SPATIAL PATTERNS IN OSCILLATOR LATTICES WITH NON-LOCAL COUPLING: APPLICATION TO AN ACTIVATOR-INHIBITOR AUTO-CATALYTIC MODEL

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1. INTRODUCTION

Spatial patterns are found in many places in Nature, from the colors of animal's skin to neural networks. Turing was one of the pioneers in the study of the pattern formation [1, 2]. He introduced a theory in which heterogeneous spatial patterns are formed from an initial stable homogeneous pattern. His theory is based on reaction-diffusion equations governing the dynamics of two reacting chemical species, called activator (*a*) and inhibitor (*h*) [3].

$$\begin{cases} \dfrac{da_k(t)}{dt} = X(a_k, h_k) + \dfrac{\mu}{2}\left(a_{k+1} - 2a_k + a_{k-1}\right) \\ \dfrac{dh_k(t)}{dt} = Y(a_k, h_k) + \dfrac{\nu}{2}\left(h_{k+1} - 2h_k + h_{k-1}\right) \end{cases}. \qquad (1)$$

where *a* and *h* is the concentration of the activator and inhibitor, respectively and *X* and *Y* represent the rate equations.

2. NON-LOCAL INTERACTIONS

We consider a one-dimensional chain of *N* coupled oscillators with periodic boundary conditions. In our work we generalize the diffusive coupling adopted in Turing's model, by using of a non-local interaction, for which the coupling strength decreases with the lattice distance (*r*) as a power-law [4].

$$\begin{cases} \dfrac{da_k(t)}{dt} = Aa_k + Bh_k - \mu a_k + \dfrac{\mu}{\kappa_0(\alpha)} \sum_{r=1}^{N'} \dfrac{a_{k-r} + a_{k+r}}{r^\alpha} \\ \dfrac{da_k(t)}{dt} = Ca_k + Dh_k - \nu h_k + \dfrac{\nu}{\kappa_0(\alpha)} \sum_{r=1}^{N'} \dfrac{h_{k-r} + h_{k+r}}{r^\alpha} \end{cases}. \qquad (2)$$

Here we already considered the linearization of the equations - it is assumed that the system is not very far from his equilibrium point. The coupling effective range is parameterized by a positive variable (α), which goes from zero to infinity, corresponding to the limit cases of the interactions: the mean-field and the diffusive couplings, respectively. The variables μ and ν represent the intensity of the interaction, and $\kappa_0(\alpha)$ it is a normalization that dependent of the α. We present analytical results on the linear stability of the homogeneous pattern, and we obtain the conditions for it loses stability, in terms of the coupling strength and effective range:
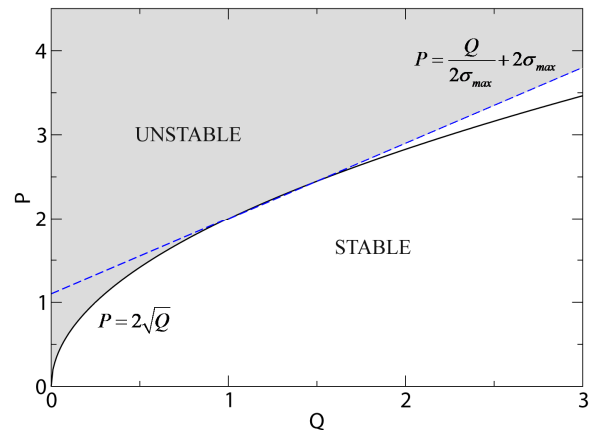
If $\alpha \neq 0$ then

$$\begin{cases} Q > 0 \\ P > 2\sqrt{Q}, \;\; 0 \leq P \leq 4\sigma_{max} \\ P > \dfrac{Q}{2\sigma_{max}} + 2\sigma_{max}, \;\; P > 4\sigma_{max} \end{cases}. \qquad (3)$$

Where

$$Q = \frac{AD - BC}{\mu\nu} \;\;, \;\; P = \frac{A}{\mu} + \frac{D}{\nu}. \qquad (4)$$

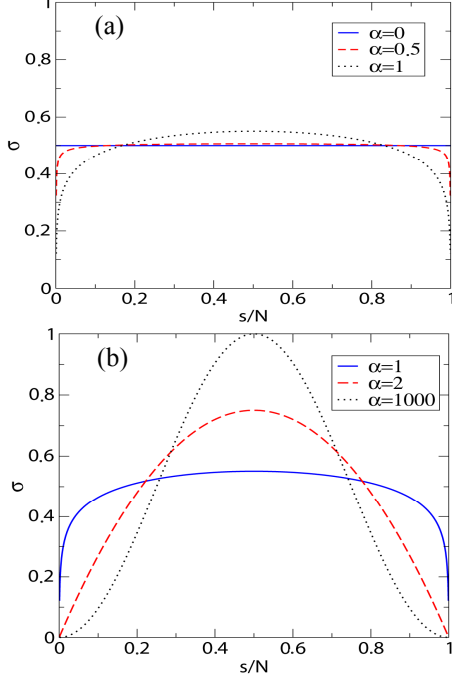These conditions define an region of instability in the space of parameters (*P,Q*) – figure (1).

Figure 1 – **Representation of the region of instability in the space of parameters *(P,Q)*.**

In the equation (3) the term σ_{max} is the maximum value of the function:

$$\sigma(s,\alpha,N)=\frac{1}{2}-\frac{1}{\kappa_0(\alpha)}\sum_{r=1}^{N'}\frac{\cos\left(\frac{2\pi sr}{N}\right)}{r^\alpha}. \qquad (5)$$

We show some characteristics of this function in the figure (2).

Figure 2 – Sigma function $\sigma(s,N,\alpha)$ as a function of s, for some values of the α.

As an example, we investigated pattern formation in a lattice of coupled Meinhardt-Gierer equations (equation (6)), which describe the interaction between an activator and an inhibitor chemical species, the former undergoing an auto-catalytic process [5, 6].

$$\begin{cases} \dfrac{da_k(t)}{dt}=\dfrac{\rho_a a_k^2}{h_k}-\mu_a a_k-D_a a_k+\dfrac{D_a}{\kappa_0(\alpha)}\sum_{r=1}^{N'}\dfrac{1}{r^\alpha}\left(a_{k-r}+a_{k+r}\right)\\ \dfrac{dh_k(t)}{dt}=\rho_h a_k^2-\mu_h h_k-D_h h_k+\dfrac{D_h}{\kappa_0(\alpha)}\sum_{r=1}^{N'}\dfrac{1}{r^\alpha}\left(h_{k-r}+h_{k+r}\right) \end{cases} \quad (6)$$

Just as in the linear case we obtained the conditions of loss of stability of the homogeneous state for the system (6). Furthermore we simulated the lattice numerically and compared with our analytical results. We showed in the illustration (1) an example of a pattern formed starting from a homogeneous state using the equations described in (6).

3. CONCLUSIONS

In short, in this work we proved to be possible the formation of patterns starting from a nonlocal interaction which the coupling intensity between cells decays in a power low fashion with the lattice distance. We also got to write analytically the necessary conditions to get the loss of stability of the system.

Additionally we applied the analytical results to a model of chemical reaction, known as model of Meinhardt-Gierer, and through computer simulations we verified our results.
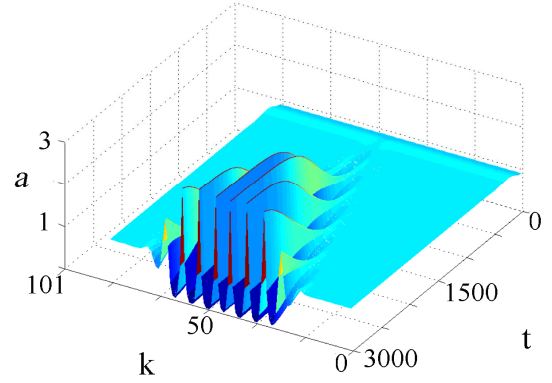
Figure 3 – A example of pattern formed starting from the system equations (3), we can observe that the pattern appears of a homogeneous state (α=1,000, D_h=0.2, D_a=0.005, μ_a=0.01, μ_h=0.02, ρ_a=0.01 and ρ_h=0.02).

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