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Chaotic Dynamics in Mathematical Economics: A Duopoly Model

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In his seminal work, Cournot [1, Chapter 7] constructed the theory of oligopoly in a mathematical framework. His model captured the strategic interactions among a small number of firms in a market for a single homogeneous commodity; each firm tries to maximize her profit by taking the output choices of the other firms as given. The equilibrium point in his model is defined as the intersection of the reaction functions. His equilibrium notion is in essence identical with the Nash equilibrium concept. In this sense Cournot anticipated Nash [2] more than a century ago. There is no disagreement on the point that Cournot's oligopoly model is the classic instance of a non-cooperative game in economics.

The purpose of the present paper is to investigate the stability of the steady state for two dynamic adjustment systems. In the first model, I deal with a system of ordinary differential equations. In the second model, I examine a system of delay differential equations, which is an extended version of the first model.

I consider a duopoly situation: the strategic interactions between firm 1 and firm 2. Let the market price be given by

$$p = a - b(x_1 + x_2), \;\; a > 0, b > 0, \tag{1}$$

where p is the price level and x_i is the output level of firm *i*. Furthermore we assume that the cost function of firm *i* is linear:

$$C_i(x_i) = c_i x_i, \;\; c_i > 0. \tag{2}$$

The profit function for firm *i* is defined as

$$\Pi_i(x_i) = p x_i - c_i x_i, \tag{3}$$

and the profit maximizing levels of output are given by

$$x_1 = R_1(x_2) = -\frac{1}{2}x_2 + \frac{a - c_1}{2b}, \tag{4a}$$

$$x_2 = R_2(x_1) = -\frac{1}{2}x_1 + \frac{a - c_2}{2b}. \tag{4b}$$

In studying the adjustment process, I assume that each firm controls the growth rate of its output according to the difference between its profit maximizing level and its actual level of output:

$$\dot{x}_1 = \alpha_1(R_1(x_2) - x_1)x_1, \;\; \alpha_1 > 0, \tag{5a}$$
$$\dot{x}_2 = \alpha_2(R_2(x_1) - x_2)x_2, \;\; \alpha_2 > 0. \tag{5b}$$

This adjustment process guarantees the global stability of the unique steady state. The proof is established by using Lyapunov's second method.

However, what has to be noticed is that time lags inherent in the adjustment process are unavoidable in the real world. To analyze the problem of time lags, I consider the following system of delay differential equations:

$$\dot{x}_1(t) = \alpha_1[R_1(x_2(t - l_1)) - x_1(t - \tau_1)]x_1(t), \tag{6a}$$
$$\dot{x}_2(t) = \alpha_2[R_2(x_1(t - l_2)) - x_2(t - \tau_2)]x_2(t), \tag{6b}$$

with the delayed arguments, l_i and $\tau_i > 0$.

This model is identical with a model developed by Shibata and Saito [3]. They examined the population dynamics of two species with fixed time lags and concluded that the system could display strange attractors for appropriate parameter values by means of numerical simulation. Thus, their results apply to the adjustment system (6).

Figure 1 summarizes one of main results in this paper: we can observe the emergence of a strange attractor in the adjustment system of a duopoly model. Note that the strange attractor is depicted as a projection onto the (ln x_1, ln x_2) plane. The trajectory of the strange attractor moves in a clockwise direction. In order to convince the exact evidence for chaos, we examine the Lyapunov characteristic exponents. Any system containing at least one positive Lyapunov characteristic exponent is said to be chaotic since it has sensitive dependence on initial conditions. In fact, we obtain the result that the largest Lyapunov exponent is 1.42. This outcome therefore suggests the existence of chaos in our numerical example.

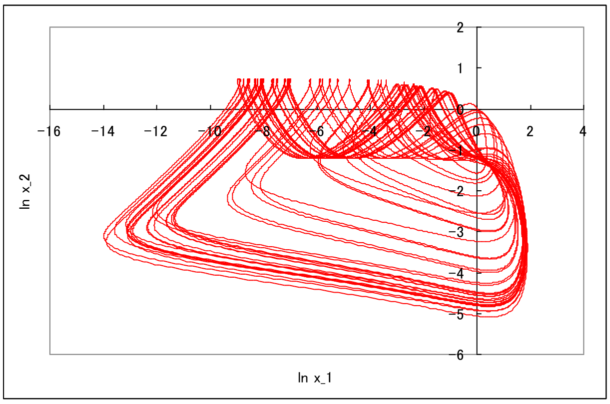

Figure 1 – Strange attractor

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