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AN ANALYTICAL SOLUTION FOR THE TEMPERATURE DISTRIBUTION OF MOUTING PLATES OF SATELLITES

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14. Abstract/Notes

An analytical solution has been obtained for the temperature distribution in a thin rectangular plate with two regions of internal heat generation. One of the sides is subjected to a constant heat flux, while the others are at specified temperatures. The problem has been resolved by using linear superposition and an image technique.

15. Remarks

This work will be presented at the 35th annual meeting of SBPC, July 6 to 13, 1983, Belem, Pará.

1 - INTRODUCTION

A typical satellite mounting plate supports several heat dissipating electronic components. For satellite thermal analysis, simulation and design, a knowledge of the thermal resistance between the various components is necessary. The problem of the temperature distribution and the associated problem of thermal resistance of two dimensional rectangular plates, with no internal heat generating components, have been dealt by Oliveira and Forslund (1974) and by Schneider, Yovanovich and Cane (1980). Temperature distributions and thermal resistance of plates, with constant and variable internal heat generation, have been dealt by Venkataraman, Cardoso and Oliveira Filho (1982), who obtained a numerical solution using finite difference methods. Here we consider the problem of a thin rectangular plate with internal heat generation produced in two distinct regions, which corresponds to the heat dissipation by two electronic modules on the mounting plate. Three of the sides have specified temperature distributions, while the remaining side has a specified flux. The objective is to determine the temperature distribution. By linear superposition and using the method of images, the problem is transformed such that all the sides are at the same temperature. The problem is then solved by a double Fourier expansion consistent with the boundary conditions. It is proposed, at a later date to make a comparison of this series solution with a numerical solution obtained by the finite element method.

2 - FORMULATION AND SOLUTION

As shown in Figure 1, we consider a thin rectangular plate of length a and width b. The coordinate system selected is shown in Figure 1.

The side $x = 0$ is subjected to a temperature distribution $f(y)$ while the side $y = 0$ is subjected to a constant heat flux q_0. The other two sides are at zero temperature. It should be noted that even if these two sides had a different temperature distribution, the problem can be reduced by linear superposition into resolving the present one. Moreover, even if the specified flux is not a constant, this presents no special problem. Internal heats (per unit volume and per second) Q_1 and Q_2 are generated in two rectangular regions defined by (x_1, y_1), (x_2, y_2) and (x_3, y_3), (x_4, y_4) respectively.

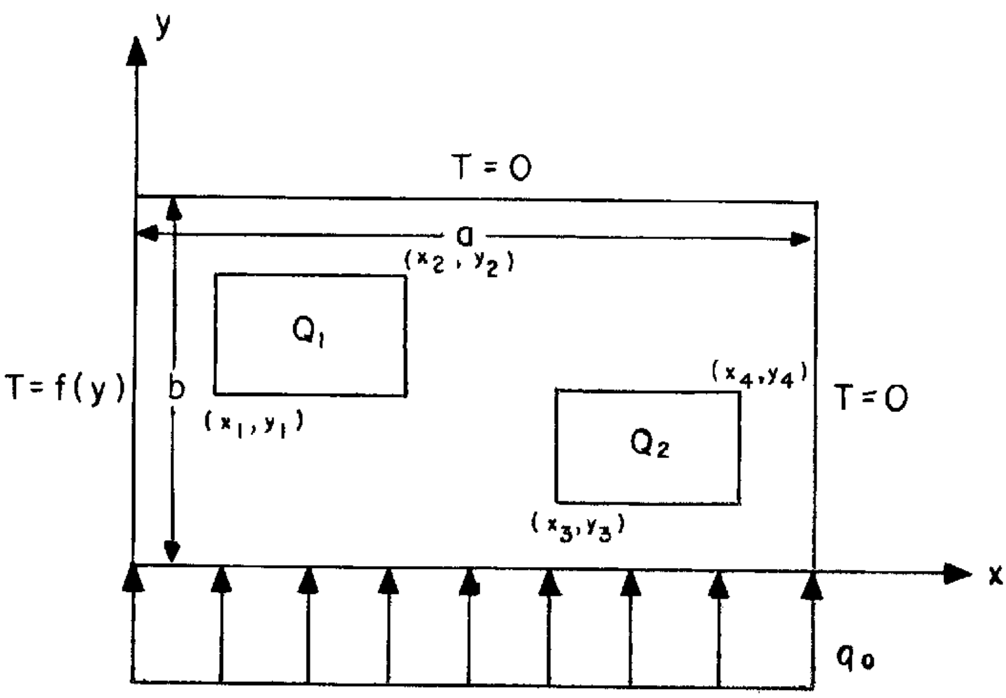

Fig. 1 - The plate geometry.

The governing equation for the steady state temperature T(x,y) is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = - \frac{Q(x,y)}{k} , \qquad (1)$$

where $Q(x,y)$ is the internal heat generation rate and k is the plate thermal conductivity. The boundary conditions are

$$T(0,y) = f(y) \quad ; \quad T(a,y) = 0 \quad ;$$

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} = q_0 \quad ; \quad T(x,b) = 0 \quad . \tag{2}$$

By using linear superposition, we can divide the problem into three simple problems, each one with one inhomogeneity. Thus, let

$$T(x,y) = T_1(x,y) + T_2(x,y) + T_3(x,y) \ ,$$

where T_1, T_2 and T_3 will take care of the inhomogeneity in the differential equation (internal heat generation), the inhomogeneous boundary condition $f(y)$ and the inhomogeneity due to heat flux q_0 respectively. T_1, T_2 and T_3 satisfy the following differential equations and boundary conditions:

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = -\frac{Q(x,y)}{k} \ ; \tag{3}$$

$$T_1(0,y) = 0 \quad ; \quad T_1(a,y) = 0 \quad ;$$

$$\left(\frac{\partial T_1}{\partial y}\right)_{y=0} = 0 \quad ; \quad T_1(x,b) = 0 \quad ; \tag{4}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0 \quad ; \tag{5}$$

$$T_2(0,y) = f(y) \quad ; \quad T_2(a,y) = 0 \quad ;$$

$$\left(\frac{\partial T_2}{\partial y}\right)_{y=0} = 0 \quad ; \quad T_2(x,b) = 0 \quad ; \tag{6}$$

$$\frac{\partial^2 T_3}{\partial x^2} + \frac{\partial^2 T_3}{\partial y^2} = 0 \; ; \tag{7}$$

$$\begin{aligned} &T_3(0,y) = 0 \quad ; \quad T_3(a,y) = 0 \; ; \\ &-k\left(\frac{\partial T_3}{\partial y}\right)_{y=0} = q_0 \; ; \quad T_3(x,b) = 0 \; . \end{aligned} \tag{8}$$

The superposition is schematically shown in Figure 2.

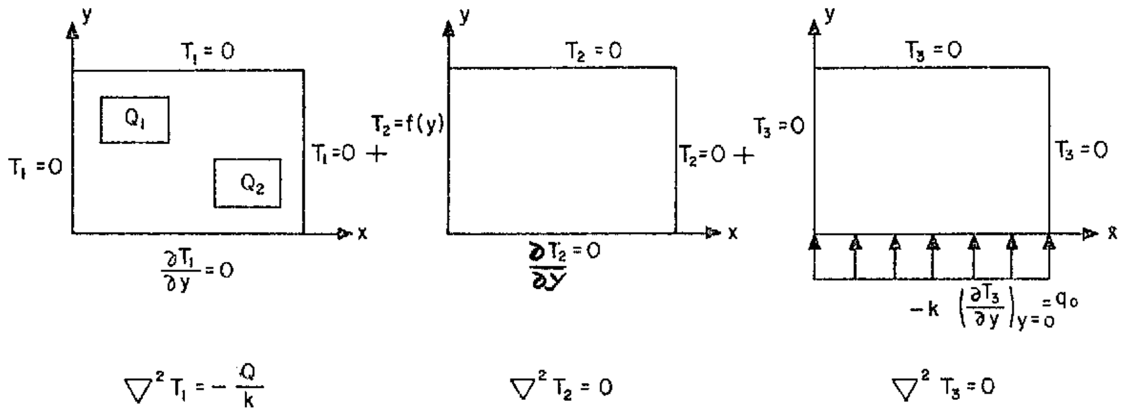

Fig. 2 - The superposition of T_1, T_2 and T_3.

The solution for $T_2(x,y)$ and $T_3(x,y)$ can be obtained by the method of separation of variables (Arpaci, 1966). For example, if $f(y)$ is a constant equal to T_0, we can get

$$T_2(x,y) = \frac{4T_0}{\pi} \sum_{n=0,1,2,\ldots} \frac{(-1)^n \sinh \lambda_n(a-x) \cos \lambda_n y}{(2n+1) \sinh \lambda_n a} , \tag{9}$$

where

$$\lambda_n = \frac{(2n+1)\pi}{2b} . \tag{10}$$

The solution for T_3 is

$$T_3(x,y) = \frac{4q_0 a}{k\pi^2} \sum_{n=1,3,5,\ldots} \frac{\sinh \frac{n\pi}{a}(y-b) \sin \frac{n\pi x}{a}}{n^2 \cosh \frac{n\pi b}{a}} . \tag{11}$$

To find $T_1(x,y)$, a mirror image is formed by reflecting the boxes about the side $y = 0$, as shown in Figure 3. The origin is shifted to the left corner of this doubled plate with the new coordinates as (x,ξ) where

$$\xi = y + b . \tag{12}$$

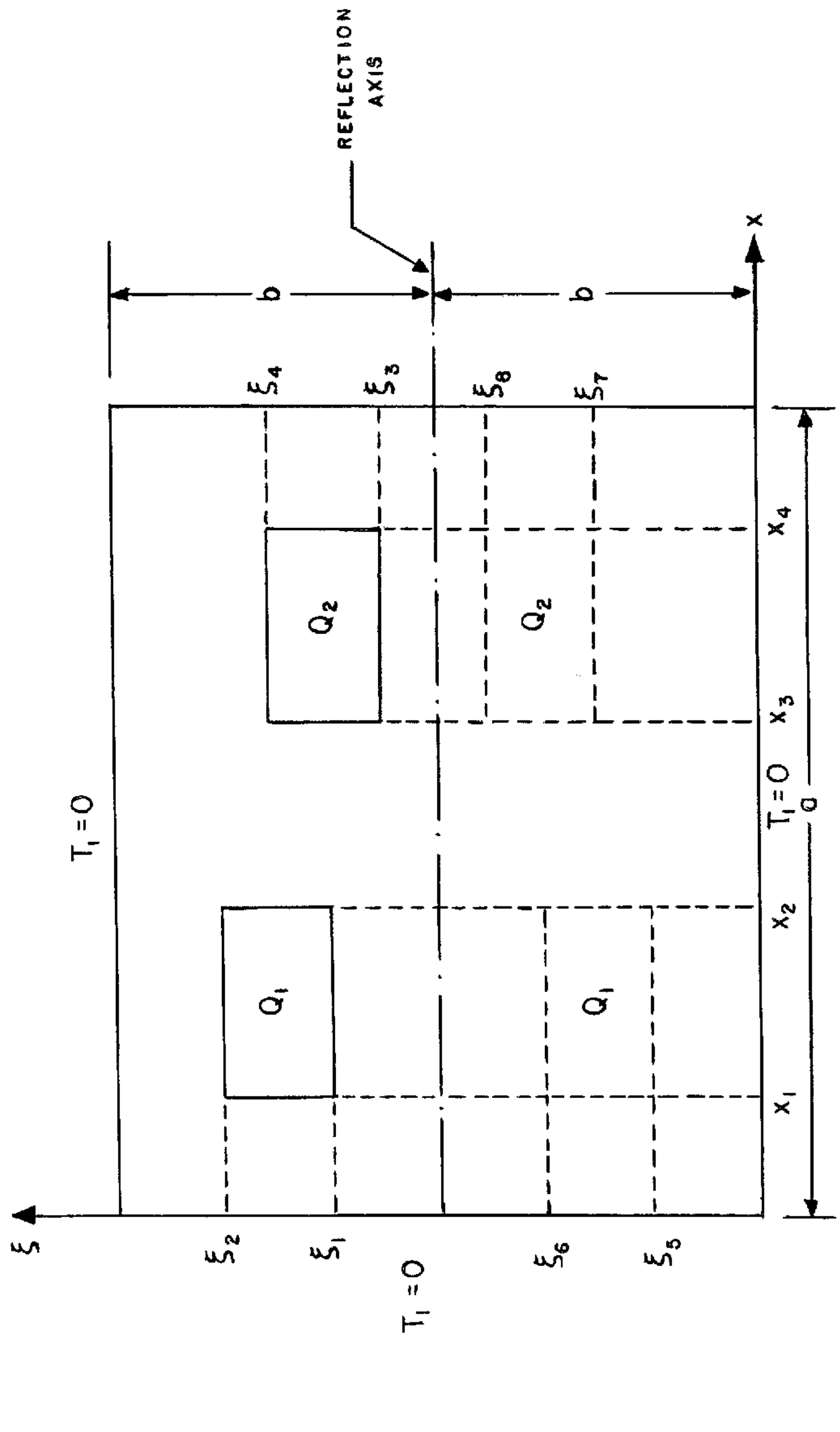

Fig. 3 - The mirror image.

The coordinates of the boxes are indicated in the Figure 3. Because of symmetry, $(\partial T_1/\partial\xi)_{\xi=b} = 0$ which signifies $(\partial T_1/\partial y)_{y=0} = 0$. Thus, the differential equation for the temperature $T_1(x,\xi)$ becomes

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial \xi^2} = - \frac{Q(x,\xi)}{k} , \tag{13}$$

with the following boundary conditions

$$\begin{aligned} &T_1(0,\xi) = 0 \quad ; \quad T_1(a,\xi) = 0 \ ; \\ &T_1(x,0) = 0 \quad ; \quad T_1(x,2b) = 0 . \end{aligned} \tag{14}$$

The solution for $T_1(x,\xi)$ is obtained by expanding $T_1(x,\xi)$ and $Q(x,\xi)$ in a double Fourier series and matching the coefficients. Let

$$T_1(x,\xi) = \sum_m \sum_n A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} \qquad m,n = 1,2,3,... \tag{15}$$

where the constants A_{mn} has to be determined. Equation 15 satisfies all the boundary conditions set forth in Equation 14. Differentiating twice, we get

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial \xi^2} = - \sum_m \sum_n A_{mn}\, \alpha_{mn}^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} , \tag{16}$$

where

$$\alpha_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{4b^2} . \tag{17}$$

Expanding $Q/k(x,\xi)$ in a double Fourier series, we get

$$\frac{Q}{K}(x,\xi) = \sum_m \sum_n B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} , \tag{18}$$

where

$$B_{mn} = \frac{2}{ab} \int_0^a \int_0^{2b} \frac{Q}{k} (x,\xi) \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} \, dx \, d\xi . \tag{19}$$

Substituting Equations 16 and 18 in Equation 13, we get

$$\sum_m \sum_n \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} (-\alpha_{mn}^2 A_{mn} + B_{mn}) = 0 . \tag{20}$$

Thus,

$$A_{mn} = \frac{B_{mn}}{\alpha_{mn}^2} . \tag{21}$$

The conditions $(\partial T_1/\partial\xi)_{y=b} = 0$ gives n= 1,3,5,... (odd).

Thus, the solution for T_1 becomes

$$T_1(x,\xi) = \sum_m \sum_n \frac{B_{mn}}{\alpha_{mn}^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi\xi}{2b} , \tag{22}$$

$m = 1,2,3,...$
$n = 1,3,5,...\text{(odd)}$

with α_{mn} and B_{mn}, given by Equations 17 and 19 respectively. In this case, with the heat generation confined to two distint regions, we get

$$B_{mn} = \frac{2}{kab}\left[Q_1 \int_{x_1}^{x_2} \left\{ \int_{\xi_5}^{\xi_6} \sin\frac{m\pi x}{a} \sin\frac{n\pi\xi}{2b}\, dx\, d\xi \; + \right.\right.$$

$$\left. + \int_{\xi_1}^{\xi_2} \sin\frac{m\pi x}{a} \sin\frac{n\pi\xi}{2b}\, dx\, d\xi \right\} + Q_2 \int_{x_3}^{x_4} \left\{ \int_{\xi_7}^{\xi_8} \sin\frac{m\pi x}{a} \sin\frac{n\pi\xi}{2b}\, dx\, d\xi \right.$$

$$\left.\left. + \int_{\xi_3}^{\xi_4} \sin\frac{m\pi x}{a} \sin\frac{n\pi\xi}{2b}\, dx\, d\xi \right\}\right] . \qquad (23)$$

These integrals can be evaluated quite easily. This completes the solution for T_1. The complete soluti-n for (x,y) is given by the superposition of T_1 (Equation 22), T_2 (Equation 9) and T_3 (Equation 11).

3 - CONCLUSION

Thus, we have obtained an analytical solution for the temperature distribution on the mounting plate using the method of images, in a series expansion. This does not necessarily mean that the precision here will be better than in a numerical solution. Later, it is proposed to make a comparison of the analytical method with a numerical solution obtained by the finite element method.

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