1. Publication Nº	2. Version	3. Date	5. Distribution
INPE-4775-PRE/1444		Dec. 1988	☐ Internal ☒ External ☐ Restricted

4. Origin	Program
DCG	*DIOCON/201405*

6. Key words - selected by the author(s)

ODE GLOBAL ERROR ESTIMATION *NUMERICAL INTEGRATION*
ORBIT PROPAGATION PROBLEMS

7. U.D.C.: 681.518

8. Title INPE-4775-PRE/1444

A STOCHASTIC APPROACH TO GLOBAL ERROR ESTIMATION IN ODE MULTISTEP NUMERICAL INTEGRATION

10. Nº of pages: *32*

11. Last page: *31*

12. Revised by

Hélio Koiti Kuga

9. Authorship

Atair Rios Neto
Kondapalli Rama Rao

Responsible author

13. Authorized by

p/ *Marco Antonio Raupp*
General Director

14. Abstract/Notes

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15. Remarks

Submitted for publication in "Journal of Computational and Applied Mathematics".

A stochastic approach to global error estimation in ODE multistep numerical integration

Atair Rios Neto
Kondapalli Rama Rao

Institute for Space Research (INPE), Dept. of Control and Guidance, Sao Jose dos Campos, Brazil

Abstract: In order to assess the quality of approximate solutions obtained in the numerical integration of ordinary differential equations related to initial value problems, there are available procedures which lead to deterministic estimates of global errors. The aim of this paper is to propose a stochastic approach to estimate the global errors, especially in the situations of integration which are often met in flight mechanics and control problems. Treating the global errors in terms of their orders of magnitude, the proposed procedure models the errors through the distribution of zero mean random variables belonging to stochastic sequences, which take into account the influence of both local truncation and round-off errors. The dispersions of these random variables, in terms of their variances, are assumed to give an estimation of the errors. The error estimation procedure is developed for Adams-Bashforth-Moulton type of multistep methods. The computational effort in integrating the variational equations to propagate the error covariance matrix associated with error magnitudes and correlations is minimized by employing a low order (1st or 2nd) Euler method. The diagonal variances of the covariance matrix, derived using the stochastic approach developed in this paper, are found to furnish reasonably precise measures of the orders of magnitude of accumulated global errors in short-term as well as long-term orbit propagations.

Keywords: ODE global error estimation, numerical integration, orbit propagation problems.

1. INTRODUCTION

In the numerical integration of ordinary differential equations (ODE) related to initial value problems, especially in the situations of very long intervals of integration, it is of fundamental importance to assess the quality of the approximate solutions obtained. Theoretical results, qualitatively relating the orders of magnitude of global and local errors or quantitatively giving upper bounds of global errors in terms of parameters dependent on local errors, step size, problem and method used, have for a long time been available in the literature (see for example, Dahlquist and Bjorck, 1974). However, only recently were produced results which translate theory into procedures of practical use (Zadunaisky, 1966, 1972, 1976, 1979; Stetter, 1979; Prothero, 1980). These procedures can be grouped into two basic types (Stetter, 1977): (i) those dealing with the generation of two primary numerical approximations having an asymptotic behaviour (Shampine and Baca, 1985) and (ii) those dealing with a primary numerical solution and a first order approximation of the associated error, calculated either by using variational equations or by a defect correction approach (Stetter, 1978). A common aspect in these procedures is that they all lead to deterministic estimates of global errors.

In this paper, a stochastic approach to estimate the global error is presented, seeking a method feasible for situations which are often met in flight mechanics and control problems, as for example state estimate predictions in statistical orbit determination or propagation of vehicle state from noisy measurements of strap-down inertial instruments. In the procedure proposed, global errors are treated with respect to their orders of magnitude. They are represented, or modelled, through the distribution of zero mean random variables belonging to stochastic sequences, which take into account the influence of both local truncation and round-off errors. The dispersions of these random variables, in terms of their variances, are assumed to give an estimation of the errors. Extending the results of a previous work, where only single-step methods were considered (Rios Neto and Cardenuto, 1983), an error estimation procedure is developed for Adams-Bashforth-Moulton (ABM) type of predictor-corrector multistep methods. The selection of this method is due to the well-known fact that it is one of the best methods availa ble to solve the orbit propagation problems (Janin, 1979; Kondapalli, 1986). To propagate the errors, the procedure makes use of the first order variational equations associated with the system being integrated; however the computational effort is minimized since only a method of much lower order - like first order Euler method or second order modified Euler method - is employed to integrate the error covariance matrix associated with error magnitudes and correlations.

Preliminary testing results are shown for three satellite orbit propagation problems, selected from a set of test problems used by several authors (Szebehely, 1967; Hull et al., 1972; Shampine and Gordon, 1975): two cases of two-body motion (circular and elliptic) and a case of the restricted three-body problem. Global error estimates made in short-term as well as in long-term propagations are given along with concluding remarks on the performance of the proposed stochastic procedure in all the cases considered.

2. PROBLEM STATEMENT

Consider a system of first order ordinary differential equations,

$$\dot{x} = f(x,t), \tag{2.1}$$

where f is a vector function of an nx1 dependent variable vector, x, and of the scalar independent variable, t. Given an initial condition $x(t_o)$, assume that the system (2.1) has a unique solution x(t). Also suppose that a discretization method is used to generate a numerical approximation x(j) at mesh points $t_j (t_o < t_j \leq t_f)$.

The aim of this work is to estimate the contribution of the truncation errors and the round-off errors to the global error $\nu(j)$, defined as:

$$\nu(j) = x(j)-x(t_j), \tag{2.2}$$

where $x(t_j)$ is the true solution at t_j, and the approximation $x(j)$ is obtained by using a zero-stable method of order k. This implies that, in the asymptotic dependence on the step size, the difference between the exponents of the local and global errors is equal to one (Dahlquist and Bjorck, 1974).

3. STOCHASTIC ESTIMATION OF GLOBAL ERROR

Since the errors involved are small, it is assumed that linear perturbations give valid approximations, allowing the use of the superposition principle after separately treating the contributions of the truncation errors and the round-off errors.

At an intermediate point t_j, being $x(j)$ a numerical approximation to the true solution $x(t_j)$, let $E(j)$ be a zero mean random variable whose dispersion in terms of its variances gives an estimate of the order of magnitude of the global error. Thus $E(j)$ belongs to a stochastic Markov sequence with given statistical properties:

$$\Gamma[E(j)] = 0, \qquad \Gamma[E(j)E^T(j)] = P(j), \tag{3.1}$$

where Γ [.] represents the expectation operator and in the methodology developed here, the standard deviations corresponding to the diagonal variances of the covariance matrix P(j) are supposed to be related to approximations of orders of magnitude of the true global errors, $\nu(j)$. One should note that, at each step, the zero mean E(j) accounts for the stepwise error accumulation through its covariance. To start the process, the same reasoning concerning the error order of magnitude is used in calculating the diagonal matrix P(0).

Motivated mainly by the objective of applying the method to orbit propagation problems, as it is well-known that the multistep integrators of type ABM are some of the most suitable ones for this kind of problems, a multistep numerical integrator has been chosen here. However, there is no hindrance to apply the procedure to other methods, as was shown by Rios Neto and Cardenuto (1983).

4. ESTIMATE OF THE CONTRIBUTION OF TRUNCATION ERRORS

Considering a multistep numerical integrator of order k-1, in a typical step, one has:

$$x^{\ell}(t_{j+1}) = T^{k-1}(x(j),\ldots,x(j-(k-1)+1)) + \varepsilon(t_{j+1}), \quad (4.1)$$

where $x^{\ell}(t_{j+1})$ represents "a local true value", generated from the numerical approximations $x(j),\ldots,x(j-(k-1)+1)$ and T^{k-1} is an operator representing the (k-1)st order integrator used. It is a usual practice to approximate the local truncation error $\varepsilon(t_{j+1})$ at (j+1)st step by the difference between the solutions obtained by two integrators of consecutive orders, k-1 and k (Shampine and Gordon, 1975). That is,

$$
\begin{aligned}
\varepsilon(t_{j+1}) &= T^{k}(x^{*}(j+1),\ldots x(j-(k-1)+1))- \\
&\quad T^{k-1}(x(j),\ldots,x(j-(k-1)+1)+e(t_{j+1}) \qquad (4.2), \\
&= \varepsilon(j+1)+e(t_{j+1}),
\end{aligned}
$$

where $x^{*}(\ldots)$ represents the numerical solution obtained by the method of lower order, $\varepsilon(j+1)$, a numerical approximation for $\varepsilon(t_{j+1})$ and $e(t_{j+1})$ is in general a locally negligible value. In a multistep method of type of ABM, the method of order k-1 is called predictor and the method of order k, corrector. Looking at the equation (4.2), one can see that an estimate of the quantity $e(t_{j+1})$ is necessary for getting a global error estimate. However, one should recall that the main idea is to estimate the error bounds but not the errors themselves. With this in mind, $e(t_{j+1})$ is modelled here as a white vector sequence of independent components $e^{m}(j+1)$, with the superscription representing a modelled variable, such that

$$
\Gamma[e^{m}(j+1)] = 0. \qquad (4.3)
$$

In modelling its dispersion, one uses the Chebychev inequality (Jazwinski, 1970; Rios Neto and Cardenuto, 1983), through which one obtains,

$$P[e^m \geq \varepsilon(j+1)] \leq var(e^m(j+1))/\varepsilon^2(j+1);$$

and then one can consider

$$var(e^m(j+1)) = \varepsilon^2(j+1)/100,$$

so as to guarantee that $e^m(j+1)$ has realizations with 99% chance of being inside the tolerance limits, given by $\varepsilon(j+1)$. As a direct consequence one gets for the covariance matrix,

$$\begin{aligned} Q_{ii}(j+1) &= \Gamma[e_i^{m^2}(j+1)] \\ &= var(e_i^m(j+1)) \\ &= \varepsilon^2(j+1)/100 \\ &= \frac{1}{100}[T_i^k(x^*(j+1),x(j),\ldots)- \\ &\qquad T_i^{k-1}(x(j),x(j-1),\ldots)]^2, \end{aligned} \tag{4.4}$$

where $i = 1,\ldots n$, n being the dimension of the vector $x(t_j)$. Following the same reasoning, $x(t_{j+1})$ is modelled by $x^m(t_{j+1})$, given by:

$$x^m(t_{j+1}) = T^{k-1}(x^m(t_j),x^m(t_{j-1}),\ldots)+\varepsilon^m(t_{j+1}), \tag{4.5}$$

where

$$x^m(t_j) = x(j)+E(j),$$
$$x^m(t_{j-1}) = x(j-1)+E(j-1), \tag{4.6}$$
etc.

Identifying the first term on the r.h.s. of the equation (4.5) as $x^{*m}(t_{j+1})$, $\varepsilon(t_{j+1})$ of the equation (4.2) is modelled as:

$$\varepsilon^m(t_{j+1}) = T^k(x^{*m}(t_{j+1}),x^m(t_j),\ldots)- T^{k-1}(x^m(t_j),x^m(t_{j-1}),\ldots)+e^m(j+1). \tag{4.7}$$

Using the equations (4.5) and (4.7), one gets:

$$x^m(t_{j+1}) = T^k(x^{*m}(t_{j+1}),x^m(t_j),\ldots)+e^m(j+1). \tag{4.8}$$

Expanding the first term on the r.h.s. of the equation (4.8) about the point $(x^{*m}(j+1),x(j),\ldots)$ up to first order, it results:

$$\begin{aligned} x^m(t_{j+1}) = {} & T^k(x^*(j+1),x(j),\ldots)+ \\ & [T^k_{x^{*m}(t_{j+1})}(x^*(j+1),x(j),\ldots). \\ & \{x^{*m}(t_{j+1})-x^*(j+1)\}+ \\ & T^k_{x^m(t_j)}(x^*(j+1),x(j),\ldots). \\ & \{x^m(t_j)-x(j)\}+\ldots]+e^m(j+1). \end{aligned} \tag{4.9}$$

Identifying the first term on the r.h.s. of the equation (4.9) as $x(j+1)$:

$$x^m(t_{j+1})-x(j+1) = \text{The term within the brackets}+e^m(j+1),$$

or, using the equations (4.6),

$$E(j+1) = \text{The term within the brackets} + e^m(j+1). \quad (4.10)$$

Now, considering the equation of the system (2.1), for a $t_j \leq t_i < t_{j+1}$, expanding the function $f(x(t_i),t_i)$ about $x(i)$ up to first order, and defining

$$\delta x(t_i) \triangleq x(t_i)-x(i),$$

there results:

$$\delta \dot{x}(t_i) = f_x(x(t_i),t_i)\big|_{x(t_i)=x(i)} \cdot \delta x(t_i). \qquad (4.11)$$

Then by using the definition of transition matrix, $\Phi(t_{j+1},t_j)$, one has:

$$\delta x(t_{j+1}) = \Phi(t_{j+1},t_j).\delta x(t_j), \qquad (4.12)$$

where $\Phi(t_{j+1},t_j)$ is as usual the solution of the equation

$$\dot{\Phi}(t,t_j) = f_x(x,t).\Phi(t,t_j), \qquad (4.13)$$

with the conditions:

$$\Phi(t_j, t_j) = I, \qquad x(t_i) = x(i).$$

As the equation (4.12) is the solution of the differential equation associated with (4.11), comparing the equation (4.12) with the equation (4.10), one can see that, in a mean square sense (Jazwinski, 1970), the numerical solution of the differential equation associated with (4.11) obtained by the integrator $T^k(\ldots)$, when E(j) is taken for the first variation in t_j, can be written as:

$$\tilde{E}(j+1) = \Phi(j+1,j)E(j), \tag{4.14}$$

which is same as the term within the brackets on the r.h.s. of the equation (4.10).

Thus one obtains from the equations (4.10) and (4.14),

$$E(j+1) = \Phi(j+1,j)E(j)+e^m(j+1). \tag{4.15}$$

It is noteworthy to mention that the transition matrix in orbit propagation problems is usually available for use in other purposes and so its use in evaluating E(j+1) is straight forward and does not require any extra computational effort. Here, one knows that the local truncation error of the predictor of order k-1 is $O(h^k)$; the local truncation error of the corrector of order k is $O(h^{k+1})$; and the local truncation error of the predictor-corrector is $O(h^{k+1})$, where h is the step size (Shampine and Gordon, 1975). As $E(j)$ is a variable created to estimate the global error after applying the predictor-corrector method, and as this estimate differs from the local truncation error by only one order of magnitude, one will have $E(j)$ with dispersions of order $O(h^k)$.

Consequently, the integration of the equation (4.13) by Euler's method of order one, which gives results with an error of $O(h^2)$, should be sufficiently accurate to take into account the term $e^m(j+1)$ of the equation (4.15), which is $O(h^{k+1})$. However, for studying the numerical behaviour of the results, the modified Euler's rule (Euler's method together with trapezoidal rule) of order 2 is also embedded in the computer program developed. Whatever be the integrator used for integrating the equation (4.13), one should note that the accuracy will not be degraded as the transition matrix is evaluated on the orbit generated by a more sophisticated integrator, to be explained in Section 6.

Then, the covariance matrix $P(j+1)$, defined in (3.1) can be deduced as:

$$P(j+1) \doteq \Phi(j+1,j).P(j).\Phi^T(j+1,j)+Q(j+1),$$

using the equations (4.15), (3.1) and (4.4).

Thus the standard deviations corresponding to the variances of the matrix P(j) furnish approximations of orders of magnitude of the true global errors $\nu(j)$.

5. ESTIMATE OF THE CONTRIBUTION OF ROUND-OFF ERRORS

Due to the limit in the number of digits that a computer could carry to represent the numbers being treated, the results obtained after a numerical process are rounded off in order to represent them within the machine capacity. Thus, each real number in the floating point (fl) range of a computer can be represented with a relative error which does not exceed the machine unit, u (Dahlquist and Bjorck, 1974), that is:

$$|fl(v \text{ op } w) - v \text{ op } w| / |v \text{ op } w| \leq u, \tag{5.1}$$

where 'op' represents one of the four basic operations (+; -; .; /) between the real variables v and w.

Considering the general formula of the Adams corrector which furnishes the value of x at the instant t_{j+1} (Gear, 1971),

$$x(j+1) = x(j) + h_j \{\beta^*_{ko} f^*(j+1) + \sum_{i=1}^{k-1} \beta^*_{ki} f(j+1-i)\}, \tag{5.2}$$

where β^*_{ki} are constant coefficients, $f^*(j+1)$ is the function value at t_{j+1} obtained from predictor and $f(j+1-i)$ is the function value at t_{j+1-i}, the representation of this formula in the arithmetic operations in floating point of a computer is given by:

$$x(j+1) = fl[x(j)+fl\{h_j(fl(\beta^*_{ko} f^*(j+1))+ fl(\sum_{i=1}^{k-1}\beta^*_{ki} f(j+1-i)))\}]. \qquad (5.3)$$

In evaluating the expression (5.3), one can easily compute the bounds for absolute errors in resulting terms which depend on the round-off errors (Dahlquist and Bjorck, 1974). Using the commutative and associative laws, the final expression for a bound for the round-off error $\varepsilon_{R_{\ell\ell}}$ in each component $x_\ell(j+1)$ can be written as:

$$\varepsilon_{R_{\ell\ell}}(j+1) \leq [|2.\tilde{x}_\ell(j)| + 7\tilde{h}_j|\tilde{\beta}^*_{ko} . \tilde{f}^*_{\ell\ell}(j+1)| + \tilde{h}_j\sum_{i=1}^{k-1} |(k+1-i). \beta^*_{ki}. \tilde{f}_{\ell\ell}(j+1-i)| + 4\tilde{h}_j\sum_{i=1}^{k-1}|\tilde{\beta}^*_{ki} \tilde{f}_{\ell\ell}(j+1-i)|]* 1.06u = r_{\ell\ell}(j+1), \qquad (5.4)$$

where the wave signs over the variables indicate the numerical values obtained after rounding.

Now, superposing the contribution of the round-off error over the contribution of the truncation error, one gets for the augmented covariance matrix of the total error:

$$P_T(j+1) = \Phi(j+1).P_T(j).\Phi^T(j+1)+Q(j+1)+R(j+1), \qquad (5.5)$$

where

$$R(j+1) = \text{diag}\ (r^2_{\ell\ell}(j+1),\ \ell=1,\ \ldots,n)$$

Here, the ℓ-th diagonal variance of P_T (j+1) furnish a measure of the order of magnitude of accumulated global error in the component x_ℓ (j+1).

6. NUMERICAL INTEGRATOR

The numerical integrator has been formulated based on the ABM multistep method. A specific fixed step algorithm of order 7(8) has been prepared after computing the β coefficients as given in Gear (1971) (see also Henrici, 1962). The proposed methodology for estimating the global errors has been embedded in the integration algorithm so that the integrator returns an estimate of global error as an output parameter (matrix $P_T(j+1)$) after each integration step. The computer used for testing the procedure is a BURROUGHS-6900 machine and the programming language is FORTRAN-IV.

7. TEST PROBLEMS

With the aim of applying the proposed procedure to orbit propagation problems of artificial satellites, three specific problems of such type have been chosen. These problems, in fact, form a part of the test problems chosen by several authors, as explained in introduction.

P1 - The first problem chosen is the well-known two-body problem which involves the integration of Newtonian equations of motion:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -x_1 / (x_1^2 + x_2^2)^{3/2} \\
\dot{x}_4 &= -x_2 / (x_1^2 + x_2^2)^{3/2},
\end{aligned}$$

with the initial conditions:

$$\begin{aligned}
x_1 (0) &= 1-e \\
x_2 (0) &= 0 \\
x_3 (0) &= 0 \\
x_4 (0) &= \sqrt{(1+e)(1-e)}
\end{aligned}$$

where e is the orbit eccentricity.

The motion in this case is periodic with period 2π, and the analytical solution of the problem is given as:

$$x_1 = \cos u - e$$
$$x_2 = \sqrt{1-e^2}\,\sin u$$
$$x_3 = -\sin u/(1-e\cos u)$$
$$x_4 = \sqrt{1-e^2}\,\cos u/(1-e\cos u),$$

where u is the solution of the Kepler's equation:

$$t = u - e \sin u.$$

In this first case, the orbit considered is circular (e=0).

P2 - The second problem chosen is the same as the first one except that a reasonably high elliptic orbit with e = 0.3 has been considered.

P3 - The third problem is the famous restricted three-body problem. The equations of motion of this classical problem are:

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = 2x_4 + x_1 - \mu^*(x_1+\mu)/r_1^3 - \mu(x_1 - \mu^*)/r_2^3$$
$$\dot{x}_4 = -2x_3 + x_2 - \mu^* x_2/r_1^3 - \mu x_2/r_2^3,$$

where

$$\mu = 1/82.45$$

$$\mu^* = 1-\mu$$

$$r_1 = ((x_1+\mu)^2 + x_2^2)^{1/2}$$

$$r_2 = ((x_1-\mu^*)^2 + x_2^2)^{1/2}.$$

For this system of equations, there is no analytical solution. However, given the initial conditions:

$$x_1(0) = 1.2$$

$$x_2(0) = 0$$

$$x_3(0) = 0$$

$$x_4(0) = -1.04935751,$$

there is a periodic solution with period T = 6.19216933.

8. RESULTS

In order to examine the influence of round-off errors on the solution, the problems P1 and P2 were initially integrated in one complete period. As it is a common practice in orbit propagation problems of low orbit earth satellites to divide the orbit in 100-150 discretization points, the orbits here were divided in 100 intervals of integration. Tables 8.1 and 8.2 show the true global errors (TE) and the estimates at few points in the orbit, both with (E2) and without (E1) inclusion of round-off errors. In the case of P1, the influence of round-off errors can clearly be seen. However, in the case of P2, the influence is practically nil. This later case can easily be explained because the global errors here are of orders greater than 1.0E-08 and it is evident that the round-off errors' influence can only be seen while working in the neighbourhood of machine accuracy, which is 1.0E-12 for the computer used here.

Then, to better test the influence of round-off errors, the number of discretization points considered in one orbit was increased from 100 to 500 in both the problems, forcing an improved accuracy in the solution. Beyond obtaining a high precision solution, the results obtained in the case of P1 did not reveal anything new. Nevertheless, the results obtained for P2, given in Table 8.3, confirmed the remark that the round-off errors have influence when working in the proximities of machine accuracy.

Now, though in the theoretical analysis of Section 4, Euler's method of order 1 was shown to be sufficient to take into account the term $e^{m}(j+1)$ of the equation (4.15), it was felt necessary to verify the results in terms of numerical behaviour by examining the effect of using modified Euler's rule of order 2 for integrating the equation (4.13). In both the problems P1 and P2, the global error estimates (E3) obtained with Euler's rule of order 2 (Tables 8.4 and 8.5) show a slight improvement over E2 (obtained with Euler's method of order 1) in the sense that they are not as conservative as the latter ones, when 100 discretization points were considered. As the effect was nil when 500 points were considered, those results are not shown.

The above results show that for short-term propagations, e.g. one orbital period, use of simple Euler's method is sufficient for solving the variational equations (4.13) if the orbit is divided into sufficient number of discretization points. In fact, with 150 discretization points, the estimates obtained with Euler's method (E4) are almost as good as the ones obtained with modified Euler's rule (E5), as shown in the Tables 8.6 and 8.7.

Having tested successfully the proposed procedure in the case of short-term propagations, the next step was to examine if the procedure furnishes reasonably accurate global error estimates in long-term propagations - the principal aim of the methodology developed. To analyze the behaviour of the procedure in these cases, the results of the Tables 8.6 and 8.7 were extended to a time period equal to 10 orbital periods. To one's dismay, the estimates in this case were found to be very much conservative. To better study this problem, some more tests were performed, now by employing modified Euler's rule in the integration of transition matrix equations. The results obtained in the case of P1 (Table 8.8), a circular orbit problem, showed again clearly the ability of the stochastic procedure to estimate satisfactorily the global errors committed in the integration process. However, in the case of problem P2, a highly eccentric orbit, a division of the orbit in 300 discretization points was required to obtain reasonably good error estimates (Table 8.9). It should be noted here that this latter case of using more number of steps per revolution to get good error estimates only shows the influence of the order of the method used in integrating the variational equations on error estimates and does not belittle the capacity of the proposed procedure in making these estimates. Through a careful analysis, one can easily infer that a use of a higher order method for the variational equations will certainly need a less number of points to give same accuracy (Rios Neto and Cardenuto, 1983).

Now coming to the problem P3, the equations were integrated by dividing the period T into intervals of 0.1. As the problem does not have an analytical solution, the solution obtained by a sophisticated variable step variable order Runge-Kutta method of order 7(8), RKF78 (Fehlberg, 1968) was taken as the true solution. Till about one-fourth of the orbital time period, the fixed step integrator used here provided solution comparable with the true solution. But later, it could not cope with the problem and the solution started deteriorating, showing that the problem of type P3 cannot be dealt with fixed step fixed order integrators. However, the manner the global error estimates agreed with the true global errors was very much impressive. Solution results obtained at few points, till the global error becomes of the order 1.0E+00 in anyone of the components, are given in Table 8.10. One can easily see that even when dealing with solution values corrupted with large errors, the procedure proposed in this paper behaved very well in estimating the global errors.

9. CONCLUSIONS

The analysis done over the results of Tables 8.1, 8.2 and 8.3 shows that, for obtaining reasonably good global error estimates, when working in the limits of machine precision, one should take the round-off errors into account.

The study on the results of Tables 8.4, 8.5, 8.6 and 8.7 indicates that in short term propagations, integration of the transition matrix equations by Euler's method of order 1 gives sufficiently accurate, and very often conservative, global error estimates.

Nevertheless, in long-term propagations, it is mandatory to use a higher order method such as modified Euler's method of order 2 in the integration of variational equations, to get reasonably good global error estimates, as seen from the Tables 8.8 and 8.9.

The methodology developed here could not be used fully in the case of the restricted three-body problem, as seen from Table 8.10, not because of its incapacity to deal with this type of problems but because of the constraints involved in the fixed step fixed order numerical integrator used here.

On the whole, one concludes that the diagonal variances of the covariance matrix, which estimates the global error and whose expression was derived using the stochastic approach developed in this paper, furnish reasonably precise measures of the orders of magnitude of accumulated global errors in short-term as well as long-term orbit propagations as confirmed in two-body circular, two-body elliptic and the restricted three-body problems, similar to those encountered in flight mechanics and control.

TABLE 8.1 - Global error comparison in the case of P1 (100 points)

Discretization point considered	Component x_1			Component x_2			Component x_3			Component x_4		
	TE	E1	E2	TE	E1	E2	TE	E1	E2	TE	E1	E2
Initial	0.0	7.3E-12	1.1E-11	1.8E-12	1.1E-12	5.3E-12	1.8E-12	1.5E-12	5.4E-12	1.8E-12	7.1E-12	1.1E-11
1/4th orbit	8.6E-11	2.8E-11	4.6E-11	5.1E-11	3.5E-11	5.9E-11	4.4E-11	2.4E-11	4.3E-11	1.5E-10	4.7E-11	7.7E-11
1/2 the orbit	1.5E-10	9.8E-11	1.6E-10	5.6E-10	1.7E-10	2.8E-10	6.0E-10	1.9E-10	3.1E-10	1.3E-10	7.3E-11	1.2E-10
3/4th orbit	1.2E-09	4.5E-10	7.3E-10	1.5E-10	1.3E-10	2.1E-10	1.2E-10	1.1E-10	1.8E-10	1.2E-09	4.0E-10	6.5E-10
End of the orbit	1.1E-10	1.5E-10	2.5E-10	1.7E-09	6.5E-10	1.0E-09	1.7E-09	5.6E-10	8.9E-10	7.3E-11	1.4E-10	2.3E-10

TABLE 8.2 - Global error comparison in the case of P2 (100 points)

Discretization point considered	Component x_1			Component x_2			Component x_3			Component x_4		
	TE	E1	E2	TE	E1	E2	TE	E1	E2	TE	E1	E2
Initial	6.4E-08	7.5E-10	7.5E-10	2.3E-08	5.2E-08	5.2E-08	1.4E-08	3.7E-07	3.7E-07	5.2E-08	1.5E-07	1.5E-07
1/4th orbit	9.8E-08	4.4E-07	4.4E-07	6.4E-07	7.1E-07	7.1E-07	1.2E-07	3.5E-07	3.5E-07	8.2E-07	1.0E-06	1.0E-06
1/2 the orbit	1.2E-06	2.0E-06	2.0E-06	2.0E-06	2.6E-06	2.6E-06	1.6E-06	2.4E-06	2.4E-06	5.6E-07	1.0E-06	1.0E-06
3/4th orbit	4.9E-06	7.7E-06	7.7E-06	6.8E-07	1.3E-06	1.3E-06	2.5E-06	4.2E-06	4.2E-06	3.2E-06	4.5E-06	4.5E-06
End of the orbit	9.9E-08	2.4E-06	2.4E-06	7.8E-06	1.1E-05	1.1E-05	1.2E-05	1.4E-05	1.4E-05	2.5E-08	2.5E-06	2.5E-06

TABLE 8.3 - Global error comparison in the case of P2 (500 points)

Discretization point considered	Component x_1			Component x_2			Component x_3			Component x_4		
	TE	E1	E2	TE	E1	E2	TE	E1	E2	TE	E1	E2
Initial	0.0	1.8E-13	5.4E-12	0.0	4.5E-14	1.5E-12	0.0	7.3E-13	2.3E-12	0.0	1.5E-12	1.1E-11
1/4th orbit	3.3E-11	5.5E-12	5.0E-11	7.3E-12	1.4E-11	1.3E-10	2.0E-11	3.4E-12	4.7E-11	2.4E-11	1.8E-11	1.7E-10
1/2 the orbit	0.0	2.8E-11	2.9E-10	1.3E-10	4.3E-11	4.2E-10	1.1E-10	3.6E-11	3.7E-10	9.1E-12	1.3E-11	1.3E-10
3/4th orbit	2.6E-10	1.1E-10	1.2E-09	1.0E-10	1.7E-11	1.6E-10	1.3E-10	5.8E-11	6.4E-10	2.2E-10	7.0E-11	7.3E-10
End of the orbit	1.3E-10	8.7E-12	1.1E-10	6.1E-10	1.8E-10	2.0E-09	1.2E-09	2.5E-10	2.9E-09	2.9E-11	1.4E-11	1.6E-10

TABLE 8.4 - Global error comparison in the case of P1(100 points)

Discretization point considered	Component x_1		Component x_2		Component x_3		Component x_4	
	TE	E3	TE	E3	TE	E3	TE	E3
Initial	0.0	1.1E-11	1.8E-12	5.3E-12	1.8E-12	5.4E-12	1.8E-12	1.1E-11
1/4th orbit	8.6E-11	4.5E-11	5.1E-11	6.2E-11	4.4E-11	4.2E-11	1.5E-10	8.1E-11
1/2 the orbit	1.5E-10	1.9E-10	5.6E-10	3.0E-10	6.0E-10	3.6E-10	1.3E-10	1.2E-10
3/4th orbit	1.2E-09	8.6E-10	1.6E-10	3.0E-10	1.2E-10	2.0E-10	1.2E-09	8.6E-10
End of the orbit	1.1E-10	3.3E-10	1.7E-09	1.5E-09	1.7E-09	1.4E-09	7.3E-11	2.2E-10

TABLE 8.5 - Global error comparison in the case of P2 (100 points)

Discretization point considered	Component x_1		Component x_2		Component x_3		Component x_4	
	TE	E3	TE	E3	TE	E3	TE	E3
Initial	6.4E-08	7.5E-10	2.3E-08	5.2E-08	1.4E-08	3.7E-07	5.2E-08	1.5E-07
1/4th orbit	9.8E-08	4.3E-07	6.4E-07	7.6E-07	1.2E-07	3.7E-07	8.2E-07	1.1E-06
1/2 the orbit	1.2E-06	2.3E-06	2.0E-06	2.7E-06	1.7E-06	2.7E-06	5.6E-07	1.0E-06
3/4th orbit	4.9E-06	8.8E-06	6.8E-07	9.4E-07	2.5E-06	5.0E-06	3.2E-06	5.5E-06
End of the orbit	9.9E-08	3.7E-07	7.8E-06	1.6E-05	1.2E-05	2.3E-05	2.5E-08	7.0E-07

TABLE 8.6 - Global error comparison in the case of P1 (150 points)

Discretization point considered	Component x_1			Component x_2			Component x_3			Component x_4		
	TE	E4	E5	TE	E4	E5	TE	E4	E5	TE	E4	E5
Initial	1.8E-12	7.9E-12	7.9E-12	0.0	3.6E-12	3.6E-12	0.0	3.6E-12	3.6E-12	1.8E-12	7.9E-12	7.9E-12
1/4th orbit	6.4E-11	4.6E-11	4.6E-11	2.2E-11	6.0E-11	6.3E-11	1.1E-11	3.8E-11	3.7E-11	4.5E-11	8.2E-11	8.5E-11
1/2 the orbit	7.3E-11	1.5E-10	1.7E-10	1.7E-10	2.8E-10	2.9E-10	1.4E-10	3.1E-10	3.4E-10	4.2E-11	1.1E-10	1.1E-10
3/4th orbit	1.5E-10	7.0E-10	8.0E-10	1.0E-10	1.9E-10	2.5E-10	5.5E-11	1.7E-10	1.9E-10	1.6E-10	6.3E-10	7.8E-10
End of the orbit	7.3E-11	2.2E-10	2.8E-10	1.7E-10	1.0E-09	1.4E-09	1.8E-10	9.1E-10	1.3E-09	2.4E-11	1.9E-10	1.9E-10

TABLE 8.7 - Global error comparison in the case of P2 (150 points)

Discretization point considered	Component x_1			Component x_2			Component x_3			Component x_4		
	TE	E4	E5	TE	E4	E5	TE	E4	E5	TE	E4	E5
Initial	2.0E-09	1.1E-09	1.1E-09	5.5E-10	2.0E-09	2.0E-09	1.3E-09	1.8E-08	1.8E-08	2.4E-09	4.3E-09	4.3E-09
1/4th orbit	5.8E-09	3.2E-08	3.1E-08	2.8E-08	4.5E-08	4.7E-08	7.6E-09	1.9E-08	1.9E-08	3.6E-08	6.1E-08	6.3E-08
1/2 the orbit	5.3E-08	1.2E-07	1.3E-07	8.5E-08	1.5E-07	1.6E-07	7.3E-08	1.4E-07	1.5E-07	2.3E-08	5.6E-08	5.7E-08
3/4th orbit	2.1E-07	4.4E-07	4.6E-07	2.9E-08	7.2E-08	5.8E-08	1.2E-07	2.5E-07	2.8E-07	1.4E-07	2.6E-07	2.9E-07
End of the orbit	3.8E-09	9.9E-08	1.8E-08	3.6E-07	6.8E-07	8.8E-07	5.6E-07	9.0E-07	1.3E-06	4.9E-09	1.0E-07	3.4E-08

TABLE 8.8 - Global error comparison in the case of P1(150 points)

Time in number of orbits	Component x_1		Component x_2		Component x_3		Component x_4	
	True error	Error estimate	True error	Error estimate	True error	Error estimate	True error	Error estimate
01	7.3E-11	2.8E-10	1.7E-10	1.4E-09	1.8E-10	1.3E-09	2.4E-11	1.9E-10
02	1.2E-10	4.1E-10	3.2E-09	3.8E-09	3.2E-09	3.7E-09	4.2E-11	2.7E-10
03	2.0E-10	5.3E-10	8.6E-09	7.2E-09	8.6E-09	7.1E-09	9.1E-11	3.4E-10
04	2.6E-10	6.5E-10	1.4E-08	1.2E-08	1.4E-08	1.2E-08	1.3E-10	4.1E-10
05	3.1E-10	7.9E-10	2.1E-08	1.7E-08	2.1E-08	1.7E-08	1.5E-10	4.9E-10
06	3.5E-10	9.7E-10	2.7E-08	2.4E-08	2.7E-08	2.3E-08	1.7E-10	5.7E-10
07	4.5E-10	1.2E-09	3.5E-08	3.3E-08	3.5E-08	3.3E-08	2.2E-10	6.8E-10
08	5.5E-10	1.5E-09	4.3E-08	4.5E-08	4.3E-08	4.5E-08	2.8E-10	8.2E-10
09	6.1E-10	1.9E-09	5.2E-08	6.0E-08	5.2E-08	6.0E-08	3.0E-10	1.0E-09
10	6.3E-10	2.4E-09	6.1E-08	7.9E-08	6.2E-08	7.9E-08	3.0E-10	1.2E-09

TABLE 8.9 - Global error comparison in the case of P2 (300 points)

Time in number of orbits	Component x_1		Component x_2		Component x_3		Component x_4	
	True error	Error estimate	True error	Error estimate	True error	Error estimate	True error	Error estimate
01	1.8E-12	1.9E-10	1.1E-09	6.5E-09	1.6E-09	9.5E-09	1.5E-11	2.9E-10
02	4.0E-11	2.7E-10	1.0E-08	1.8E-08	1.6E-08	2.6E-08	1.5E-11	4.6E-10
03	2.4E-11	3.4E-10	2.5E-08	8.3E-08	3.7E-08	5.0E-08	8.7E-11	5.8E-10
04	4.4E-11	3.9E-10	3.8E-08	5.4E-08	5.8E-08	8.1E-08	2.0E-10	7.0E-10
05	2.2E-10	4.5E-10	4.9E-08	8.0E-08	7.4E-08	1.2E-07	1.2E-10	8.2E-10
06	2.1E-10	5.1E-10	5.9E-08	1.1E-07	8.9E-08	1.7E-07	1.0E-10	9.5E-10
07	2.3E-10	5.7E-10	6.7E-08	1.5E-07	1.0E-07	2.3E-07	8.7E-11	1.1E-09
08	1.8E-10	6.4E-10	7.7E-08	2.0E-07	1.2E-07	3.0E-07	4.4E-11	1.3E-09
09	1.0E-10	7.3E-10	8.7E-08	2.6E-07	1.3E-07	3.9E-07	0.0	1.5E-09
10	6.2E-11	8.4E-10	9.9E-08	3.4E-07	1.5E-07	5.1E-07	2.9E-11	1.8E-09

TABLE 8.10 - Global error comparison in the case of P3

Time instant	Component x_1		Component x_2		Component x_3		Component x_4	
	True error	Error estimate	True error	Error estimate	True error	Error estimate	True error	Error estimate
Initial	0.0	2.9E-12	1.8E-12	5.1E-12	1.8E-12	1.8E-10	0.0	7.3E-11
0.4	5.8E-11	1.1E-10	2.4E-11	8.2E-11	5.6E-11	3.7E-10	5.3E-11	3.4E-10
0.8	7.1E-11	2.4E-10	6.9E-11	2.4E-10	5.5E-11	3.4E-10	1.6E-10	5.8E-10
1.2	3.3E-11	3.6E-10	1.3E-10	5.6E-10	6.5E-11	6.4E-10	6.0E-11	1.5E-09
1.42	2.6E-06	6.2E-07	2.6E-08	4.3E-08	5.3E-05	8.1E-05	3.2E-05	1.1E-05
1.45	2.1E-04	2.8E-05	1.8E-04	1.4E-05	4.3E-03	7.0E-03	2.4E-02	1.1E-02
1.46	2.8E-04	1.5E-04	3.3E-03	1.9E-04	8.1E-01	1.4E-01	5.6E-01	8.8E-01

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Address for correspondence:

Kondapalli Rama Rao
Instituto de Pesquisas Espaciais
Depto. de Controle e Guiagem
Cx.P. 515
12201 - São José dos Campos - SP
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AUTOR(ES)

Atair Rios Neto

Kondapalli Rama Rao

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