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DESCRIBING A PHASE TRANSITION IN THE DYNAMICS OF A PARTICLE MOVING IN A TIME-DEPENDENT POTENCIAL

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Abstract: Some dynamical properties for a classical particle confined in an infinitely deep box of potential containing a periodically oscillating square well are studied. The dynamics of the system is described by a two dimensional nonlinear area preserving mapping for the variables energy and time. The phase space is mixed. We will find the Lyapunov exponents and show that the system has chaotic components.

Dynamical systems described by mappings have been considered widely along the last years. In special and for the most simple cases, i.e. for systems with 1 and $1/2$ degrees of freedom, that correspond to a time perturbation in a system with one-degree of freedom, the description of Hamiltonian systems lead many times to two dimensional non-linear area preserving mappings. Many different applications of the formalism of two-dimensional area preserving mappings are observed, particularly in the study of magnetic field lines in toroidal plasma devices with reversed shear (like tokamaks) [1], waveguide, Fermi acceleration, billiards and many others generalizations.

The dynamics of the model is described by a two-dimensional non-linear area preserving mapping for the variables energy and time. The phase space of the model is of mixed type in the sense that Kolmogorov-Arnold-Moser (KAM) islands are observed surrounded by a chaotic sea which is characterized by a positive Lyapunov exponent. The *size* of the chaotic sea depends on the control parameters and is limited by a set of invariant tori (also called as invariant spanning curves) which prevents the energy of the particle to growth unlimited. Thus, if the law which controls the time perturbation of the moving well is smooth enough, Fermi acceleration (unlimited energy growth of the particle) is not observed.

The model consists of classical particle which is confined inside a box of an infinitely deep potential which contains an oscillating square well in the middle. A typical sketch of the potential is shown in Fig. 1.

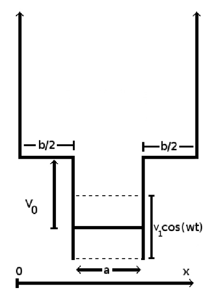

Figure 1 – Sketch of the potential considered.

The potential $V(x,t)$ is given by

$$V(x,t) = \begin{cases} \infty, \text{ if } x \leq 0 \text{ or } x \geq (a+b) \\ V_0, \text{ if } 0 < x < \frac{b}{2} \text{ or } (a+\frac{b}{2}) < x < (a+b) \\ V_1\cos(\omega t), \text{ if } \frac{b}{2} \leq x \leq (a+\frac{b}{2}) \end{cases}, \quad (1)$$

where the control parameters a, b, V_0, V_1 and ω are constants.

We can see that there are too many control parameters which are not all relevant to describe the dynamics, five in total, namely a, b, V_0, V_1 and ω. Defining dimensionless variables we obtain $\delta = V_1/V_0$, $r = b/a$, $e_n = E_n/V_0$, $N_c = \omega/(2\pi)\ (a/\sqrt{2V_0/m})$ and measure the time in terms of the number of oscillations of the moving well, $\phi = \omega t$. With this set of new control parameters, the mapping is written as

$$T : \begin{cases} e_{n+1} = e_n + \delta[\cos(\phi_n + i\Delta\phi_a) - \cos\phi_n] \\ \phi_{n+1} = \phi_n + i\Delta\phi_a + \Delta\phi_b \ \ mod 2\pi \end{cases}, \quad (2)$$

where the auxiliary variables are given by

$$\Delta\phi_a = \frac{2\pi N_c}{\sqrt{e_n - \delta\cos(\phi_n)}} \ , \ \Delta\phi_b = \frac{2\pi N_c r}{\sqrt{e_{n+1} - 1}}.$$

Since the determinant of the Jacobian matrix is equal to the unity, the mapping (2) is area preserving. The phase space of the model is mixed containing both KAM islands, chaotic sea and invariant spanning curves, as one can see in Fig. 2.

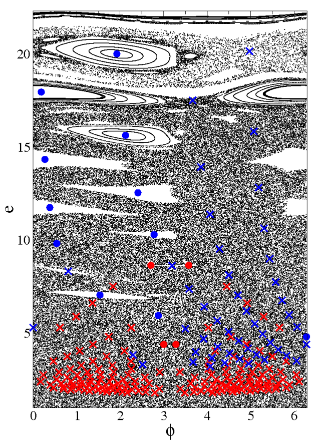

Figure 2 – (Color online) Phase space for the mapping (2). The circles correspond to the elliptical fixed points while the crosses denote the hyperbolic ones. The red (gray) color is used to the first treatment and the blue (black) color to the second treatment. The control parameters used were $r = 1$, $N_c = 33.18$ and $\delta = 0.5$.

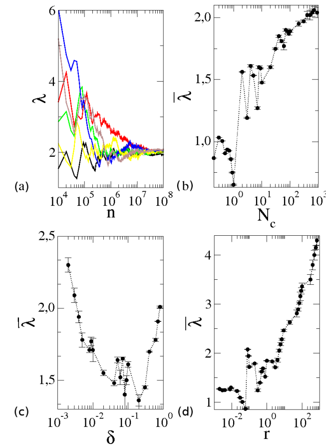

Figure 3 – (Color online) (a) Time evolution of the positive Lyapunov exponent for six different initial conditions evolved up to 5×10^8 times. The control parameters used were $r = 1$; $N_c = 500$ and $\delta = 0.5$. (b) Plot of $\bar{\lambda} \times N_c$ for fixed $r = 1$ and $\delta = 0.5$. (c) Plot of $\bar{\lambda} \times \delta$ for fixed $r = 1$ and $N_c = 33.18$. (d) Plot of $\bar{\lambda} \times r$ for fixed $N_c = 33.18$ and $\delta = 0.5$.

Let us now discuss our numerical results. We start the discussion presenting the Lyapunov exponents characterizing the chaotic properties of the chaotic sea at low energy. It is well known that the Lyapunov exponents are widely used for the characterization of chaotic properties in dynamical systems. Basically, the procedure to obtain the Lyapunov exponents consists in verify if two nearby initially trajectories diverge exponentially for an infinitely long time. If the system exhibits at least one positive Lyapunov exponent, then it has chaotic components. The Lyapunov exponents can be obtained by

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \ln |\Lambda_j{}^{(n)}|, \; j = 1, 2 \qquad (3)$$

where $\Lambda_j{}^{(n)}$ are the eigenvalues of the matrix $M = \prod_i^n J_i(e_i, \phi_i)$ and J_i is the Jacobian matrix of our system.

It is shown in Fig. 3(a) a plot of the positive Lyapunov exponent as function of n for six different initial conditions randomly chosen along the chaotic sea. The control parameters used were $r = 1$, $N_c = 500$ and $\delta = 0.5$. After an initial fluctuation, the positive Lyapunov exponent converges to a constant value for large enough n. It is also important to obtain the behavior of λ as function of the control parameters N_c, r and δ. Figure 3(b) shows a plot of $\bar{\lambda} \times N_c$ for fixed $r = 1$ and $\delta = 0.5$. One can see that the positive Lyapunov exponent varies from $\bar{\lambda} \approx 0.5$ for $N_c = 1$ up to $\bar{\lambda} \approx 2$ for $N_c = 10^3$. It also has a monotonic tendency to growth as function of N_c. Note however that increasing N_c corresponds to raising the number of oscillations of the well and consequently increasing the *randomness* of the system, therefore leading to an increase in the Lyapunov exponent. A plot of $\bar{\lambda} \times \delta$ is shown in Fig. 3(c). The control parameters used were $r = 1$ and $N_c = 33.18$. One can also see that small values of δ, which correspond to very small fluctuations of the oscillating square well produce a large Lyapunov exponent. A minimum value of $\bar{\lambda} \approx 1.4$ was observed for $\delta \approx 0.2$. Finally, a plot of $\bar{\lambda} \times r$ is shown in Fig. 3(d) for fixed $\delta = 0.5$ and $N_c = 33.18$. Since the control parameter $r = b/a$, enlarging r for a fixed N_c corresponds to enlarging b thus increasing the distance from the well up to the box of potential. Such an increase leads to a long flight of the particle until next entrance in the oscillating square well therefore yielding in an increase of the number of oscillations of the moving well and consequently increasing the *randomness* of the system. The sudden jumps in the behavior of the Lyapunov exponent are explained as the destruction of invariant spanning curves leading to a joint of different chaotic regions (see for example Ref. [2] for a discussion in Fermi-Ulam model and Ref. [3] for time dependent square well).

1. ACKNOWLEDGMENTS

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