derivatives.

There is a large number of possible application of alternative Hamiltonization procedure as the linearization of the Hamilton-Jacobi equation [1, 2, 4].

3. DIRECT HAMILTONIZATION

The generalization of the alternative Hamiltonization is given by direct Hamiltonization for mechanical systems [7]. From the Hamiltonization procedure developed previously it can be stated that any function with the form

$$H = p_{\,i}A^{i}(q,t), \tag{9}$$

will be a Hamiltonian function since the canonical equations of motion is satisfied. The variational procedure sets that the function $A^{i}(q,t)$ is defined by the solutions of the system obtained form the first group of equations de Hamilton as

$$A^{i}(q,t) = \dot{q}^{\,i} = \partial H/\partial p_{\,i}, \tag{10}$$

and the equations of motion that describes the system

$$\Phi_i(\ddot{q},\ \dot{q},q,t) = 0, \tag{11}$$

i.e.,

$$\Phi_i(\dot{A},\ A,q,t) = 0. \tag{12}$$

The momenta of the system is given by second set of canonical equations of motion as

$$\dot{p}_{\,i} = -\partial H/\partial q^{\,i}. \tag{13}$$

It can be easily proved that the addition of an arbitrary function $f(q,\dot{q},t)$ to the Hamiltonian function, for example, the Lagrangian one, do not change the Hamiltonian function, implying only in the redefinition of the conjugate momenta of the system.

4. FINAL REMARKS

The direct Hamiltonization is a generalization of the alternative Hamiltonization and recover this procedure when to the Hamiltonian function is added the Lagrangian one.

As in this procedure there are no restriction about the number of equations of motion describing the system it can be applied to non Lagrangian, Nambu [8], non holonomic [5] and dynamical systems.

It is interesting emphasize that the alternative Hamiltonization procedure applied to the non holonomic systems induced the procedure of direct Hamiltonization, as for these systems there is no Lagrangian function and an auxiliar one is used.

An important result of the direct Hamiltonization procedure is obtained when it is applied to the Nambu Mechanics proving that this Mechanics is an usual Hamiltonian one, and not a generalization [8].

There is a lot of possible applications and extensions of the direct Hamiltonization procedure. One of the most important is the development of this procedure to field theories and the following direct Hamiltonization of gravitational fields. Another one is the Hamiltonization of dynamical systems.

It is interesting draw a comparison between the quantizations of the different Hamiltonian functions obtained from the alternative and the direct Hamiltonization procedures.

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