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14. Abstract/Notes

Fixed point theorems have been an important tool used by mathematicians in the study of the existence of a solution to nonlinear differential equations, and nonlinear operator equations in general, since the beginning of the century. In the last seven years we have seen the appearance of some papers which extended the basic idea of this tool to solve problems of control, state estimation and parameter identification of nonlinear systems. The present paper discuss the several fixed point techniques used in control of nonlinear systems in papers already published on this subject. These techniques consider a space of functions $\mathcal{U}$ as the set of admissible functions. Some new techniques which allow cases where the set of admissible functions $\mathcal{U}_{ad}$ is a bounded, closed and convex subset of $\mathcal{U}$ are also presented here. The semigroup approach is used so that distributed parameter systems are considered. A brief discussion on the various types of fixed point theorems are given, as well as some references on this subject.

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CONTROL OF NONLINEAR SYSTEMS USING FIXED POINT THEOREMS

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Abstract. Fixed point theorems have been an important tool used by mathematicians in the study of the existence of a solution to nonlinear differential equations, and nonlinear operator equations in general, since the beginning of the century. In the last seven years we have seen the appearance of some papers which extended the basic idea of this tool to solve problems of control, state estimation and parameter identification of nonlinear systems. The present paper discuss the several fixed point techniques used in control of nonlinear systems in papers already published on this subject. These techniques consider a space of functions $\mathcal{U}$ as the set of admissible functions. Some new techniques which allow cases where the set of admissible functions $\mathcal{U}_{ad}$ is a bounded, closed and convex subset of $\mathcal{U}$ are also presented here. The semigroup approach is used so that distributed parameter systems are considered. A brief discussion on the various types of fixed point theorems are given, as well as some references on this subject.

INTRODUCTION

In this paper we consider semilinear systems of the type

$$\dot{z} = Az + Nz + Bu, \quad z(0) = z_0, \tag{1}$$

where A is a linear operator on an appropriate Banach space Z (the state space), N a nonlinear operator, B a linear operator from an input space U to Z, and $u \in \mathcal{U}$ the control ($\mathcal{U}$ being a space of functions from the interval [0,T] to the input space U of the system).

It is assumed that the dynamics of the linearized system

$$\dot{z} = Az, \quad z(0) = z_0, \tag{2}$$

can be described in terms of a strongly continuous semigroup S(t) on z, so that the above formulation includes distributed parameter systems and delay systems, as well as lumped parameter systems.

The problem of controllability is to find a control $u^* \in \mathcal{U}$ which drives system (1) from z_0 at $t = 0$ to a given desired state $z_d \in Z$ at $t = T$. We shall call u^* a wanted control.

System (1) may be derived from the linearization of a system described by a nonlinear evolution equation such as

$$\dot{z} = f(z,u,t), \quad z(0) = z_0.$$

In fact this is shown in: Carmichael, Pritchard and Quinn (1980); Magnusson, Pritchard and Quinn (1981) and Felippe De Souza (1983c).

Equation (1) is to be interpreted in the mild sense

$$z(t) = S(t)z_0 + \int_0^t S(t-\tau)Nz(\tau)d\tau + \int_0^t S(t-\tau)Bu(\tau)d\tau, \tag{3}$$

$$z(0) = z_0.$$

We shall discuss the fixed point techniques used in control of nonlinear distributed parameter systems in papers already published on this subject, as well as to bring some new techniques. In order to draw comparisons we shall try to maintain the notation introduced here, even when results are quoted from papers which adopt different terminology.

EXAMPLES

In this section, the nonlinear control problem (1) is illustrated with two examples of systems described by partial differential equations.

Nonlinear Parabolic System

Consider the following equation which describes the control of the flux of neutrons in a nuclear reactor (p. 95 of Henry, 1981)

$$\frac{\partial z}{\partial t} = k\,\frac{\partial^2 z}{\partial x^2} + \lambda z - \eta z^2 + u,$$

with boundary and initial conditions

$$z_x(0,t) = z_x(\ell,t) = 0, \quad z(x,0) = z_0(x),$$

where k, λ, $\eta > 0$.

This system can be expressed in the form (1) by setting $Z = L^2(0,\ell)$, A the linear operator on $L^2(0,\ell)$

$$A = k\,\frac{\partial^2}{\partial x^2} + \lambda I,$$

$$\mathcal{D}(A) = \left\{ z \in L^2(0,\ell) : \frac{\partial^2 z}{\partial x^2} \in L^2(0,\ell),\ \frac{\partial z}{\partial x} = 0 \text{ at } x = 0,\ell \right\},$$

$U = Z$, $B = I$ (identity on Z) and N the nonlinearity

$$Nz = -\eta z^2.$$

It can be shown (Curtain and Pritchard, 1978) that A generates the strongly continuous semigroup $S(t)$ on $L^2(0,\ell)$ given by

$$S(t)z = \sum_{n=1}^{\infty} e^{\lambda+(n-1)^2\pi^2 k}\ \phi_n <z,\phi_n>,$$

where $\phi_n(x) = (1/\sqrt{2})\cos(n-1)\pi x$, $n > 1$, $\phi_1(x) = 1$ and $<\cdot,\cdot>$ is the inner product in $L^2(0,\ell)$.

Nonlinear Hyperbolic System

Consider the following nonlinear wave equation which appears in quantum mechanics (Lions, 1969)

$$\frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - |w|^\rho w + u,$$

with boundary conditions

$$w(0,t) = w(\ell,t) = 0,$$

and initial conditions

$$w(x,0) = \dot{w}(x,0) = 0,$$

where $\gamma \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$.

Set $w' = \dot{w} = \partial w/\partial t$ and A the linear operator on $L^2(0,\ell)$

$$A = -\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A) = H^2(0,\ell) \cap H_0^1(0,\ell),$$

where L^2; H^2 and H_0^1 denote Lebesgue and Sobolev spaces on $(0,\ell)$ as usual (Lions, 1969).

Now this system can be written in the form (1) with z, z_0, A, N, and the state space Z chosen as follows

$$z = \begin{pmatrix} w \\ w' \end{pmatrix}, \quad z_0 = \begin{pmatrix} w_0 \\ w_0' \end{pmatrix},$$

$$Az = A\begin{pmatrix} w \\ w' \end{pmatrix} = \begin{bmatrix} 0 & I \\ -A & \gamma I \end{bmatrix} \begin{bmatrix} w \\ w' \end{bmatrix},$$

$$Nz = N\begin{pmatrix} w \\ w' \end{pmatrix} = \begin{bmatrix} 0 \\ -|w|^\rho w \end{bmatrix},$$

and

$$Z = \mathcal{D}(A^{1/2}) \times L^2(0,\ell) = H_0^1(0,\ell) \times L^2(0,\ell).$$

So, the topology on Z is given by the norm

$$\|z\|_Z^2 = \|w\|^2_{H_0^1(0,\ell)} + \|w'\|^2_{L^2(0,\ell)}.$$

A generates a strongly continuous semigroup on Z. This is shown on page 24 of Curtain and Pritchard (1978).

FIXED POINT THEOREMS

Fixed point theorems have been used since the beginning of the century to show local existence theorems for differential equations. Recently, several papers have been published using fixed point theorems to provide existence of the solutions to the problems of controllability and state estimation (Pritchard, 1982; Felippe De Souza, 1986). These papers usually provide mappings ϕ or F defined on some space X of functions from [0,T] to Z (e.g., $X = C(0,T;Z)$ or $X = L^p(0,T;Z)$ for some $p \geq 1$ etc.). The solutions to the problems of nonlinear controllability and state estimation are obtained via the fixed points of such mappings.

So, existence of a solution to the problem of nonlinear controllability or state estimation is transformed into existence of fixed points of these mappings.

Types of Fixed Point Theorems

The papers referred to above used fixed point theorems of contractive type, topological type or contractive type with perturbations.

Contractive type. A classical example of a contractive type fixed point theorem is the Banach contraction principle (Banach, 1922). Other fixed point theorems of this type may be found in Belluce and Kirk (1969) or Nashed and Wong (1969). However, the latter may neither provide an iterative procedure for reaching the fixed point nor guarantee uniqueness of such fixed point, two things which are peculiar to Banach's result.

Topological fixed points. Among the several topological fixed point theorems, we refer to Brouwer's fixed point theorems (Dunford and Schwartz, 1963) and Schauder fixed point theorem (Leray and Schauder, 1934).

Contractive type with perturbations. These are mappings F of the type

$$F = F_1 + F_2,$$

where F_1 is a contractive type (such as a contraction) and F_2 is either compact or completely continuous. Among the numerous more recent papers which develop fixed point theorems of this type, we mention Nussbaum (1969), Belluce and Kirk (1969) and Petryshyn (1973).

CONTROL PROBLEM

Let X denote the space of the trajectories (e.g., $X = C(0,T;Z)$ or $X = L^p(0,T;Z)$ for some $p \geq 1$ etc.) and $L(t)$ be the linear operator defined on $\tilde{X} \supseteq X$ for each $t \in [0,T]$ by

$$L(t)x(\cdot) = \int_0^t S(t-s)x(s)ds.$$

So, equation (3), the mild form of (1), can be rewritten as

$$z(\cdot) = S(\cdot)z_0 + L(\cdot)Nz(\cdot) + L(\cdot)Bu, \quad z(0) = z_0. \tag{4}$$

Now let $G: \mathcal{U} \to Z$ be the operator

$$Gu = \int_0^T S(T-s)Bu(s)ds = L(T)Bu.$$

The Mappings ϕ

Here we assume that the initial state is $z = 0$ (the origin). Set $\tilde{X} = \text{kernel } (G)$ and $\tilde{G}: \mathcal{U}/\tilde{X} \to Z$ defined by $\tilde{G}[u] = Gu$ for all equivalence classes $[u] \in \mathcal{U}/\tilde{X}$ so that $\tilde{G}^{-1}$ always exists.

Suppose now that we know an actual trajectory $z^*(t)$, $t \in [0,T]$ which takes system (1) from the initial state $z_0 = 0$ (the origin) to the desired state z_d on the interval $[0,T]$. Thus, it is easy to verify that $u^* \in U$ defined by

$$\begin{aligned} u^* &= \tilde{G}^{-1}[z_d - \int_0^T S(T-\rho)Nz^*(\rho)d\rho] \\ &= \tilde{G}^{-1}[z_d - L(T)Nz^*(\cdot)] \end{aligned} \tag{5}$$

is a wanted control which steers system (1) from the origin to z_d on $[0,T]$.

So, the wanted control u^* as given in (5) depends on the knowledge of the actual trajectory $z^*(\cdot)$ of the system. Substituting $z(\cdot)$ and u in (4) by $z^*(\cdot)$ and u^* given in (5), respectively, we obtain

$$z^*(\cdot) = L(\cdot)B\tilde{G}^{-1}[z_d - L(T)Nz^*(\cdot)] + L(\cdot)Nz^*(\cdot),$$

which is an expression of the actual trajectory depending on itself. Clearly, $z^*(\cdot)$ is a fixed point of the mapping $\phi: X \to X$ defined by

$$\phi z(\cdot) = L(\cdot)B\tilde{G}^{-1}[z_d - L(T)Nz(\cdot)] + L(\cdot)Nz(\cdot). \tag{6}$$

That is, $z^*(\cdot) = \phi(z^*(\cdot))$. The problem of controllability of system (1) (in other words, the problem of the existence of a wanted control u^*) is transformed into the existence of a fixed point for ϕ. Moreover, the problem of finding a wanted control u^* is transformed into finding a fixed point for ϕ. This approach was used in Magnusson, Pritchard and Quinn (1981); Carmichael, Pritchard and Quinn (1981); and Pritchard (1981). In fact, Magnusson, Pritchard and Quinn (1981) and Carmichael, Pritchard and Quinn (1981), using a contraction mapping theorem and a fixed point theorem for mappings of contractive type with perturbations, respectively, have shown that the diffusion process

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + Nz + u, \tag{7}$$

with the nonlinearity $Nz = z^2$, is controllable to any z_d within a ball of radius r in Z (r being dependent on the norm on X of $S(\cdot)z_1$, $z_1 \in Z$).

Actually, the work of Ichiwaka and Pritchard (1979) was one of the first to use a similar approach to solve the existence and uniqueness of evolution equations. Mappings similar to ϕ in (6) were used in Carmichael and Quinn (1982) to solve an optimal control problem and also in Kassara and El Jai (1983) in an attempt to construct an algorithm for solving the control problem.

Felippe De Souza (1982, 1983c) showed that the mapping ϕ in (6) does not provide necessary conditions for us to obtain the wanted controls u^* which drive the system from $z_0 = 0$ to z_d in the interval $[0,T]$. That is, there may be wanted controls u^* which can never be found via the fixed points $z^*(\cdot)$ of ϕ.

Using the Generalized Inverse of G

First take $z_0 = 0$ again. Pritchard (1982) found that the mapping ϕ of (6) could be simplified by using $G^\dagger$ (the generalized inverse of G) instead of $\tilde{G}^{-1}$, as long as Range (G) is closed in Z. Moreover, Pritchard (1982) assumed that $\mathcal{U}$ and/or Z could be adjusted in order to the operator G to have closed range in Z. Later Felippe De Souza (1983a, Chapter 5 of 1983c) showed that this adjustment of $\mathcal{U}$ and Z is always possible. Furthermore, Felippe De Souza (1983b) presented an iterative procedure for this adjustment to be done.

Felippe De Souza and Pritchard (1985) and Felippe De Souza (1985) extended this result to cases where the initial state is $z_0 \neq 0$. The mapping ϕ then becomes

$$\phi z(\cdot) = S(\cdot)z_0 + L(\cdot)Nz(\cdot) + L(\cdot)BG^\dagger[z_d - S(T)z_0 - L(T)Nz(\cdot)]$$

and, if $z^*(\cdot)$ is a fixed point of ϕ, then a wanted control $u^* \in \mathcal{U}$ is given by

$$u^* = G^\dagger[z_d - S(T)z_0 - L(T)Nz^*(\cdot)].$$

Felippe De Souza and Pritchard (1985) also showed that the diffusion process (7) with nonlinearities Nz such as z^4, z_x^2, zz_x etc. can be controlled to any z_d within a ball of radius r (r being dependent on the norm on X of $S(\cdot)z_1$, $z_1 \in Z$).

The mapping ϕ with $G^\dagger$ was an improvement in the simplicity but, on the other hand, it neither provides necessary nor sufficient conditions for us to obtain the wanted controls u^*. Actually, a fixed point $z^*(\cdot)$ of ϕ must satisfy

$$[z_d - L(T)Nz^*(\cdot)] \in \text{Range (G)}$$

in order to u^*, given by (5), be a wanted control for system (1). This condition was called "check of consistency" in Pritchard (1982). Check of consistency has also been used in Felippe De Souza and Pritchard (1985) with similar mappings ϕ for which sufficient conditions for us to obtain a wanted control u^* does not hold either.

A New Approach

A more sophisticated approach to solve the control problem for system (1) was introduced in Felippe De Souza (1982). This new approach involved a pair $z = (z(\cdot), z_f)$ consisting of the trajectory $z(\cdot) \in X$ and the final state $z_f \in Z$. Instead of $\phi: X \to X$, the mapping used was $F: M \to M$; M being the cross product between X and Z.

Assume that the state space Z is a Hilbert space and the trajectory $z(\cdot)$ on [0,T] lies in X

$$X = L^2(0,T;Z).$$

Define the Hilbert space

$$M = X \times Z$$

with the inner-product given by

$$\left\langle \begin{pmatrix} z(\cdot) \\ z_f \end{pmatrix}, \begin{pmatrix} z'(\cdot) \\ z_f' \end{pmatrix} \right\rangle_M = \langle z(\cdot), z'(\cdot)\rangle_X + \langle z_f, z_f' \rangle_Z.$$

Define the operators $\mathcal{S}:Z \to M$ and $L:X \to M$ by

$$\mathcal{S}z_f = (S(\cdot)z_f,\ S(T)z_f),$$

$$Lx(\cdot) = (L(\cdot)x(\cdot),\ L(T)x(\cdot)).$$

Now system (4) can be represented in the compact form

$$z = \mathcal{S}z_0 + LNz(\cdot) + LBu, \qquad z(0) = z_0. \tag{8}$$

It is assumed that Range (LB) is closed in M. If this is not the case for a particular choice $(\mathcal{U},Z)$ of space of input functions $\mathcal{U}$ and state space Z, it is necessary to reframe the system in a restricted state space Z' and/or an enlarged space of input functions $\mathcal{U}'$ such that the assumption will hold for the choice $(\mathcal{U}',Z')$. Here again the work of Felippe De Souza (1983a, 1983b, 1983c) on the adjustment of the topology of the spaces can be applied.

The mapping $F:M \to M$ used in Felippe De Souza (1982) was

$$F(z) = \gamma + (I - P)LNz(\cdot) + Pz - LBG^{\dagger}z_f, \tag{9}$$

where $\gamma = [LBG^{\dagger}z_d + (I - P)\mathcal{S}z_0] \in M$ is a fixed element, I is the identity on M and P is any continuous projection onto Range (LB).

Felippe De Souza (1982) showed that if there is a wanted control u^* in $\mathcal{U}$ which drives the system from z_0 at $t = 0$ to the desired state z_d at $t = T$, then u^* can be obtained via the fixed points of F. In other words, F was the first mapping with necessary conditions for us to obtain the wanted controls u^*. This was possible only because of the approach of the pair $(z(\cdot),z_f)$ consisting of the trajectory and final state.

Also note that F in (9) is in fact a family of mappings, since P is any continuous projections onto Range (LB). Several examples of continuous projections onto Range (LB) are shown in Chapter 4 of Felippe De Souza (1983c). It is also shown that when $z_0 = 0$ and for a particular projection $P = \Pi$, where $\Pi = \Pi_1 + \Pi_2$ was defined in Felippe De Souza (1982), F in (9) becomes

$$F(z) = (\phi z(\cdot), (\phi z(\cdot))(T)) + \Pi_2(z - LNz(\cdot)),$$

where ϕ is as before. That is, ϕ can be regarded as the particular case of F when $P = \Pi_1$. Also, $\Pi_2(z - LNz(\cdot))$ is the missing term in ϕ which gives necessity to F.

Later, Felippe De Souza (1983c, 1984) developed mappings $F:M \to M$ which provide necessary and sufficient conditions. That is: if z^* is a fixed point of F, then we can obtain a wanted control u^*, and if u^* is a wanted control, then it can be obtained using a fixed point z^* of F.

Actually, Felippe De Souza (1983c) showed that such mappings (with necessary and sufficient conditions) could not be obtained without considering the approach of the space M (i.e., the approach of the pair $(z(\cdot),z_f)$ consisting of the trajectory and final state).

New Results

We shall now present two new results (Theorem 1 and 2) which illustrate this new class of mappings F which provide necessary and sufficient conditions. Theorem 1 is in fact a more general version of a result shown in Felippe De Souza (1984). Theorem 2 is only being introduced here and admits a set of admissible controls $\mathcal{U}_{ad} \subset \mathcal{U}$, a case which has been ignored by the authors so far.

Let P be as before (i.e., any continuous projection onto Range (LB), $\xi:M \to M$ be the mapping

$$\xi(z) \doteq z - \mathcal{S}z_0 - LNz(\cdot),$$

$\bar{x}$ be any fixed element of M which satisfies

$$\bar{x} \in \text{Range (LB) and } \bar{x} \neq 0,$$

and q be any functional on Z which satisfies

$$q(z_f) = 0 \quad \text{if and only if} \quad z_f = z_d. \tag{10}$$

Several examples of functionals q are given in Felippe De Souza (1983c).

Theorem 1. The control $u^* \in \mathcal{U}$ given by

$$u^* = (LB)^{\dagger}\xi(z^*) \tag{11}$$

drives system (1) from z_0 at $t=0$ to z_d at $t=T$ if and only if $z^*=(z^*(\cdot),z_f^*) \in M$ is a fixed point of the mapping $F:M \to M$

$$F(z) = \mathcal{S}z_0 + LNz(\cdot) + P\xi(z) + q(z_f)\bar{x} \tag{12}$$

for all $z = (z(\cdot),z_f) \in M$. □

Proof. Necessity: If u^* given by (11) drives system (1) from z_0 at $t=0$ to z_d at $t=T$, then, using (8),

$$\xi(z^*) = z^* - \mathcal{S}z_0 - LNz^*(\cdot) = LBu^*$$

and therefore $\xi(z^*) \in$ Range (LB), which implies that

$$P\xi(z^*) = \xi(z^*), \tag{13}$$

and $z^*=(z^*(\cdot),z_f^*)$ is the pair consisting of the actual trajectory $z^*(\cdot)$ and the final state

$$z^*(T) = z_f^* = z_d. \tag{14}$$

Then $q(z_f^*) = 0$. Now, using (13),

$$F(z^*) = Sz_0 + LNz^*(\cdot) + \xi(z^*) = z^*.$$

Sufficiency: If $z^* = F(z^*)$, then

$$z^* = Sz_0 + LNz^*(\cdot) + P\xi(z^*) + q(z_f^*)\bar{x}, \tag{15}$$

and hence

$$\xi(z^*) = z^* - Sz_0 - LNz^* = P\xi(z^*) + q(z_f^*)\bar{x}. \tag{16}$$

But, since both $P\xi(z^*)$ and $q(z_f^*)\bar{x}$ lie in Range (LB), we have that $\xi(z^*)$ also lies in Range (LB) and therefore $\xi(z^*) = P\xi(z^*)$. So, using (16),

$$q(z_f^*)\bar{x} = 0. \tag{17}$$

Hence, since $\bar{x} \neq 0$, $z_f^* = z_d$. Equations (15) and (17) imply that $(z^*(\cdot), z_f^*)$ satisfies the dynamic equation (8) with u^* given by (11). So, $z^* = (z^*(\cdot), z_f^*)$ is again the pair consisting of the actual trajectory $z^*(\cdot)$ and the final state $z^*(T) = z_f^*$ and the control u^* drives system (1) from z_0 at $t = 0$ to z_d at $t = T$. □

Now suppose that $\mathcal{U}_{ad}$, the set of admissible controls, is either

$$\mathcal{U}_{ad} = \mathcal{U} \quad \text{(the space of all input functions)}, \tag{18}$$

(which are the cases admitted so far), or

$\mathcal{U}_{ad}$ is a bounded, closed and convex subset of $\mathcal{U}$.

Let now P be any continuous projection onto $LB(\mathcal{U}_{ad})$. Several examples of such projections are given in Chapter 4 of Felippe De Souza (1983c).

Also define the mapping $\zeta_d : M \to M$ by

$$\zeta_d(z) = (0, z_1 - z_d) \quad \text{for all } z = (z(\cdot), z_1) \in M.$$

Theorem 2. There exists a control $u^* \in \mathcal{U}_{ad}$ given by

$$u^* = (LB)^{\dagger}\xi(z^*(\cdot), z_d) \tag{19}$$

which drives system (1) from z_0 at $t = 0$ to z_d at $t = T$ if and only if there exists $z^* = (z^*(\cdot), z_1) \in M$ which is a fixed point of $F : M \to M$

$$F(z) = \zeta_d(z) + Sz_0 + LNz(\cdot) + P\xi(z(\cdot), z_d) \tag{20}$$

for all $z = (z(\cdot), z_1) \in M$. □

Proof. Necessity: If there exists $u^* \in \mathcal{U}_{ad}$ given by (19), then, using (8), we can show, similarly to Theorem 1, that $\xi(z^*(\cdot), z_d) \in LB(\mathcal{U}_{ad})$ and hence

$$P\xi(z^*(\cdot), z_d) = \xi(z^*(\cdot), z_d).$$

Now since

$$\xi(z^*(\cdot),z_d) = \xi(z^*) - \zeta_d(z^*), \tag{21}$$

$$F(z^*) = \xi(z^*) + Sz_0 + LNz^* = z^*.$$

Sufficiency: If $z^* = F(z^*)$, then

$$z^* = \zeta_d(z^*) + Sz_0 + LNz^*(\cdot) + P\xi(z^*(\cdot),z_d).$$

Hence,

$$\xi(z^*) - \zeta_d = P\xi(z^*(\cdot),z_d),$$

and thus, using (21) again, $\xi(z^*(\cdot),z_d) = P\xi(z^*(\cdot),z_d)$ which implies that $\xi(z^*(\cdot),z_d) \in LB(U_{ad})$. So, $(z^*(\cdot),z_d)$ satisfies (8) with u^* given by (19) and is the pair trajectory-final state. So u^* drives system (1) from z_0 at $t=0$ to z_d at $t=T$. □

Remarks

Note that a fixed point $z^* = (z^*(\cdot),z_f^*)$ of F in (12) always satisfies (14), namely

$$z^*(T) = z_f^* = z_d.$$

This is not necessarily the case for F in (20).

The mapping F given in (12) for Theorem 1 cannot be used for the case $U_{ad} \subsetneq U$. This is actually shown in Felippe De Souza (1983c).

Conclusions

The main contribution of the present paper is the extension of the fixed point techniques to cases where the set of admissible functions U_{ad} is not the whole space of input functions U. However, the history of the previous work done on this subject and the brief discussion on the various fixed point theorems presented here are also valuable contributions.

REFERENCES

Banach, S. (1922). Sur les opērations dans les ensembles abstraits et leur applications aux ēquation integrales. *Fund. Math.*, 3, 133-181.

Belluce, L.P., and W.A. Kirk (1969). Fixed point theorems for certain classes of non-expansive mappings. *Proc. Amer. Math. Soc.*, 20, 141-146.

Carmichael, N., A.J. Pritchard, and M.D. Quinn (1980). State and parameter estimation problems for nonlinear systems. *Control Theory Centre Report*, 89, University of Warwick, Coventry, England.

Carmichael, N., A.J. Pritchard, and M.D. Quinn (1981). Control and state estimation of non-linear systems. In Kappel and Schappacher (Ed.), *Evolution Equations and their Applications*. Proc. Conference on Differential Equations and Applications. University of Graz, Austria. Pitman Advanced Publishing Program, Boston. pp.30-51.

Carmichael, N., and M.D. Quinn (1982). Notes on optimal control and estimation results of nonlinear systems. *Proc. Third IFAC Symposium on Control of Distributed Parameter Systems*, Toulouse, France.

Curtain, R.F., and A.J. Pritchard (1978). In Balakrishnan and Thoma (Ed.), *Infinite Dimensional Linear System Theory*. Lecture Notes in Control and Information Sciences, Vol. 8, Springer-Verlag, Berlin.

Dunford, N., and J.T. Schwartz (1963). *Linear Operators*, Part II, Willey (Interscience), New York.

Felippe De Souza, J.A.M. (1982). The application of projections and fixed points to nonlinear control and estimation. *Control Theory Centre Report*, 106, University of Warwick, Coventry, England.

Felippe De Souza, J.A.M. (1983a). Some aspects of linear operators in inner-product spaces. *Control Theory Centre Report*, 111, University of Warwick, Coventry, England.

Felippe De Souza, J.A.M. (1983b). The generation of complete matched sets. *Control Theory Centre Report*, 112, University of Warwick, Coventry, England.

Felippe De Souza, J.A.M. (1983c). *Some Application of Projections in Nonlinear Control and Estimation*. Ph.D. Thesis, Control Theory Centre, University of Warwick, Coventry, England.

Felippe De Souza, J.A.M. (1984). Nonlinear control and estimation using fixed point theorems. In Tzafestas and Hamza (Ed.), *Methods and Applications of Measurements and Control*, Acta Press, Anaheim, California, Vol. 1, pp. 138-141.

Felippe De Souza, J.A.M. (1985). Control of nonlinear distributed parameter systems. In Ibarra-Zannatha (Ed.), *Proc. IV Coloquio de Control Automatico*, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional de Mexico, Mexico, Vol. I, pp. 37-43.

Felippe De Souza, J.A.M. (1986). Control, state estimation and parameter identification of nonlinear distributed parameter systems using fixed point techniques: a survey. *Proc. IV IFAC Symposium on Control of Distributed Parameter Systems*, Pasadena, California, USA.

Felippe De Souza, J.A.M., and A.J. Pritchard (1985). Control of linear and semilinear distributed parameter systems. In Felippe De Souza (Ed.), *Telecommunication and Control, Proc. 2nd. IASTED Conference on Telecommunication and Control (TELECON'85)*, Rio de Janeiro, Brazil, pp. 158-162.

Henry, D. (1981). In A. Dold and B. Eckmann (Ed.), *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin.

Ichikawa, A., and A.J. Pritchard (1979). Existence, uniqueness and stability of nonlinear evolution equations. *J. Math. Anal. Appl.*, 68, 454-476.

Kassara, K. and A. El Jai (1983). Algorithmes pour la commande d'une classe de systemes a parametres repartis non lineaires. *Journal Marrocain D'Automatique D'Informatique et de Traitement du Signal.*

Leray, J., and J. Schauder (1934). Topologie et equations fonctionelles. *Ann. Sci. Ecole Norm. Sup.*, 51, 45-78.

Lions, J.L. (1969). *Quelques méthodes de résolution des problemes aux limites non lineaires.* Dunod, Paris.

Magnusson, K., A.J. Pritchard, and M.D. Quinn (1981). The application of fixed point theorems to global nonlinear controllability problems, *Proc. Semester on Control Theory*, Banach International Mathematical Centre, Warsaw, Poland.

Nashed, M.Z., and J.S.W. Wong (1969). Some variants of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations. *J. Math. Mech.*, 18, 767-777.

Nussbaum, R.D. (1969). The Fixed Point Index and Fixed Point Theorems for k-set Contraction. Ph.D. Thesis, University of Chicago, Chicago.

Petryshyn, W.V. (1973). Fixed point theorems for various classes of 1-set contractive and 1-ball contractive mappings in Banach space. *Trans. Amer. Math. Soc.*, 182, 323-352.

Pritchard, A.J. (1981). The application of a fixed point theorem to nonlinear systems theory. *Proc. Third IMA Conference on Control Theory*, University of Sheffield, Sheffield, England, 775-792.

Pritchard, A.J. (1982). Nonlinear infinite dimensional system theory. *Proc. Third IFAC Symposium on Control of Distributed Parameter Systems*, Toulouse, France.