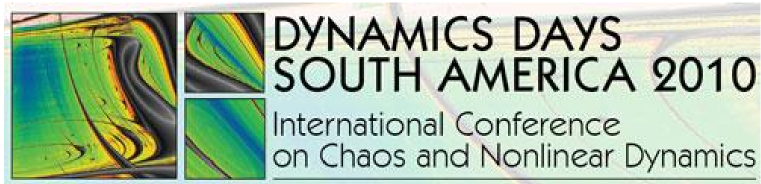

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BIFURCATIONS OF SMOOTH AND LURCHING WAVES IN A THALAMIC NEURONAL NETWORK

Jaime E. Cisternas[1], *Thomas M. Wasylenko*[2], *Carlo R. Laing*[3], *Ioannis G. Kevrekidis*[4]

[1]Universidad de los Andes, Santiago, Chile, jecisternas@miuandes.cl
[2]Massachusetts Institute of Technology, Cambridge, U.S.A., twasylen@mit.edu
[3]Massey University, Auckland, New Zeland, c.r.laing@massey.ac.nz
[4]Princeton University, Princeton, U.S.A., yannis@princeton.edu

Abstract: In this work we consider a one-dimensional lattice of neurons, where each lattice point has both an excitatory thalamocortical and an inhibitory reticular thalamic neuron. Such networks are known to support “lurching” waves, for which propagation is not smooth but rather progresses in a saltatory fashion. We show that these lurching waves are fixed points of appropriately defined Poincaré maps, and follow these fixed points as parameters are varied. By doing this we are able to explain the observed transitions in behavior. Our analysis is quite general and could be applied to other spatially extended systems which show non-trivial forms of wave propagation.

keywords: Applications of Nonlinear Sciences, Bifurcation theory and analysis, Neuronal Dynamics.

1. INTRODUCTION

Several models of thalamic networks have been proposed, most of them based on two distinct populations of neurons. Here we based our work on the model proposed by Rinzel, Terman, and coworkers [1, 2], which exhibits smooth propagating pulses (Fig. 1(a)) and other waves (Fig. 1(b,c,d)) that propagate by “lurching” a certain number of sites in the lattice.

These waves persist as parameters are changed, resulting in “branches” of solutions; their existence and stability can be analyzed numerically using bifurcation theory.

2. METHODS

The model considers a one dimensional lattice, where each site $i = 1 \dots N$ consists of two neurons. The first cell is an excitatory thalamocortical (TC) neuron and its state is described by the variables v^{TC} (voltage) and h^{TC} (relative suppressive influence). The second cell at the site is an inhibitory reticular (RE) thalamic neuron and is described by the variables v^{RE} and h^{RE}. We have considered a symmetric

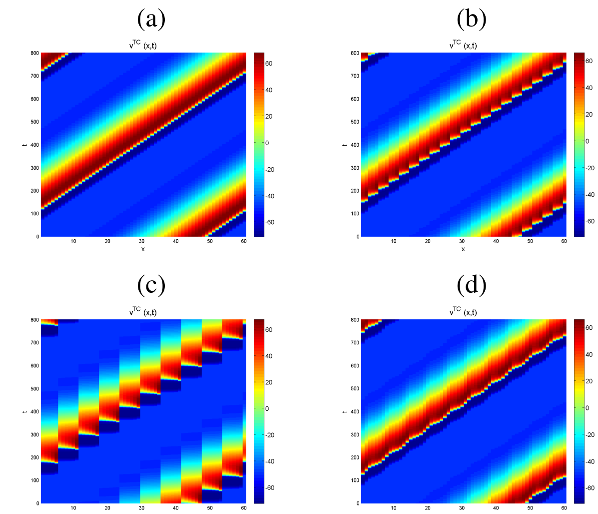

Figure 1 – Spacetime diagrams that show representative behaviors of the network: (a) smooth travelling wave; (b) 3-lurcher; (c) 6-lurcher; and (d) quasiperiodic wave.

synaptic footprint (Fig. 2). The equations of the model:

$$\begin{aligned}
\frac{dv_i^{\mathrm{TC}}}{dt} &= -g_L^{\mathrm{TC}}(v_i^{\mathrm{TC}} - e_L^{\mathrm{TC}}) \\
&\quad -g_{\mathrm{Ca}}(m_\infty(v_i^{\mathrm{TC}}))^3 h_i^{\mathrm{TC}}(v_i^{\mathrm{TC}} - e_{\mathrm{Ca}}) \\
&\quad -g_{\mathrm{TC}} s_\infty(v_i^{\mathrm{RE}})(v_i^{\mathrm{TC}} - e_{\mathrm{syn}}^{\mathrm{RE}}), \qquad (1)\\
\frac{dh_i^{\mathrm{TC}}}{dt} &= \epsilon^{\mathrm{TC}} \frac{h_\infty(v_i^{\mathrm{TC}}) - h_i^{\mathrm{TC}}}{\tau_h(v_i^{\mathrm{TC}})}, \qquad (2)\\
\frac{dv_i^{\mathrm{RE}}}{dt} &= -g_L^{\mathrm{RE}}(v_i^{\mathrm{RE}} - e_L^{\mathrm{RE}}) \\
&\quad -g_{\mathrm{Ca}}(m_\infty(v_i^{\mathrm{RE}}))^3 h_i^{\mathrm{RE}}(v_i^{\mathrm{RE}} - e_{\mathrm{Ca}}) \qquad (3)\\
&\quad -\left(\frac{\sum_{j=-\omega}^{\omega} g^{\mathrm{RE}} s_\infty(v_{i+j}^{\mathrm{TC}})}{2\omega + 1}\right)(v_i^{\mathrm{RE}} - e_{\mathrm{syn}}^{\mathrm{TC}}),\\
\frac{dh_i^{\mathrm{RE}}}{dt} &= \epsilon^{\mathrm{RE}} \frac{h_\infty(v_i^{\mathrm{RE}}) - h_i^{\mathrm{RE}}}{\tau_h(v_i^{\mathrm{RE}})}, \qquad (4)
\end{aligned}$$

Functions $m_\infty(v)$, $h_\infty(v)$, $s_\infty(v)$, $\tau_\infty(v)$ are bounded and have a sigmoidal shape.

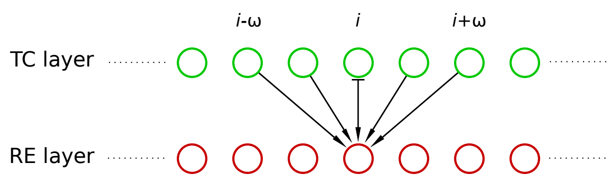

Figure 2 – Inter-neuronal couplings in the 1-d network.

We define a Poincaré map $P_{d,\tau} = \Phi_\tau \circ S_d$ that is the combined action of a spatial shift of d sites to the left (for right-traveling waves), and evolution for time τ. Using this 'shift-and-run' map we could characterize all the solutions as periodic or quasiperiodic orbits. Linear stability can be extracted from the Floquet multipliers μ, that must lie inside the unit circle $|\mu| \leq 1$ for stable solutions. The continuation of the branches (using nondimensional parameter s) was performed using pseudo-arclength method.

3. RESULTS

Pseudo-arclength continuation captured branches of solutions and multiple bifurcations of periodic orbits: saddle-node (SN), Hopf (H) (see Fig. 3(a,b)), period-doubling (PD), as well as saddle-saddle (SS). The Hopf bifurcation connects the traveling wave with quasiperiodic torus (Fig. 1(a,d)). There are two PD bifurcations: a *forward* period-doubling (FPD) connecting a stable 3-lurcher with an unstable 6-lurcher; and a *backward* period-doubling (BPD) connecting a stable 6-lurcher with a stable 12-lurcher.

Further analysis showed clear signs of frequency locking: the torus deforms as it gets closer (as parameter is varied) to a pair of stable/unstable periodic orbits (3-lurchers) born in a saddle-node bifurcation, as depicted in Fig. 3(c,d).

In Fig. 4 a qualitative summary of our findings is presented, showing stable (solid blue) and unstable branches (dashed red). Up to three stable solutions exist for given parameter value, including two types of tori and two 6-lurchers. GB means a global bifurcation: the breakup of torus A is associated to a saddle-node bifurcation of periodic orbits.

4. DISCUSSION AND CONCLUSIONS

Waves in a 1-d neuronal network can be analyzed using a Poincaré map: smooth traveling and lurching waves are represented by periodic orbits, and quasiperiodic waves show up as tori. This approach enabled us to compute stabilities and characterize transitions. Using pseudo-arclength continuation we followed unstable branches and found nontrivial connections between solutions, as well as additional stable solutions.

The existence of lurching waves, both periodic and quasiperiodic, are manifestations of more generic codimension-2 behavior: the interaction of resonance horns

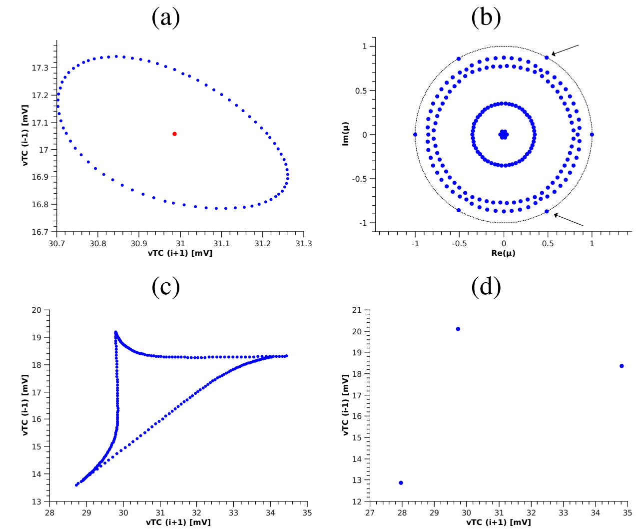

Figure 3 – (a) Poincaré map for torus (blue) born from Hopf bifurcation and unstable periodic point (red). (b) Floquet multipliers of traveling wave close to Hopf bifurcation, arrows indicate bifurcating multipliers. (c) Poincaré map of quasiperiodic torus. (d) Periodic solution corresponding to 3-lurcher.

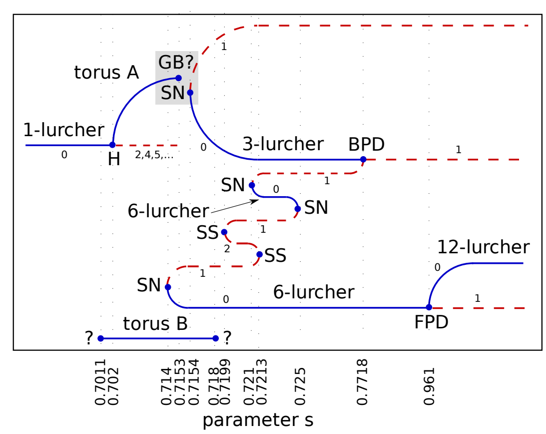

Figure 4 – Schematic summarizing current understanding.

being born from Hopf bifurcations.

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