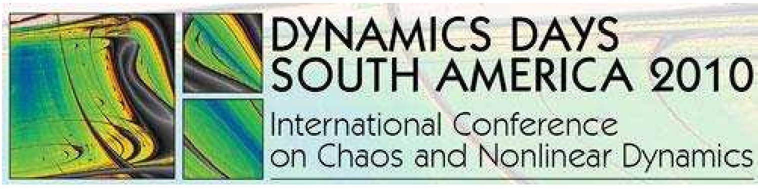

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Reconstruction and Predictability of Stochastic Processes

J. Carvalho[1], *M. Haase*[2], *D. Kleinhans*[3], *P. Lind*[1,4], *J. Peinke*[5], *F. Raischel*[1], *V. Vasconcelos*[1], *M. Wächter*[5]

[1]Center for Theoretical and Computational Physics, University of Lisbon, 1649-003 Lisbon, Portugal, Email: raischel@cii.fc.ul.pt
[2]Institute for High Performance Computing, University of Stuttgart, D-70569 Stuttgart, Germany
[3]Institute for Marine Ecology, University of Gothenburg, SE-405 30 Gothenburg, Sweden
[4]Departamento de Física, Faculdade de Ciências da Universidade de Lisboa, 1649-003 Lisboa, Portugal
[5]Institute of Physics, University of Oldenburg, D-26111 Oldenburg, Germany

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In time-series analysis, complex systems can often be described by Langevin equations [1], the parameters of which can be extracted directly from the data [2]. This approach has been applied successfully to a number of problems [3]. We present here a set of procedures that permit an improved analysis of stochastic time series. The first set of methods allows to extract the stochastic parameters and the noise amplitude present in a Langevin type stochastic time series afflicted with strong measurement noise [4]. The second set deals with transformations of multivariate Langevin type stochastic equations into a different set of variates, one of which displays reduced stochasticity, thus improving the predictability of this component.

We first consider a univariate stochastic time series with an underlying Langevin dynamics

$$\frac{dx}{dt} = D_1(x) + \sqrt{D_2(x)}\Gamma_t, \tag{1}$$

where Γ_t represents a Gaussian δ-correlated dynamical noise. Functions $D_1(x)$ and $D_2(x)$ are the drift and diffusion coefficients, or Kramers-Moyal (KM) coefficients, given as

$$D_n(x) = \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}M_n(x,\tau)\,. \tag{2}$$

As was first shown in [5], the conditional moments

$$M_n(x,\tau) =< \,[x(t+\tau)-x(t)]^n > |_{x(t)=x} \tag{3}$$

can be calculated directly from the time series.

If we assume that the measurement of the time series is accompanied by an additional Gaussian δ-correlated measurement noise, we obtain the measured time series of observations $y(t) = x(t) + \sigma\zeta(t)$, where σ denotes the amplitude of the measurement noise.

The presence of measurement noise does not allow the application of standard methods for the extraction of the KM coefficients [2]. In the case of large amplitudes of measurement noise, approximation procedures [6] for determining the stochastic coefficients and the noise amplitude break down.

The procedure presented here is based on a first order approximation of the noisy KM coefficients. The measured expansion functions can then be compared to their calculated counterparts derived from a quadratic Ansatz. The functional distance between measured and calculated functions is then minimized by an iterative Levenberg-Marquardt method [7]. As a result, a parameter free reconstruction of the Langevin process and the measurement noise amplitude is obtained.

We also present results for the minimization of two-dimensional Langevin-type equations:

$$\begin{aligned}\frac{dX_1}{dt} &= h_1(\vec{X},t) + g_{11}(\vec{X},t)\Gamma_1(t) + g_{12}(\vec{X},t)\Gamma_2(t)\,,\\ \frac{dX_2}{dt} &= h_2(\vec{X},t) + g_{21}(\vec{X},t)\Gamma_1(t) + g_{22}(\vec{X},t)\Gamma_2(t)\,.\end{aligned} \tag{4}$$

The goal here is to find a transform $\vec{X}' = F(\vec{X})$ such that stochasticity in one direction $i = 1, 2$ is minimized, in order to increase the predictability of the time series. We highlight the contrast between a previous approach [8], where a proper functional was minimized by means of an Euler-Lagrange approach, to a new minimization based on a polynomial Ansatz for the Lagrange coefficients and the subsequent minimization through a simulated annealing procedure. The findings of this procedure is then compared to the calculation of the eigenvectors of g_{ij} in the system of coupled Langevin Equations, Eq. (4).

The one-dimensional recovery approach for 1d Langevin equations with strong measurement noise is tested by applying the method to a general nonOrnsteinUhlenbeck process with the drift and diffusion coefficients being linear and quadratic forms, i. e. $D_1(x) = d_{10} + d_{11}x\,, D_2(x) = d_{20} + d_{21}x + d_{22}x^2$. The coefficients and the noise amplitude are shown to be recovered correctly, see Fig. .

In the case where we want to reduce stochasticity in one direction of the multidimensional case, Eq. (4), we focus on

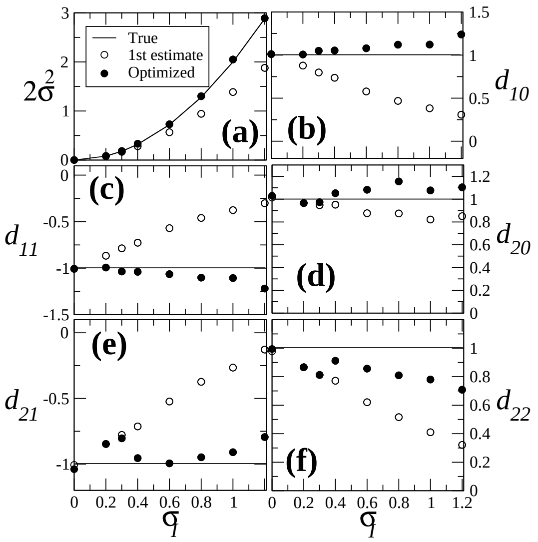

Figure 1 – Optimized parameter values and measurement noise amplitude of a reconstructed Ornstein Uhlenbeck process as a function of noise strength σ_I.

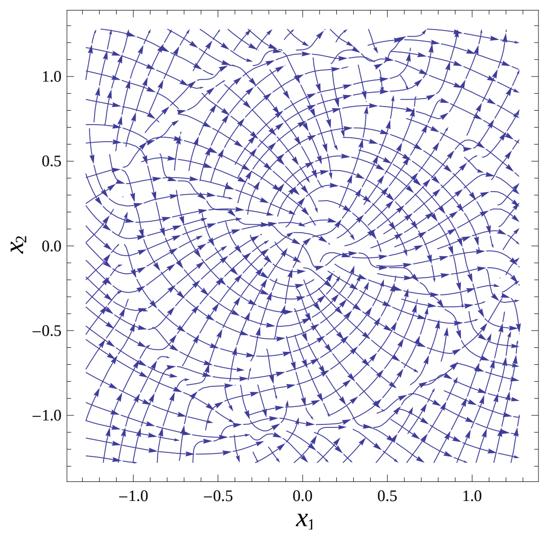

Figure 2 – Stream field of the eigenvectors for a coupled Langevin system taken by deriving the diffusion matrix for the transformed data set $(\mathbf{x_1}, \mathbf{x_2}) = (\mathbf{r}\cos\theta, \mathbf{r}\sin\theta)$ ($\mathbf{10^7}$ data-points). Here $\mathbf{g_{12} = g_{21} = 0}$, $\mathbf{g_{22} = 0.005}$ and $\mathbf{g_{11} = 0.5r}$.

a Hopf oscillator with dynamical noise,

$$\begin{aligned} \frac{dr}{dt} &= r(1-r^2) + g_{11}\Gamma_1 \\ \frac{d\theta}{dt} &= 1 + r^2 + g_{22}\Gamma_2\,, \end{aligned} \tag{5}$$

which for $g_{11} = g_{22} = 0$ shows a limit cycle at $r = 1$ and an unstable focus at $r = 0$.

Here, we consider a stochastic force which near the limit cycle is much larger in the radial direction than in the angular direction, $g_{11}(r) \gg g_{22}(r)\,, r \to 1$. For simplicity we set the coefficients for the mixing of the stochastic forces in Eq. (4) $g_{12} = g_{21} = 0$. Then, transforming the system from polar coordinates to Cartesian coordinates, the new matrix $\{g'_{ij}\}$ should yield locally (for each point (x_1, x_2)) two eigenvalues, one large with an eigenvector in the radial direction, and another smaller one pointing perpendicularly to it. Figure shows these two eigenvectors for a particular choice of functions g_{ij} and for the two joint time-series $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. It is evident in this stream plot that the radial structure is recovered remarkably well.

Our results have to be seen in comparison to previous works on obtaining a parameter free reconstruction of 1d noise-free [2] and low-noise[6] Langevin equations, where the method presented here allows the reconstruction even in the presence of strong measurement noise.

Comparing the optimization of stochasticity in the 2d case via a Euler-Lagrange form [8], the presented approaches allow a semi-analytical treatment of the problem via eigenvectors, which successfully recovers the radial orientation in our example. The simulated annealing routine works without calculating derivatives and with fewer numerical requirements, while reducing the risk of optimizing merely to a local minimum of the parameter space.

References

[1] H. Risken. *The Fokker-Planck Equation*. Springer, 1996.

[2] R. Friedrich, S. Siegert, J. Peinke, S. Lück, M. Siefert, M. Lindemann, J. Raethjen, G. Deuschl, and G. Pfister. Extracting model equations from experimental data. *Phys. Lett. A*, 271(3):217 – 222, 2000.

[3] R. Friedrich, J. Peinke, and M. R. R. Tabar. *Encyclopedia of Complexity and Systems Science*, chapter Importance of Fluctuations:Complexity in the View of Stochastic Processes. Springer, 2009.

[4] P. G. Lind, M. Haase, F. Böttcher, J. Peinke, D. Kleinhans, and R. Friedrich. Extracting strong measurement noise from stochastic series: Applications to empirical data. *Phys. Rev. E*, 81:041125, 2010.

[5] R. Friedrich and J. Peinke. Description of a turbulent cascade by a fokker-planck equation. *Phys. Rev. Lett.*, 78(5):863–866, 1997.

[6] M. Siefert, A. Kittel, R. Friedrich, and J. Peinke. On a quantitative method to analyze dynamical and measurement noise. *Europhys. Lett.*, 61:466, 2003.

[7] W. Press et al. *Numerical Recipes in C*. Cambridge University Press, 2nd edition, 1992.

[8] P. G. Lind, A. Mora, J. A. C. Gallas, and M. Haase. Reducing stochasticity in the north atlantic oscillation index with coupled langevin equations. *Phys. Rev. E*, 72:056706, 2005.