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# SELF-ORGANIZATION AND PATTERN FORMATION IN COUPLED LORENZ OSCILLATORS UNDER A DISCRET SYMMETRIC TRANSFORMATION

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**Abstract:** We present a spatial array of Lorenz oscillators, with each cell lattice in the chaotic regime. This system, shows spatial ordering due to self-organization of chaos synchronization after a bifurcation. It is shown that an array of such oscillators transformed under a discrete symmetry group, does not maintain the global dynamics, altough each transformed unit cell is locally identical to its precursor.

**keywords:** Complex Networks, Spatial ordering, self-organization.

## 1. INTRODUCTION

A covering dynamical system is a system with a discrete symmetry, and the equations of motion that describes it are left unchanged by a discrete symmetry group. By means of a local diffeomorphism, is posible to map it to a locally equivalent dynamical system without symmetry [1]. The dynamical properties of this *new* system, such as Lyaponuv exponent and fractal dimensions are left unchanged [2], although the resulting system (*image*) is topologically different from the former (*cover*). Due to the sensitivity of chaotic trajectories to their initial conditions and the consecuent unpredictability of long term evolution of the orbits, one can hardly anticipate that chaotic units can construct patterns with strictly ordered space structures. In this work, we show how locally similar dynamical systems can exhibit different pattern formations, and that the parametric changes needed to perform the bifurcation have global properties. We also investigate how a discrete equivariant transformation affect the dynamic behavior of an array of chaotic oscillators, and how their global characteristics, like bifurcations, are changed.

## 2. MODEL SYSTEM

We take a two-dimensional lattice of coupled Lorenz oscillators, with fixed boundary conditions:

$$\begin{aligned}
\dot{x}_{ij} &= -\sigma(x_{ij}-y_{ij}) + \epsilon \bigtriangledown^2 x \qquad (1)\\
\dot{y}_{ij} &= -x_{ij}\, z_{ij} + r\, x_{ij} - y_{ij} + \epsilon \bigtriangledown^2 y\\
\dot{z}_{ij} &= x_{ij}\, y_{ij} - \beta z_{ij} + \epsilon \bigtriangledown^2 z
\end{aligned}$$

where

$$\bigtriangledown^2 w = w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j} \qquad (2)$$

is a two-dimensional difussive coupling, and $i,j = 1,2,\cdots,N$, where $N$ is the dimension of the $N \times N$ lattice. The parameter settings are $\sigma = 10$, $\beta = 8/3$ and $r = 27$, so that each single independent Lorenz oscillator is chaotic, and are left unchanged for the following. The spatial orderings of the system are obtained varying homogeneuosly the control parameter $\epsilon$, and in order to study this transition, we use the following error functions [3]:

$$\begin{aligned}
E_{min(max)} &= min\big(E_1(m), E_2(m)\big) \qquad (3)\\
E_{max} &= max\big(E_1(m), E_2(m)\big) \qquad (4)
\end{aligned}$$

where

$$\begin{aligned}
E_1(m) &= \frac{1}{N}\sum_{i}^{N}\left\| x(i,\frac{N}{2}+m) - x(i,\frac{N}{2}-m)\right\| \qquad (5)\\
E_2(m) &= \frac{1}{N}\sum_{j}^{N}\left\| x(\frac{N}{2}+m,j) - x(\frac{N}{2}-m,j)\right\| \qquad (6)
\end{aligned}$$

and $m$ is a spatial delay; its value is chosen in the range $0 < m < N/2$. With this definitions, we can identify three types of behavior: when $E_{min} > 0$, $E_{max} > 0$ there is no pattern in the array; when $E_{min} = 0$, $E_{max} > 0$ there is a mirror symmetry respect to one of the axis $i = N/2$ or $j = N/2$. Finally, when $E_{min} = 0$, $E_{max} = 0$, the symmetry is central, *i.e.*, there is a mirror symmetry for each axis. The mapping of the system (1) into its *image* system [4] is

$$\begin{aligned}
\dot{u}_{ij} =& (-\sigma-1)u_{ij} + (\sigma-r)v_{ij} + v_{ij}w_{ij} \qquad (7)\\
&+(1-\sigma)\rho + \epsilon \bigtriangledown^2 u\\
\dot{v}_{ij} =& (r-\sigma)u - (\sigma+1)v_{ij} - u_{ij}w_{ij} + (r+\sigma)\rho\\
&-w_{ij}\rho + \epsilon \bigtriangledown^2 v\\
\dot{w}_{ij} =& -\beta w_{ij} + 0.5v_{ij} + \epsilon \bigtriangledown^2 w,
\end{aligned}$$

where $\rho = \sqrt{u^2+v^2}$. The system (7) is topologically similar to a Rössler oscillator.

## 3. RESULTS

In the Figure (1) the two regimes of the system (1) are plotted for times greater than the transient. The only dynamical transition present is from disorder to a central symmetry.

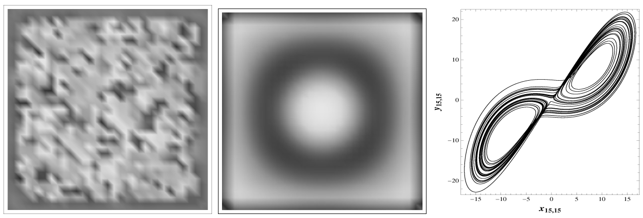

**Figure 1 – Space pattern of a Lorenz oscillator lattice. Left: $\epsilon = 0.5$. Center: $\epsilon = 0.9$. In both cases $N = 31$; Right: chaotic evolution with $\epsilon = 0.9$ of cell $(15, 15)$.**

By means of the error functions (3), we observe that the dependency of the parameter $\epsilon_b$ where the bifurcation occur with the size of the array is roughly parabolic, different from the exponential dependence as proposed in [3]. Altough each central configuration at which the system arrives after the bifurcation depends on the initial conditions, the same cualitative behavior is observed for long enough times. The results

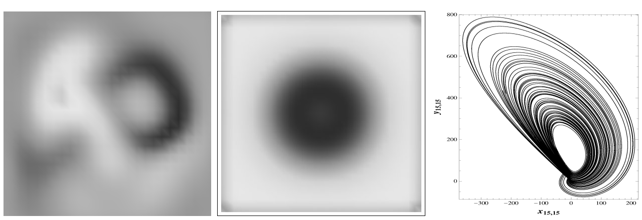

**Figure 2 – Space pattern of an image Lorenz oscillator lattice. Left: $\epsilon = 0.4$. Center: $\epsilon = 2.3$. In both cases $N = 31$. Right: chaotic evolution with $\epsilon = 2.3$ of cell $(15, 15)$.**

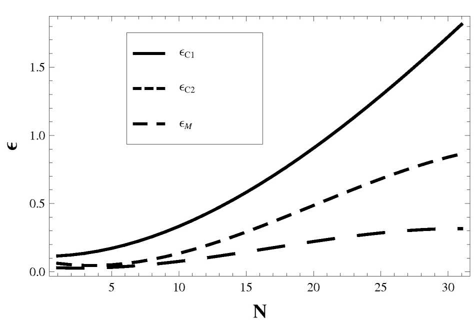


**Figure 3 – Variation of the control paramenter $\epsilon$ with the lattice dimension.**

for the image system are plotted in (2). We can see a very different behavior: in this system, made up of locally equivalent units of (1), the array present two bifurcations: one that goes from disorder to a mirror symmetry pattern, and from there to a central symmetry. We found that the most common mirror symmetry present in this array is about the diagonal axis, so a redefinition of the error functions was done in order to cuantify it. This behavior is observed for systems without discrete symmetries, like Rössler, and the change of the control parameter with the dimension of the lattice is almost linear for both bifurcation points; the change of $\epsilon$ for the two systems considered are shown in the Figure (3). The continuos line shows the $\epsilon_{C1}$ where the random-to-central symmetry bifurcation appears in the system (1); the dotted lines correspond to bifurcation points in the transformed system (7): the upper line is the mirror to central bifurcation parameter, $\epsilon_M$, and the lower is the *none to mirror* bifurcation parameter.

## 4. CONCLUSION

Altough identical dynamic information can be gathered from a cover system and his image (Lyapunov exponent, fractal dimensions), their behavior in phase space is topologically different. The system without symmetry is more easy to study, since there is a reduction of the manifolds in the *asymmetry* space. However we observed that the information that can be recovered from the mapped lattice is different in a esential way from the former: *i.e*, with the tools available to study spatiotemporal chaos, is imposible to claim for the origin of the double-bifucated system: an observer can not say if the array is made with Rössler oscillators, or with Lorenz *images* oscillators. More important is the fact that altough the phase space dynamics of a individual cell image is *simpler*, because in the case of a Lorenz oscillator it is a one-lobe attractor, the dynamic of the lattice is more complicated: it seems to show globally a pattern that recalls the individual cell phase space dynamics. We found that for large enough values of $\epsilon$, the systems reach a stable state with no chaotic unit-cell behavior. Finally we show numerically that the dependence of the control parameter with the dimension of an array of one-lobe coupled systems is almost linear, and is posible to study finite but high dimensional lattice without the problem of high computational efforts. More work on cuantitative diferentiation lattice dynamics is needed, altought a small insight has been gained through this work.

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