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OPTIMAL CONTROL IN NOISY CHAOTIC SYSTEMS

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It is well known that a large number of natural [1, 2] and technological [3, 4] systems behave chaotically for some ranges of their parameters. Ott, Grebogi and Yorke (OGY) [5] illustrated not only that chaotic systems described by maps may be controlled, but that the richness of possible behaviors in chaotic systems may be exploited to enhance the performance of a dynamical system in a manner that would not be possible had the system's evolution not been chaotic. Shortly thereafter, Ditto *et. al.* [6] reported a successful laboratory implementation of the control strategy outlined in Ref. [5], demonstrating that controlling chaos is not just a theory, but is physically attainable as well [7]. Pyragas [8] developed these ideas in continuous dynamical systems using a delayed feedback control strategy for unstable periodic orbit (UPO) [9]. For our purpose, it is worth mentioning the study of the effect of an external noise on the controlled system [10], and the search for optimal control strategies using a periodic driving [11], but in which the control strength does not remain small.

In this work we continue along the lines set forth by Pyragas [8]. We begin by describing a fairly simple method to estimate the Lyapunov spectrum of a known UPO. This method will allow us to map the task of controlling the UPO to an optimization problem, as suggested in Refs. [11] and [12]. We will use a driving term that is natural for these types of systems, as suggested by Pyragas [8], and that can converge to a small control effort, consistent with a given noise level. Special attention is given to the problem of handling noise, which can affect considerably the estimation of Lyapunov exponents and the strength of the force required to keep the orbit close to the UPO. This particular exponent estimation method allows us to develop a cleaning strategy for the exponents based on singular value decomposition [13]. Under these noisy conditions it becomes relevant to find optimal control strategies that minimize the effect of noise on the orbit close to the UPO. We illustrate these ideas with Lorenz, Rossler, and the Van der Pol systems, that have a single Lyapunov exponent with a positive real part, and a hyperchaotic system with two Lyapunov exponents with positive real parts [14].

Let us consider a nonlinear dynamical system described by

$$\dot{\vec{x}} = \mathbf{f}(\vec{x}) \; , \tag{1}$$

where $\vec{x}$ is a vector in $\mathbb{R}^d$ and $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear and at least C^1 function. For this paper, controlling an UPO of period τ in this system implies: (a) reducing the asymptotic average distance to the UPO for a given τ; and (b) converging to an asymptotically small average control effort, consistent with a given noise level. Following [12], we can define the instantaneous distance of a trajectory to the UPO as

$$D^2(\vec{x}, t) = |\vec{x} - \vec{x}^*(t)|^2$$

and we take the control as $(1/2)\mathbf{A}\vec{\nabla}_x D^2 \sim \mathbf{A}(\vec{x} - \vec{x}^*(t))$, using an optimization analogy [13]. Hence, for this work we will assume a feedback control scheme

$$\dot{\vec{x}} = \mathbf{f}(\vec{x}) + \mathbf{A}(\vec{x} - \vec{x}^*(t)) \; , \tag{2}$$

where the control is taken as $\vec{C}(\vec{x}, t) = \mathbf{A}(\vec{x} - \vec{x}^*(t))$. Let us note that this is the form suggested by Pyragas [8].

We now describe a numerical method to compute the spectrum λ_i using the linearization of the above problem. There exists a fundamental matrix $\mathbf{B}(t)$ associated with the linearized problem such that a given vector $\vec{v}(t)$ evolves as,

$$\vec{v}(t) = \mathbf{B}(t)\, \vec{v}(0) \; , \tag{3}$$

for any initial vector $\vec{v}(0)$, which requires that $\mathbf{B}(0) = \mathbf{1}$. Now let us take an arbitrary initial basis $\{\vec{v}_1(0), \vec{v}_2(0), \ldots, \vec{v}_d(0)\}$, so that an arbitrary perturbation vector $\vec{\eta}(0)$ around the UPO, can be written as $\vec{\eta}(0) = \sum_{j=1}^{d} c_j \vec{v}_j(0)$, where the set $\vec{c} = \{c_1, \ldots, c_d\}$ satisfies $\sum_{j=1}^{d} c_j = 1$. Hence we need to determine the set of coefficients that corresponds to $\vec{\eta}_i$. The above expression implies that

$$\vec{\eta}(\tau) = \mathbf{B}(\tau) \sum_{j=1}^{d} c_j \vec{v}_j(0) = \sum_{j=1}^{d} c_j \vec{v}_j(\tau) \; , \tag{4}$$

where $\vec{v}_j(t)$ can be found numerically. The above relations can be rewritten in matrix form

$$\left[\mathbf{V}(\tau)\mathbf{V}(0)^{-1} - e^{\lambda\tau}\mathbf{1}\right]\vec{c} = 0\,, \qquad (5)$$

where $\mathbf{V}(t)$ is the known matrix $\{\vec{v}_1(t), \ldots, \vec{v}_d(t)\}$. Notice that $\mathbf{B}(t) = \mathbf{V}(t)\mathbf{V}^{-1}(0)$. Solving the eigensystem (5) yields all the $\vec{\eta}_i$ vectors and their corresponding λ_i exponents.

To handle noisy system, define $H(\vec{x}_0, \tau) = |\vec{x}(\tau) - \vec{x}_0|^2$, which is the function to be minimized. Now we will resort to singular value decomposition (SVD), that is generally used in signal processing of images [13]. We start by constructing the matrix

$$M^i_{k,j} = x^k_i(j\Delta t)\,,$$

where $\Delta t = \tau/N_t$ for some integer N_t. The SVD transformation of $\mathbf{M} = \mathbf{V\Sigma U}^T$. The first column of $\mathbf{U}$ should be proportional to the cleaned orbit, which can be rescaled by $x_i(0)$. For a close neighborhood of the UPO, we repeat this analysis for $i = 1, \ldots, d$, from which we can obtain the value of $H(\vec{x}_0, \tau)$. Then we search for a local minimum of $H(\vec{x}_0, \tau)$ over $\vec{x}_0$ and τ [14].

The second step is to compute the Lyapunov exponents. Since noise is present, standard estimation procedures will give considerable fluctuations. Instead, we estimate finite Lyapunov exponents from Eq. (2), by integrating numerically an initially small, not necessarily infinitesimal, perturbation $\vec{x}(t) = \vec{x}^*_S(t) + \vec{v}^{(k)}(t)$ from $t = 0 \to \tau$, where $\vec{x}^*_S(t)$ comes from the SVD procedure. Next we start all the trajectories with $|\vec{v}^{(k)}(0)| = \delta_0$. We will resort again to singular value decomposition to clean the exponents. In order to compute the matrix $\mathbf{B}(\tau) = \mathbf{V}(\tau)\mathbf{V}^{-1}(0)$ we need to invert the matrix $\mathbf{V}(0)$. We can take a non-square matrix of initial conditions

$$\mathbf{V}(0) = \{\vec{v}_1(t), \ldots, \vec{v}_N(t)\}\,,$$

with $N \geq d$, with d the dimension of the system. These N initial conditions can be chosen at random, or taken from the dynamics of the system each time the trajectory passes close to the UPO, in the case of experiments. Even though $\mathbf{V}(0)$ is a non-square matrix, we can compute its pseudo-inverse [13] and estimate a square $d \times d$ matrix $\mathbf{B}(\tau) = \mathbf{V}(\tau)\mathbf{V}(0)^{-1}$. As an example, we assume a control strategy based on a matrix with the single non-null element $A_{11} = \alpha$, the result is shown in Fig. 1. For further discussion see [14].

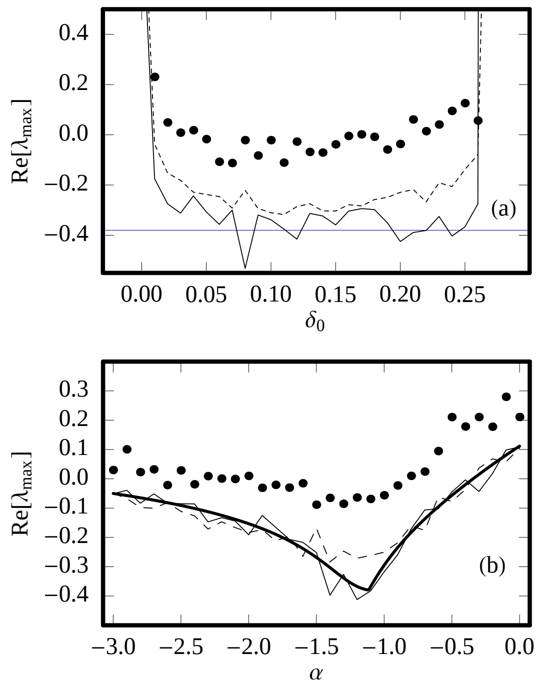

Figure 1 – Maximum exponent $\lambda_{\max}$ calculated for the Rossler system with intrinsic noise. (a) The value of $\lambda_{\max}$ at $\alpha_R = -1.1$ as a function of δ_0 using: the average of 40 sets of $N = 3$ initial conditions (dotted line), the average of 12 sets of $N = 10$ initial conditions (dashed line), and the result of 1 set of $N = 120$ initial conditions (thin line). The estimated value of $\lambda_{\max}$ using the infinitesimal approach is shown as the horizontal thick line. (b) The estimated maximum exponent as a function of α using the SVD cleaning procedure for the same sets as before. We take $\sigma = 0.1/\sqrt{3}$ with $\omega = 100$ Hz. For details see [14].

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