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# DYNAMIC VALID MODELS FOR THE CONSERVATIVE HÉNON-HEILES SYSTEM

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## 1. INTRODUCTION

When dealing with the solution of conservative nonlinear differential equations, several problems such as energy loss and symmetry break can occur. In order to avoid such problems as much as possible, there are many numerical integration methods available in literature [1]. Although the primary objective of these methods is the solution itself, a possible by-product is a difference equation that hopefully reproduces the same behavior as the one generated by the original differential equations.

In [2] Mickens proposes one such a discretization scheme. The authors of [3, 4] successfully obtain discretized models for a dissipative system using this method. The models found are topologically equivalent to the original continuous system, except for a small displacement in the parameter space. However, for conservative systems, [5] shows that the symmetry is not conserved and numerical instabilities may occur when the discretization step increases.

Monaco e Normand-Cyrot propose a different discretization scheme in [6], which was studied in [7] in the context of the behavior of chaotic dynamical systems. The purpuse of this work is to find valid discretized models, using this discretization scheme, for the Hénon-Heiles system, conserving the energy and symmetry of the solution.

## 2. THE HÉNON-HEILES SYSTEM

The Hénon-Heiles system is a well known nonlinear conservative system and widely studied [1, 5]. It can be described by four ordinary differential equations, as follows:

$$\left\{\begin{array}{lcl} \dot{x} & = & u \\ \dot{y} & = & v \\ \dot{u} & = & -x - 2\,x\,y \\ \dot{v} & = & -x^2 + y^2 - y \end{array}\right. \tag{1}$$

The initial condition $(x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)$, as used in [5], results in the energy equal to $H = 0.128546999$, leading to a chaotic behavior. The Poincaré section under such condition is shown in Figure $1a$. Symmetry, a feature of conservative systems, occurs for $v_n = 0$.

For the variable $x$, Fourier analysis shows that the maximum frequency is $f_{max} \approx 0.75Hz$. Therefore, the maximum discretization step must be less than $0.67s$ to satisfy the Nyquist criteria. As long as the Nyquist criteria is satisfied, numerical instabilities are avoided and discretized models generate solutions which are equivalent to the one of the original system except for a small displacement in the parameter space [3–5].

## 3. MONACO AND NORMAND-CYROT DISCRETIZATION SCHEME

Consider the dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right) \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ are state variables, $\mathbf{f}(.)$ are analytic functions of appropriate dimentions. The derivative of $\mathbf{x}$ with respect to time is denoted by $\dot{\mathbf{x}}$.

The discrete model of eq (2) is given by:

$$\mathbf{x}_{k+1} = \mathbf{g}\left(\mathbf{x}_k, h\right) \tag{3}$$

where $\mathbf{x}_k \in \mathbb{R}^m$ are dynamic variables at time $t = t_0 + kh$, and $h$ is the discretization step.

In [5, 7] it is shown that the discretization, originally proposed by Monaco e Normand-Cyrot [6], can be accomplished by the Lie exponential expansion of eq. (2), as follows:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sum_{n=1}^{\eta} \frac{h^n}{n!} L_f^n\left(\mathbf{x}_k\right) \tag{4}$$

where $\eta$ is the expansion order. The Lie derivative is given by:

$$L_f\left(\mathbf{x}_k\right) = \sum_{j=1}^{m} f_j \frac{\partial \mathbf{x}}{\partial x_j} \tag{5}$$

where $f_j$ represents the $j-th$ component in the vector field. Higher derivative orders can be calculated recursively by:

$$L_f^n\left(\mathbf{x}_k\right) = L_f\left(L_f^{n-1}\left(\mathbf{x}_k\right)\right) \tag{6}$$

Expansion order of eq. (4) should be truncated to avoid an excessive amount of terms, making computational simulations unfeasible.

## 4. RESULTS: HÉNON-HEILES DISCRETIZATION

Poincaré section of the Hénon-Heiles discretized model using the approach proposed by Mickens is shown in Figure $1b$. Symmetry is not conserved, as already observed in [5].

Several discretized models were obtained using the Monaco e Normand-Cyrot method. Each one was simulated with different discretization steps. Low order models, such as $\eta = 3$, were able to reproduce successfully the original continuous Poincaré section for a small discretization step, $h < 0.02s$, without breaking the conversation of the energy and the symmetry. As the discretization order increases, greater discretization steps can be accomplished preserving the original structure similarity. The $12^{th}$ order discretized model is able to successfully reproduce the attractor of the original continuous system for large values of the discretization step. Even though the Poincaré section for $h = 0.67s$, Figure $1c$, is smaller due to a slightly energy loss, and lacks details due to border effect, it is still topologically equivalent to the original one.

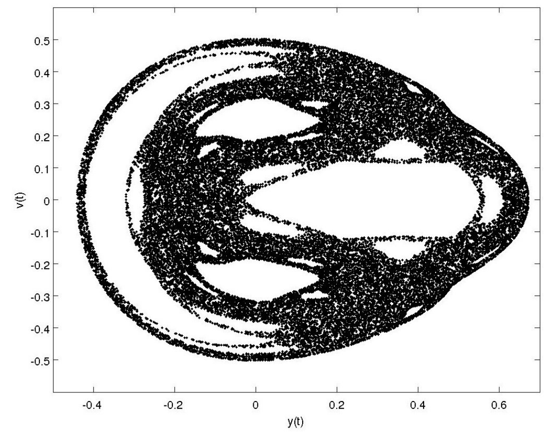
## 5. CONCLUSION

The discretization scheme proposed by Monaco and Normand-Cyrot is a direct and simple method that provides robust discretized models.

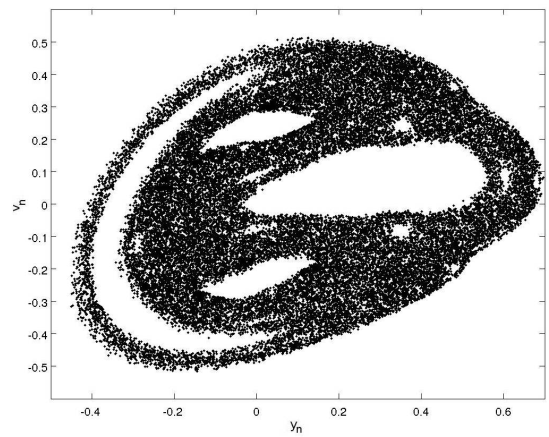
Higher order discretized models successfully reproduce the Hénon-Heiles Poincaré section similar to the original continuous one, even for large discretization steps. Energy was nearly constant till the Nyquist frequency. The symmetry of the conservative system was preserved in all simulations, even for discretization steps above Nyquist criteria. This was not observed in simulations with Mickens discretized models.

Low order discretized models can generate valid dynamic models, but only for small discretization steps.

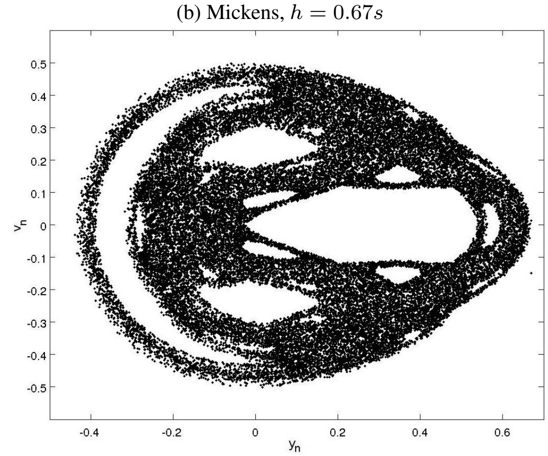
The increase in the discretized model order demands more computational effort. One should consider studying the necessary model approximation and the discretization step used to generate a less computationally demanding model.




(a) Continuous




(b) Mickens, $h = 0.67s$




(c) Monaco and Normand-Cyrot, $12^{th}$ order, $h = 0.67s$

**Figure 1 – Poincaré section of the Hénon-Heiles system, defined by $P \equiv \{(y_n, u_n, v_n) \in \mathbb{R}^3 \mid x_n = 0\}$, with initial condition** $(x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)$**,** $0 \leq t \leq 120000$

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