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16. Summary/Notes

This ninth chapter, on the Fundamentals of Plasma Physics, presents a derivation of the macroscopic transport equations for a plasma, considered as a single conducting fluid. The equations of conservation of mass, of momentum, and of energy, for the plasma as a whole, are derived starting from the corresponding equations for each individual species constituting the plasma. These equations are complemented by a set of electrodynamic equations for a conducting fluid. A detailed derivation is given for the generalized Ohm's law. The complete set of simplified magnetohydrodynamic (MHD) equations, normally used in MHD problems, are also presented and the approximations involved are discussed.

17. Remarks

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CHAPTER 9

MACROSCOPIC EQUATIONS FOR A CONDUCTING FLUID

CHAPTER 9

MACROSCOPIC EQUATIONS FOR A CONDUCTING FLUID

1. MACROSCOPIC VARIABLES FOR A PLASMA AS A CONDUCTING FLUID

A plasma can also be considered as a conducting fluid, without specifying its various individual species. The macroscopic transport equations, derived in the previous chapter, describe the macroscopic behavior of each individual species in the plasma (electrons, ions, neutral particles). We will determine now the set of transport equations which describe the behavior of the plasma as a whole. Each macroscopic variable is combined, by adding the contributions of the various particle species in the plasma. This procedure yields the *total* macroscopic parameters of interest, such as the total mass and charge density, the total mass and charge current density (ou flux), the total kinetic pressure dyad and the total heat flux vector.

The *mass density* is the mass per unit volume of fluid and is given by

$$\rho = \sum_{\alpha} \rho_{\alpha} = \sum_{\alpha} n_{\alpha} m_{\alpha} \tag{1.1}$$

The *electric charge density* represents the electric charge per unit volume of fluid,

$$\rho_c = \sum_\alpha n_\alpha q_\alpha \tag{1.2}$$

The *mean fluid velocity*, $\underset{\sim}{u}$, is defined such that the momentum density is the same as if each particle was moving at the mean fluid velocity, according to

$$\rho \underset{\sim}{u} = \sum_\alpha \rho_\alpha \underset{\sim}{u}_\alpha \tag{1.3}$$

The mean velocity of the plasma, $\underset{\sim}{u}$, is therefore a weighted mean value, where each species is weighted proportionally to its mass density. The mean velocity of each particle species, when considered in a reference system moving with the global mean velocity $\underset{\sim}{u}$ of the plasma, is called the *diffusion velocity* $\underset{\sim}{w}_\alpha$,

$$\underset{\sim}{w}_\alpha = \underset{\sim}{u}_\alpha - \underset{\sim}{u} = \underset{\sim}{u}_\alpha - \frac{1}{\rho} \sum_\alpha \rho_\alpha \underset{\sim}{u}_\alpha \tag{1.4}$$

The *mass current density*, or *mass flux* is given by

$$\underset{\sim}{J}_m = \sum_\alpha n_\alpha m_\alpha \underset{\sim}{u}_\alpha = \rho \underset{\sim}{u} \tag{1.5}$$

and the *electric current density*, or *charge flux* is expressed as

$$\underset{\sim}{J} = \sum_{\alpha} n_{\alpha} q_{\alpha} \underset{\sim}{u}_{\alpha} = \rho_c \underset{\sim}{u} + \sum_{\alpha} n_{\alpha} q_{\alpha} \underset{\sim}{w}_{\alpha} \tag{1.6}$$

Note that in Eq. (1.5) we have $\sum_{\alpha} \rho_{\alpha} \underset{\sim}{w}_{\alpha} = 0$, in virtue of Eq. (1.4), which defines the diffusion velocity $\underset{\sim}{w}_{\alpha}$.

The *kinetic pressure dyad* for each species of particles in the plasma is defined in Eq. (6.6.2) as

$$\underset{\approx}{p}_{\alpha} = \rho_{\alpha} < \underset{\sim}{c}_{\alpha} \underset{\sim}{c}_{\alpha} > \tag{1.7}$$

where $\underset{\sim}{c}_{\alpha} = \underset{\sim}{v} - \underset{\sim}{u}_{\alpha}$ is the peculiar or random velocity of the particles of type α. Note that the pressure is defined as the time rate in which momentum is transported by the particles of type α through a surface element moving with the mean velocity of the particles of type α. For the plasma as a whole it is necessary to define a peculiar velocity $\underset{\sim}{c}_{\alpha o}$, for the particles of type α, relative to the global mean velocity of the plasma, $\underset{\sim}{u}$, that is

$$\underset{\sim}{c}_{\alpha o} = \underset{\sim}{v} - \underset{\sim}{u} \tag{1.8}$$

Thus, the total pressure is defined as the time rate of transfer of momentum, due to all particles in the plasma, through a surface element moving with the global mean velocity $\underset{\sim}{u}$. The *total kinetic pressure dyad*, $\underset{\approx}{p}$, is, therefore, given by

$$\underset{\approx}{p} = \sum_{\alpha} \rho_{\alpha} < \underset{\sim}{c}_{\alpha o} \; \underset{\sim}{c}_{\alpha o} > \tag{1.9}$$

To relate $\underset{\approx}{p}$, given in (1.9), with $\underset{\approx}{p}_{\alpha}$, given in (1.7) ,we substitute $\underset{\sim}{u}$ by $\underset{\sim}{u}_{\alpha}$ - $\underset{\sim}{w}_{\alpha}$ and $\underset{\sim}{v}$ by $\underset{\sim}{c}_{\alpha}$ + $\underset{\sim}{u}_{\alpha}$ in (1.8), which gives

$$\underset{\sim}{c}_{\alpha o} = \underset{\sim}{c}_{\alpha} + \underset{\sim}{w}_{\alpha} \tag{1.10}$$

Consequently,

$$\underset{\approx}{p} = \sum_{\alpha} \rho_{\alpha} < (\underset{\sim}{c}_{\alpha} + \underset{\sim}{w}_{\alpha})(\underset{\sim}{c}_{\alpha} + \underset{\sim}{w}_{\alpha}) > \tag{1.11}$$

and expanding this expression,

$$\underset{\approx}{p} = \sum_{\alpha} \rho_{\alpha}(< \underset{\sim}{c}_{\alpha} \underset{\sim}{c}_{\alpha} > + < \underset{\sim}{c}_{\alpha} \underset{\sim}{w}_{\alpha} > + < \underset{\sim}{w}_{\alpha} \underset{\sim}{c}_{\alpha} > + < \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha} >) \tag{1.12}$$

From the definition of $\underset{\sim}{w}_{\alpha}$ we see that $< \underset{\sim}{w}_{\alpha} > = \underset{\sim}{w}_{\alpha}$, since it is a macroscopic variable, and,therefore, $< \underset{\sim}{c}_{\alpha} \underset{\sim}{w}_{\alpha} > = < \underset{\sim}{c}_{\alpha} > \underset{\sim}{w}_{\alpha} = 0$. Thus , Eq.(1.12) becomes

$$\underset{\approx}{p} = \sum_{\alpha} \underset{\approx}{p}_{\alpha} + \sum_{\alpha} \rho_{\alpha} \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha} \tag{1.13}$$

Note that $\underset{\approx}{p}_{\alpha}$ is a pressure relative to $\underset{\sim}{u}_{\alpha}$, whereas $\underset{\approx}{p}$ is relative to the global mean velocity $\underset{\sim}{u}$.

The *total scalar pressure*, p, is defined as one third the trace of $\underset{\approx}{p}$, that is,

$$p = \frac{1}{3} \sum_i p_{ii} = \frac{1}{3} \sum_i \sum_\alpha \rho_\alpha < c_{\alpha oi} \; c_{\alpha oi} >$$

$$= \frac{1}{3} \sum_\alpha \rho_\alpha < c^2_{\alpha o} > \tag{1.14}$$

Using Eq. (1.13) we can write

$$p = \sum_\alpha p_\alpha + \frac{1}{3} \sum_\alpha \rho_\alpha \, w^2_\alpha \tag{1.15}$$

Finally, we define the *total heat flux vector*, $\underset{\sim}{q}$, as

$$\underset{\sim}{q} = \sum_\alpha \frac{1}{2} \rho_\alpha < c^2_{\alpha o} \, \underset{\sim}{c}_{\alpha o} > \tag{1.16}$$

and the *thermal energy density* of the plasma as a whole as

$$\frac{3}{2} p = \sum_\alpha \frac{1}{2} \rho_\alpha < c^2_{\alpha o} > \tag{1.17}$$

It is useful to relate $\underset{\sim}{q}$, defined in (1.16), with the heat flux vector $\underset{\sim}{q}_\alpha$ for the particles of type α,

$$\underset{\sim}{q}_\alpha = \frac{1}{2} \rho_\alpha < c^2_\alpha \, \underset{\sim}{c}_\alpha > \tag{1.18}$$

For this purpose, we substitute $\underset{\sim}{c}_{\alpha o}$, in (1.16), by $\underset{\sim}{c}_\alpha + \underset{\sim}{w}_\alpha$, and expand the resulting expression, obtaining

$$\underset{\sim}{q} = \sum_\alpha \frac{1}{2} \rho_\alpha \Big[< c_\alpha^2 \, \underset{\sim}{c}_\alpha > + w_\alpha^2 < \underset{\sim}{c}_\alpha > + 2 < (\underset{\sim}{w}_\alpha . \underset{\sim}{c}_\alpha) \underset{\sim}{c}_\alpha > +$$

$$+ < c_\alpha^2 > \underset{\sim}{w}_\alpha + w_\alpha^2 \, \underset{\sim}{w}_\alpha + 2 \, (< \underset{\sim}{c}_\alpha > . \underset{\sim}{w}_\alpha) \, \underset{\sim}{w}_\alpha \Big] \qquad (1.19)$$

The second and sixth terms in the right-hand side of this equation are equal to zero, since $< \underset{\sim}{c}_\alpha > = 0$. Therefore,

$$\underset{\sim}{q} = \sum_\alpha \frac{1}{2} \rho_\alpha \Big[< c_\alpha^2 \, \underset{\sim}{c}_\alpha > + 2 \, \underset{\sim}{w}_\alpha . < \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha > + < c_\alpha^2 > \underset{\sim}{w}_\alpha + w_\alpha^2 \, \underset{\sim}{w}_\alpha \Big] \qquad (1.20)$$

Using (1.18), (1.7) and the relation $p_\alpha = \rho_\alpha < c_\alpha^2 > /3$,

we can write (1.20) as

$$\underset{\sim}{q} = \sum_\alpha (\underset{\sim}{q}_\alpha + \underset{\sim}{w}_\alpha \cdot \underset{\approx}{p}_\alpha + \frac{3}{2} p_\alpha \, \underset{\sim}{w}_\alpha + \frac{1}{2} \rho_\alpha \, w_\alpha^2 \, \underset{\sim}{w}_\alpha) \qquad (1.21)$$

In particular, for the isotropic case in which $\underset{\approx}{p}_\alpha = p_\alpha \underset{\approx}{1}$, we have $\underset{\sim}{w}_\alpha \cdot \underset{\approx}{p}_\alpha = \underset{\sim}{w}_\alpha p_\alpha$, so that (1.21) becomes

$$\underset{\sim}{q} = \sum_\alpha \left(\underset{\sim}{q}_\alpha + \frac{5}{2} p_\alpha \underset{\sim}{w}_\alpha + \frac{1}{2} \rho_\alpha w_\alpha^2 \underset{\sim}{w}_\alpha \right) \tag{1.22}$$

2. CONTINUITY EQUATION

To obtain the continuity equation for the plasma as a whole, we add the continuity equation (8.3.2) over all species of particles in the plasma,

$$\sum_\alpha \frac{\partial \rho_\alpha}{\partial t} + \sum_\alpha \underset{\sim}{\nabla} \cdot (\rho_\alpha \underset{\sim}{u}_\alpha) = \sum_\alpha S_\alpha \tag{2.1}$$

which gives

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0 \tag{2.2}$$

with ρ and $\underset{\sim}{u}$ given by Eqs. (1.1) and (1.3), respectively. The collision term S_α, when summed over all particle species must certainly vanish, as a consequence of the conservation of the total mass of the system. It is of interest to note that, using the total time derivate

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}.\underset{\sim}{\nabla} \tag{2.3}$$

the continuity equation (2.2) can also be written in the form

$$\frac{D\rho}{Dt} + \rho\, \underset{\sim}{\nabla}.\underset{\sim}{u} = 0 \tag{2.4}$$

3. <u>EQUATION OF MOTION</u>

Similarly, adding the equation of conservation of momentum (8.4.9) over all particle species in the plasma, yields

$$\sum_\alpha \rho_\alpha \left[\frac{\partial \underset{\sim}{u}_\alpha}{\partial t} + (\underset{\sim}{u}_\alpha.\underset{\sim}{\nabla})\, \underset{\sim}{u}_\alpha \right] = \sum_\alpha n_\alpha q_\alpha \underset{\sim}{E} + \sum_\alpha n_\alpha q_\alpha \underset{\sim}{u}_\alpha \times \underset{\sim}{B} + \sum_\alpha \rho_\alpha \underset{\sim}{g} -$$

$$- \sum_\alpha \underset{\sim}{\nabla}.\, \underset{\approx}{p}_\alpha + \sum_\alpha \underset{\sim}{A}_\alpha - \sum_\alpha \underset{\sim}{u}_\alpha S_\alpha \tag{3.1}$$

Since the total momentum of the particles in the plasma is conserved, the collision term for momentum transfer vanishes when summed over all species. Using the definitions (1.1), (1.2) and (1.6), and the relation (1.13), we can write Eq. (3.1) as

$$\sum_\alpha \rho_\alpha \left[\frac{\partial \underset{\sim}{u}_\alpha}{\partial t} + (\underset{\sim}{u}_\alpha.\underset{\sim}{\nabla})\, \underset{\sim}{u}_\alpha \right] = \rho_c \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} + \rho\, \underset{\approx}{g} - \underset{\sim}{\nabla}.\underset{\approx}{p} +$$

$$+ \sum_{\alpha} \underset{\sim}{\nabla} \cdot \left(\rho_{\alpha} \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha} \right) - \sum_{\alpha} \underset{\sim}{u}_{\alpha} S_{\alpha} \tag{3.2}$$

The term involving S_{α} can be eliminated using the equation of conservation of mass, that is,

$$\sum_{\alpha} \underset{\sim}{u}_{\alpha} S_{\alpha} = \sum_{\alpha} \underset{\sim}{u}_{\alpha} \left[\frac{\partial \rho_{\alpha}}{\partial t} + \underset{\sim}{\nabla} \cdot \left(\rho_{\alpha} \underset{\sim}{u}_{\alpha} \right) \right] \tag{3.3}$$

Combining this expression with the terms in the left-hand side of (3.2), results in the expression

$$\sum_{\alpha} \left[\frac{\partial}{\partial t} \left(\rho_{\alpha} \underset{\sim}{u}_{\alpha} \right) + \underset{\sim}{\nabla} \cdot \left(\rho_{\alpha} \underset{\sim}{u}_{\alpha} \underset{\sim}{u}_{\alpha} \right) \right] \tag{3.4}$$

We can now substitute the mean velocity $\underset{\sim}{u}_{\alpha}$ by $\underset{\sim}{w}_{\alpha} + \underset{\sim}{u}$ and expand the result. Noting that

$$\sum_{\alpha} \rho_{\alpha} \underset{\sim}{w}_{\alpha} = \sum_{\alpha} \rho_{\alpha} \left(\underset{\sim}{u}_{\alpha} - \underset{\sim}{u} \right) = \rho \underset{\sim}{u} - \rho \underset{\sim}{u} = 0 \tag{3.5}$$

we can express (3.4) as

$$\sum_{\alpha} \left[\frac{\partial}{\partial t} \left(\rho_{\alpha} \underset{\sim}{u}_{\alpha} \right) + \underset{\sim}{\nabla} \cdot \left(\rho_{\alpha} \underset{\sim}{u}_{\alpha} \underset{\sim}{u}_{\alpha} \right) \right] = \frac{\partial}{\partial t} \left(\rho \underset{\sim}{u} \right) + \underset{\sim}{\nabla} \cdot \left(\rho \underset{\sim}{u} \underset{\sim}{u} \right) +$$

$$+ \sum_{\alpha} \underset{\sim}{\nabla} \cdot (\rho_{\alpha} \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha}) = \rho \left[\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} . \underset{\sim}{\nabla}) \underset{\sim}{u} \right] + \underset{\sim}{u} \left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) \right] +$$

$$+ \sum_{\alpha} \underset{\sim}{\nabla} \cdot (\rho_{\alpha} \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha}) = \rho \frac{D\underset{\sim}{u}}{Dt} + \sum_{\alpha} \underset{\sim}{\nabla} \cdot (\rho_{\alpha} \underset{\sim}{w}_{\alpha} \underset{\sim}{w}_{\alpha}) \tag{3.6}$$

where we have used the continuity equation (2.2) and the total time derivative (2.3). Taking this result back into the equation of motion (3.2), we obtain the following momentum equation for the plasma as a whole,

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \rho_c \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} + \rho \underset{\sim}{g} - \underset{\sim}{\nabla} \cdot \underset{\approx}{p} \tag{3.7}$$

This equation is an expression of Newton's second law.

4. ENERGY EQUATION

To obtain the equation of conservation of energy, for the plasma as a conducting fluid, we start from the energy equation (8.5.4) for the particles of type α, and add this equation over all species in the plasma,

$$\sum_{\alpha} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_{\alpha} < v^2 >_{\alpha} \right) + \sum_{\alpha} \underset{\sim}{\nabla} \cdot \left[\frac{1}{2} \rho_{\alpha} < v^2 \underset{\sim}{v} >_{\alpha} \right] -$$

$$- \sum_{\alpha} n_{\alpha} < \underset{\sim}{F} . \underset{\sim}{v} >_{\alpha} = 0 \tag{4.1}$$

where the collision term M_α vanishes when summed over all species of particles. We substitute now, $\underset{\sim}{v}$ by $\underset{\sim}{c}_{\alpha o} + \underset{\sim}{u}$, and expand each term of (4.1). For the *first term* we have

$$\frac{\partial}{\partial t}\Big(\sum_\alpha \frac{1}{2}\rho_\alpha \langle \underset{\sim}{v} \cdot \underset{\sim}{v}\rangle_\alpha\Big) = \frac{\partial}{\partial t}\left[\sum_\alpha \frac{1}{2}\rho_\alpha(\langle c^2_{\alpha o}\rangle + u^2 + 2\,\underset{\sim}{w}_\alpha \cdot \underset{\sim}{u})\right]$$

$$= \frac{\partial}{\partial t}\Big(\sum_\alpha \frac{1}{2}\rho_\alpha \langle c^2_{\alpha o}\rangle\Big) + \frac{\partial}{\partial t}\Big(\frac{1}{2}\rho\, u^2\Big)$$

$$= \frac{\partial}{\partial t}\Big(\frac{3}{2}p\Big) + \frac{\partial}{\partial t}\Big(\frac{1}{2}\rho\, u^2\Big) \qquad (4.2)$$

where we have used the definition (1.17) and the fact that $\sum_\alpha \rho_\alpha \underset{\sim}{w}_\alpha = 0$. For the *second term* we note initially that

$$\langle v^2 \underset{\sim}{v}\rangle_\alpha = \langle (c^2_{\alpha o} + u^2 + 2\,\underset{\sim}{c}_{\alpha o} \cdot \underset{\sim}{u})(\underset{\sim}{c}_{\alpha o} + \underset{\sim}{u})\rangle$$

$$= \langle c^2_{\alpha o}\, \underset{\sim}{c}_{\alpha o}\rangle + u^2\, \underset{\sim}{w}_\alpha + 2\langle \underset{\sim}{c}_{\alpha o}\, \underset{\sim}{c}_{\alpha o}\rangle \cdot \underset{\sim}{u} +$$

$$+ \langle c^2_{\alpha o}\rangle\, \underset{\sim}{u} + u^2\, \underset{\sim}{u} + 2\,(\underset{\sim}{w}_\alpha \cdot \underset{\sim}{u})\, \underset{\sim}{u} \qquad (4.3)$$

since $\underset{\sim}{c}_{\alpha o} = \underset{\sim}{c}_\alpha + \underset{\sim}{w}_\alpha$ and $\langle \underset{\sim}{c}_\alpha \rangle = 0$. Therefore,

$$\underset{\sim}{\nabla}\cdot\Big(\sum_\alpha \frac{1}{2}\rho_\alpha \langle v^2 \underset{\sim}{v}\rangle_\alpha\Big) = \underset{\sim}{\nabla}\cdot\Big(\sum_\alpha \frac{1}{2}\rho_\alpha \langle c^2_{\alpha o}\, \underset{\sim}{c}_{\alpha o}\rangle\Big) + \underset{\sim}{\nabla}\cdot\Big(\sum_\alpha \rho_\alpha \langle \underset{\sim}{c}_{\alpha o}\, \underset{\sim}{c}_{\alpha o}\rangle \cdot u\Big) +$$

$$+ \underset{\sim}{\nabla} \cdot \left(\sum_{\alpha} \frac{1}{2} \rho_{\alpha} < c^2_{\alpha o} > \underset{\sim}{u}\right) + \underset{\sim}{\nabla} \cdot \left(\sum_{\alpha} \frac{1}{2} \rho_{\alpha} u^2 \underset{\sim}{u}\right) \qquad (4.4)$$

Using the definitions of the total heat flux vector, $\underset{\sim}{q}$, and of the total kinetic pressure dyad, $\underset{\approx}{p}$, we can write (4.4) as

$$\underset{\sim}{\nabla} \cdot \left(\sum_{\alpha} \frac{1}{2} \rho_{\alpha} < v^2 \underset{\sim}{v} >_{\alpha}\right) = \underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \underset{\sim}{\nabla} \cdot (\underset{\approx}{p} \cdot \underset{\sim}{u}) + \underset{\sim}{\nabla} \cdot \left(\frac{3}{2} p \underset{\sim}{u}\right) +$$

$$+ \underset{\sim}{\nabla} \cdot \left(\frac{1}{2} \rho u^2 \underset{\sim}{u}\right) \qquad (4.5)$$

For the *third term* of (4.1) we have

$$\sum_{\alpha} n_{\alpha} < \underset{\sim}{F} \cdot \underset{\sim}{v} >_{\alpha} = \sum_{\alpha} n_{\alpha} \Big[q_{\alpha} < \underset{\sim}{E} \cdot \underset{\sim}{v} >_{\alpha} + q_{\alpha} < (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{v} >_{\alpha} +$$

$$+ m_{\alpha} < \underset{\sim}{g} \cdot \underset{\sim}{v} >_{\alpha} \Big] \qquad (4.6)$$

where we have considered external forces due to electromagnetic and gravitational fields. Since $< \underset{\sim}{v} >_{\alpha} = \underset{\sim}{u}_{\alpha}$ and since, for any vector $\underset{\sim}{v}$, we have $(\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \underset{\sim}{B} = 0$, we obtain

$$\sum_{\alpha} n_{\alpha} < \underset{\sim}{F} \cdot \underset{\sim}{v} >_{\alpha} = \underset{\sim}{J} \cdot \underset{\sim}{E} + \underset{\sim}{J}_m \cdot \underset{\sim}{g} \qquad (4.7)$$

where we have used the definitions (1.5) and (1.6), and where $\underset{\sim}{E}$

and $\underset{\sim}{q}$ are smoothed macroscopic fields.

Combining the results contained in Eqs. (4.2), (4.5) and (4.7), the energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p\right) + \underset{\sim}{\nabla} \cdot \left(\frac{3}{2}p\ \underset{\sim}{u}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, u^2\right) + \underset{\sim}{\nabla} \cdot \left(\frac{1}{2}\rho\, u^2\, \underset{\sim}{u}\right) + \underset{\sim}{\nabla} \cdot \underset{\sim}{q} +$$

$$+ \underset{\sim}{\nabla} \cdot \left(\underset{\approx}{p} \cdot \underset{\sim}{u}\right) - \underset{\sim}{J} \cdot \underset{\sim}{E} - \underset{\sim}{J}_m \cdot \underset{\sim}{g} = 0 \qquad (4.8)$$

This equation can be further simplified as follows. The third and fourth terms of (4.8) can be combined as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, u^2\right) + \underset{\sim}{\nabla} \cdot \left(\frac{1}{2}\rho\, u^2\, \underset{\sim}{u}\right) \equiv \frac{1}{2}u^2 \left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{u})\right] +$$

$$+ \underset{\sim}{u} \cdot \left(\rho \frac{D\underset{\sim}{u}}{Dt}\right) \qquad (4.9)$$

and using the continuity equation (2.2) and the equation of motion (3.7), we can express (4.9) as

$$\rho_c\, \underset{\sim}{u} \cdot \underset{\sim}{E} + \underset{\sim}{u} \cdot (\underset{\sim}{J} \times \underset{\sim}{B}) + \underset{\sim}{J}_m \cdot \underset{\sim}{g} - \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\approx}{p}) \qquad (4.10)$$

Taking this result back into the energy equation (4.8), yields

$$\frac{D}{Dt}\left(\frac{3}{2}p\right)+\frac{3}{2}p\,\nabla\cdot\underset{\sim}{u}+\nabla\cdot\underset{\sim}{q}+(\underset{\approx}{p}\cdot\nabla)\cdot\underset{\sim}{u}=\underset{\sim}{J}\cdot\underset{\sim}{E}-\underset{\sim}{u}\cdot(\underset{\sim}{J}\times\underset{\sim}{B})-\rho_c\,\underset{\sim}{u}\cdot\underset{\sim}{E} \tag{4.11}$$

where we have used the total time derivative (2.3).

The first term in the left-hand side of (4.11) represents the time rate of change of the total thermal energy density of the plasma, 3 p/2, in a frame of reference moving with the global mean velocity $\underset{\sim}{u}$. The second term contributes to this rate of change through the thermal energy transfered to this volume element, as a consequence of the particle motions. The third term represents the heat flux, and the fourth term the work done on the volume element by the pressure forces(normal and tangential). The terms in the right-hand side of (4.11) represent the work done on the volume element by the electric field existing in the frame of reference moving with the global mean velocity $\underset{\sim}{u}$. These last terms can be combined as follows. We note, initially, that the charge current density consists of two parts

$$\underset{\sim}{J}=\sum_\alpha n_\alpha q_\alpha \underset{\sim}{u}_\alpha=\sum_\alpha n_\alpha q_\alpha \underset{\sim}{w}_\alpha+\sum_\alpha n_\alpha q_\alpha \underset{\sim}{u}$$

$$=\underset{\sim}{J}'+\rho_c\,\underset{\sim}{u} \tag{4.12}$$

where $\rho_c\,\underset{\sim}{u}$ is the *convection* charge current density, which represents the flux of the space charge with velocity $\underset{\sim}{u}$, and $\underset{\sim}{J}'$ is the *conduction* charge current density, which represents the charge current density in the frame of reference moving with the global mean velocity $\underset{\sim}{u}$. On the other hand, we can write

$$\underset{\sim}{u} \cdot (\underset{\sim}{J} \times \underset{\sim}{B}) = - \underset{\sim}{J} \cdot (\underset{\sim}{u} \times \underset{\sim}{B}) = - \underset{\sim}{J}' \cdot (\underset{\sim}{u} \times \underset{\sim}{B}) \tag{4.13}$$

Substituting Eqs. (4.13) and (4.12) into the energy equation (4.11), we obtain, finally,

$$\frac{D}{Dt}\left(\frac{3}{2} p\right) + \frac{3}{2} p \underset{\sim}{\nabla} \cdot \underset{\sim}{u} + \underset{\sim}{\nabla} \cdot \underset{\sim}{q} + (\underset{\approx}{p} \cdot \underset{\sim}{\nabla}) \cdot \underset{\sim}{u} = \underset{\sim}{J}' \cdot \underset{\sim}{E}' \tag{4.14}$$

where $\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}$ is the electric field existing in the reference system moving with the global mean velocity $\underset{\sim}{u}$. The term $\underset{\sim}{J}' \cdot \underset{\sim}{E}'$ represents, therefore, the rate of change in the energy density due to Joule heating.

5. ELECTRODYNAMIC EQUATIONS FOR A CONDUCTING FLUID

In the previous sections we have derived the macroscopic transport equations for conservation of mass, momentum and energy in a conducting fluid. As mentioned before, this set of equations does not constitute a complete system, and it is necessary to truncate the hierarchy of macroscopic equations at some stage and to make some simplifying assumptions. The continuity equation relates the mass density, ρ, with the global mean velocity $\underset{\sim}{u}$; the equation of motion, which specifies the variation of $\underset{\sim}{u}$, involves also the total kinetic pressure dyad, $\underset{\approx}{p}$; the energy equation, which specifies the rate of change of the total thermal energy density (3 p/2), includes also the heat flux vector, $\underset{\sim}{q}$ (a more general energy equation would give as the variation of the total kinetic pressure dyad $\underset{\approx}{p}$, which would include the total heat flow triad $\underset{\approx}{Q}$). We can continue taking moments of higher order and obtain, for example, the transport equation governing the variation of the heat flow triad $\underset{\approx}{Q}$. To obtain a complete system it is essential, therefore, to truncate the hierarchy of transport equations at some point. However, even after this truncation, the remaining equations include the following electrodynamic variables: electric field $\underset{\sim}{E}$, magnetic induction $\underset{\sim}{B}$, charge current density $\underset{\sim}{J}$, and charge density ρ_c. Besides the hydrodynamic transport equations, we need, therefore, ten electrodynamic equations which must relate the variations in $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{J}$ and ρ_c. These equations are considered next.

5.1 - Maxwell curl equations

The following Maxwell equations

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \partial \underset{\sim}{B} / \partial t \tag{5.1}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o (\underset{\sim}{J} + \varepsilon_o \, \partial \underset{\sim}{E}/\partial t) \tag{5.2}$$

provide six component equations, which can be considered as the equations governing the variations of the electromagnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

5.2 - Conservation of electric charge

The equation of conservation of charge can be obtained by multiplying the equation of conservation of mass (8.3.2) by q_α/m_α, and adding over all species,

$$\frac{\partial}{\partial t}\left(\sum_\alpha n_\alpha q_\alpha\right) + \underset{\sim}{\nabla}\cdot\left(\sum_\alpha n_\alpha q_\alpha \underset{\sim}{u}_\alpha\right) = \sum_\alpha (q_\alpha/m_\alpha)\, S_\alpha \qquad (5.3)$$

Using the definitions of ρ_c and $\underset{\sim}{J}$, and noting that the *total* electric charge does not change as a result of collisions, we obtain

$$\frac{\partial \rho_c}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{J} = 0 \qquad (5.4)$$

It is worth noting here that Eq. (5.4) can also be derived, in a independent way, considering Maxwell curl equation (5.2) and the Maxwell divergence equation

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \rho_c/\varepsilon_o \qquad (5.5)$$

Taking the divergence of Eq. (5.2), yields

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{J} + \varepsilon_o\, \partial\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{E})/\partial t = 0 \qquad (5.6)$$

since the divergence of the curl of a vector field vanishes identically. This last equation, combined with Eq. (5.5), yields the equation of conservation of charge (5.4). Eqs. (5.4) and (5.5) cannot, therefore, be considered as independent. As we have just shown, Maxwell equations (5.2) and (5.5) imply in conservation of electric charge.

Another interesting aspect of Maxwell equations can be seen by taking the divergence of (5.1), which gives

$$\frac{\partial}{\partial t} (\underset{\sim}{\nabla} \cdot \underset{\sim}{B}) = 0 \tag{5.7}$$

or

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \text{constant} \tag{5.8}$$

Therefore, the Maxwell equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \tag{5.9}$$

can be considered as an *initial condition* for Eq. (5.1), since if we take $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$ initially, Eq. (5.1) implies that this condition will remain satisfied for all subsequent times.

5.3 - Generalized Ohm's law

To obtain a differential equation governing the variation of the charge current density $\underset{\sim}{J}$, we proceed in a way analogous to the derivation of Eq (5.4). To this end, we multiply the equation of conservation of momentum (8.4.9) by q_α / m_α, and add over all particle species. This procedure leads to

$$\sum_\alpha n_\alpha q_\alpha \frac{\partial \underset{\sim}{u}_\alpha}{\partial t} + \sum_\alpha n_\alpha q_\alpha (\underset{\sim}{u}_\alpha \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}_\alpha = \sum_\alpha n_\alpha \frac{q_\alpha}{m_\alpha} < \underset{\sim}{F} >_\alpha - \underset{\sim}{\nabla} \cdot (\sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\approx}{p}_\alpha) +$$

$$+ \sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\sim}{A}_\alpha - \sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\sim}{u}_\alpha \; S_\alpha \tag{5.10}$$

We define now the *electrokinetic pressure dyad*, $\underset{\approx}{p}^E_\alpha$, for the particles of type α, by

$$\underset{\approx}{p}^E_\alpha = \frac{q_\alpha}{m_\alpha} \underset{\approx}{p}_\alpha = n_\alpha q_\alpha < \underset{\sim}{c}_\alpha \underset{\sim}{c}_\alpha > \tag{5.11}$$

Consequently, for the plasma as a conducting fluid, we have the following relation analogous to (1.13)

$$\underset{\approx}{p}^E = \sum_\alpha \underset{\approx}{p}^E_\alpha + \sum_\alpha n_\alpha q_\alpha \underset{\sim}{w}_\alpha \underset{\sim}{w}_\alpha \tag{5.12}$$

The *second term* in the right-hand side of (5.10) becomes, therefore,

$$-\underset{\sim}{\nabla} \cdot \left(\sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\approx}{p}_\alpha\right) = - \underset{\sim}{\nabla} \cdot \underset{\approx}{p}^E + \underset{\sim}{\nabla} \cdot \left(\sum_\alpha n_\alpha q_\alpha \underset{\sim}{w}_\alpha \underset{\sim}{w}_\alpha\right) \tag{5.13}$$

Using the continuity equation (8.3.2), and substituting $\underset{\sim}{u}_\alpha$ by $\underset{\sim}{w}_\alpha + \underset{\sim}{u}$, the *last term* in the right-hand side of (5.10) can be written

$$-\sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\sim}{u}_\alpha S_\alpha = - \sum_\alpha \underset{\sim}{w}_\alpha \frac{\partial}{\partial t} (n_\alpha q_\alpha) - \sum_\alpha \underset{\sim}{w}_\alpha \left[\underset{\sim}{\nabla} \cdot (n_\alpha q_\alpha \underset{\sim}{w}_\alpha)\right] -$$

$$- \sum_\alpha \underset{\sim}{w}_\alpha \left[\underset{\sim}{\nabla} \cdot (n_\alpha q_\alpha \underset{\sim}{u})\right] - \underset{\sim}{u} \frac{\partial \rho_c}{\partial t} - \underset{\sim}{u} (\underset{\sim}{\nabla} \cdot \underset{\sim}{J}) \tag{5.14}$$

Similarly, the *first* and *second terms* in the left-hand side of (5.10) can be combined in the form

$$\sum_\alpha n_\alpha q_\alpha \frac{\partial \underset{\sim}{w}_\alpha}{\partial t} + \sum_\alpha (n_\alpha q_\alpha \underset{\sim}{w}_\alpha \cdot \underset{\sim}{\nabla}) \underset{\sim}{w}_\alpha + \sum_\alpha (n_\alpha q_\alpha \underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{w}_\alpha +$$

$$+ \rho_c \frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{J} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} \tag{5.15}$$

We can now substitute expressions (5.13), (5.14) and (5.15) into Eq (5.10) and simplify the result. Making use of the following identity for two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$$\underset{\sim}{\nabla} . (\underset{\sim}{a} \, \underset{\sim}{b}) = \underset{\sim}{b} (\underset{\sim}{\nabla} . \underset{\sim}{a}) + (\underset{\sim}{a} . \underset{\sim}{\nabla}) \underset{\sim}{b} \tag{5.16}$$

and the relation (4.12), we obtain

$$\frac{\partial \underset{\sim}{J}}{\partial t} + \underset{\sim}{\nabla} . (\underset{\sim}{u} \, \underset{\sim}{J}' + \underset{\sim}{J} \, \underset{\sim}{u}) + \underset{\sim}{\nabla} . \underset{\approx}{p}^{E} = \sum_{\alpha} n_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} < \underset{\sim}{F} >_{\alpha} + \sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} \underset{\sim}{A}_{\alpha} \tag{5.17}$$

Equations (5.1), (5.2), (5.4) and (5.17) constitute ten component equations which complement the equations of conservation of mass, momentum and energy for a conducting fluid. Eq.(5.17) , however, is still in a very general form of little practical value. A very useful and simple expression exists for the case of a completely ionized plasma consisting of electrons and only one type of ions. In what follows, we simplify Eq. (5.17) for this case.

The electric charge current density, $\underset{\sim}{J}$, and the electric charge density, ρ_c, for a completely ionized plasma containing only electrons and one type of ions of charge e are given, respectively, by

$$\underset{\sim}{J} = \sum_{\alpha} n_{\alpha} q_{\alpha} \underset{\sim}{u}_{\alpha} = e (n_i \underset{\sim}{u}_i - n_e \underset{\sim}{u}_e) \tag{5.18}$$

$$\rho_c = \sum_\alpha n_\alpha q_\alpha = e\,(n_i - n_e) \qquad (5.19)$$

The global mean velocty $\underset{\sim}{u}$, defined in (1.3), becomes

$$\underset{\sim}{u} = \frac{1}{\rho}\,(\rho_e\,\underset{\sim}{u}_e + \rho_i\,\underset{\sim}{u}_i) \qquad (5.20)$$

where $\rho = \rho_e + \rho_i$. Combining this last equation with (5.18), gives

$$\underset{\sim}{u}_i = \frac{\mu}{\rho_i}\left(\frac{\rho \underset{\sim}{u}}{m_e} + \frac{\underset{\sim}{J}}{e}\right) \qquad (5.21)$$

$$\underset{\sim}{u}_e = \frac{\mu}{\rho_e}\left(\frac{\rho \underset{\sim}{u}}{m_i} - \frac{\underset{\sim}{J}}{e}\right) \qquad (5.22)$$

where $\mu = m_e\, m_i\, /\, (m_e + m_i)$ denotes the reduced mass.

We assume now that the mean velocity of the electrons and ions, relative to the global mean velocity $\underset{\sim}{u}$, (that is, the diffusion velocities $\underset{\sim}{w}_e$ and $\underset{\sim}{w}_i$) are small compared with the thermal velocities. This condition being satisfied, Eq.(5.12) becomes

$$\underset{\sim}{p}^E = \underset{\sim}{p}_i^E + \underset{\sim}{p}_e^E = e\left(\frac{1}{m_i}\,\underset{\sim}{p}_i - \frac{1}{m_e}\,\underset{\sim}{p}_e\right) \qquad (5.23)$$

Considering the conducting fluid immersed in an electromagnetic field, the term containing the external force in Eq. (5.17) becomes

$$\sum_{\alpha} n_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} < \mathbf{F} >_{\alpha} = \sum_{\alpha} n_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} [q_{\alpha}(\mathbf{E} + \mathbf{u}_{\alpha} \times \mathbf{B})]$$

$$= e^2 \left(\frac{n_i}{m_i} + \frac{n_e}{m_e} \right) \mathbf{E} +$$

$$+ e^2 \left(\frac{n_i}{m_i} \mathbf{u}_i + \frac{n_e}{m_e} \mathbf{u}_e \right) \times \mathbf{B} \tag{5.24}$$

Substituting the relations (5.21) and (5.22) in this last equation and simplifying, yields

$$\sum_{\alpha} n_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} < \mathbf{F} >_{\alpha} = e^2 \left(\frac{n_i}{m_i} + \frac{n_e}{m_e} \right) \mathbf{E} + e^2 \left(\frac{n_i}{m_e} + \frac{n_e}{m_i} \right) \mathbf{u} \times \mathbf{B} +$$

$$+ e \left(\frac{1}{m_i} - \frac{1}{m_e} \right) \mathbf{J} \times \mathbf{B} \tag{5.25}$$

It is convenient at this moment to simplify this equation by making one additional approximation. Since the ion mass m_i is much larger than the electron mass m_e (for protons and electrons, for example, $m_i/m_e \cong 1836$) and assuming macroscopic charge neutrality, that is, $n_e = n_i = n$, we can take

$$\frac{1}{m_i} - \frac{1}{m_e} \simeq - \frac{1}{m_e} \tag{5.26}$$

$$\frac{n_i}{m_i} + \frac{n_e}{m_e} \simeq \frac{n}{m_e} \tag{5.27}$$

$$\frac{n_i}{m_e} + \frac{n_e}{m_i} \simeq \frac{n}{m_e} \tag{5.28}$$

Consequently, from (5.23) we have $\underset{\approx}{p}^E = - e\, \underset{\approx}{p}_e / m_e$, and from (5.25)

$$\sum_\alpha n_\alpha \frac{q_\alpha}{m_\alpha} < \underset{\sim}{F} >_\alpha = \frac{ne^2}{m_e} \left(\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} \right) - \frac{e}{m_e} \underset{\sim}{J} \times \underset{\sim}{B} \tag{5.29}$$

For the collision term in Eq. (5.17), we make use of expression (8.4.11), that is

$$\underset{\sim}{A}_e = - \rho_e \nu_{ei} \left(\underset{\sim}{u}_e - \underset{\sim}{u}_i \right) \tag{5.30}$$

$$\underset{\sim}{A}_i = - \rho_i \nu_{ie} \left(\underset{\sim}{u}_i - \underset{\sim}{u}_e \right) \tag{5.31}$$

From Eq. (8.4.13) we have $\rho_i \nu_{ie} = \rho_e \nu_{ei}$,so that

$$\sum_\alpha \frac{q_\alpha}{m_\alpha} \underset{\sim}{A}_\alpha = e \rho_e \nu_{ei} \left(\underset{\sim}{u}_e - \underset{\sim}{u}_i \right) \left(\frac{1}{m_i} + \frac{1}{m_e} \right)$$

$$= - \nu_{ei} \underset{\sim}{J} \qquad \text{5.32)}$$

where we have used Eq. (5.18) for $\underset{\sim}{J}$, and the approximations $m_i >> m_e$ and $n_e = n_i = n$.

We can now substitute the results contained in Eqs. (5.23), (5.29) and (5.32), into Eq. (5.17), to obtain

$$\frac{\partial \underset{\sim}{J}}{\partial t} + \underset{\sim}{\nabla} \cdot (\underset{\sim}{u}\, \underset{\sim}{J} + \underset{\sim}{J}\, \underset{\sim}{u}) - \frac{e}{m_e} \underset{\sim}{\nabla} \cdot \underset{\approx}{p}_e =$$

$$= \frac{ne^2}{m_e} (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) - \frac{e}{m_e} \underset{\sim}{J} \times \underset{\sim}{B} - \nu_{ei} \underset{\sim}{J} \qquad (5.33)$$

Note that, since we assumed $n_e = n_i$, we must have $\rho_c = 0$ and $\underset{\sim}{J}' = \underset{\sim}{J}$. In some situations in which $\underset{\sim}{J}$ and $\underset{\sim}{u}$ can be considered as small perturbations, the nonlinear terms involving their product may be neglected compared to the other terms. With this simplifying approximation and using the notation

$$\sigma_o = \frac{ne^2}{m_e \nu_{ei}} \qquad (5.34)$$

which represents the longitudinal electrical conductivity, we get for (5.33)

$$\frac{m_e}{ne^2}\frac{\partial \underset{\sim}{J}}{\partial t} - \frac{1}{ne}\underset{\sim}{\nabla}\cdot \underset{\approx}{p}_e = \underset{\sim}{E} + \underset{\sim}{u}\times\underset{\sim}{B} - \frac{1}{ne}\underset{\sim}{J}\times\underset{\sim}{B} - \frac{1}{\sigma_o}\underset{\sim}{J} \tag{5.35}$$

This equation is known as the *generalized Ohm's law*. The terms is the right - hand side are the ones normally retained in magnetohydrodynamics while all the others are neglected. The omission of the terms in the left-hand side of (5.35) is, generally, not always justifiable.

For cases in which $\underset{\sim}{J}$ does not vary with time, that is, under steady state conditions, we have $\partial\underset{\sim}{J} / \partial t = 0$. If we consider also that the pressure term in Eq. (5.35) is negligible, that is, $\underset{\sim}{\nabla}\cdot\underset{\approx}{p}_e = 0$, then Eq. (5.35) simplifies to

$$\underset{\sim}{J} = \sigma_o\left(\underset{\sim}{E} + \underset{\sim}{u}\times\underset{\sim}{B}\right) - \frac{\sigma_o}{ne}\underset{\sim}{J}\times\underset{\sim}{B} \tag{5.36}$$

The last term in this equation is related to a phenomenon called the *Hall effect* in magnetohydrodynamic flow problems, and, for this reason, it is normally called the Hall effect term. This term is small when $(\sigma_o |\underset{\sim}{B}|/ne) \ll 1$, that is, when $\omega_{ce} \ll \nu_{ei}$.

Thus, when the collision frequency is much larger than the magnetic gyrofrequency, the Hall effect term can be neglected and (5.36) reduces to

$$\underset{\sim}{J} = \sigma_o \, (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) \tag{5.37}$$

In the absence of an external magnetic field, Eq. (5.37) reduces further to

$$\underset{\sim}{J} = \sigma_o \, \underset{\sim}{E} \tag{5.38}$$

which is the expression commonly known as *Ohm's law.*

6. SIMPLIFIED MAGNETOHYDRODYNAMIC EQUATIONS

In the last two sections we have shown that the set of macroscopic transport equations for each individual species in the plasma can be substituted by transport equations for the whole plasma as a conducting fluid, complemented by the electrodynamic equations. These total macroscopic equations for a conducting fluid are known as the magnetohydrodynamic (MHD) equations. In their most general form they are essentially equivalent to the set of equations for each individual particle species. In practice, however, the MHD equations are seldom used in their general form. Several simplifying approximations are normally considered, based on physical arguments which permit the elimination of some of the terms in the equations. For steady state situations, or slowly varying problems, the MHD equations are very convenient and, in many cases, lead to results which would not be easily obtained from the individual equations for each species of particles.

One of the approximations normally used in MHD consists in neglecting the term $\varepsilon_o \, \partial \underset{\sim}{E} / \partial t$ in the Maxwell equation (5.2). To analyse the validity of this approximation it is convenient to use dimensional analysis, as follows. We can express, in general, the charge current density as $\underset{\sim}{J} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E}$, so that, dimensionally, we have

$$J \simeq \sigma E \tag{6.1}$$

$$\sigma_o \, | \, \partial \underset{\sim}{E} / \partial t \, | \simeq \varepsilon_o \, E / \tau \tag{6.2}$$

where τ represents a characteristic time for changes in the electric field and σ represents a characteristic conductivity. The ratio of the two terms in the right-hand side of (5.2) becomes, therefore,

$$\frac{\varepsilon_o \mid \partial \underset{\sim}{E} / \partial t \mid}{J} \simeq \frac{\varepsilon_o}{\sigma \tau} \tag{6.3}$$

For most of the fluids normally used in MHD problems, σ is typically greater than 1 mho/m, whereas ε_o is of the order of 10^{-11} Farad/m. Consequently,

$$\frac{\varepsilon_o \mid \partial \underset{\sim}{E} / \partial t \mid}{J} \approx \frac{10^{-11}}{\tau} \tag{6.4}$$

with τ in seconds, which shows that this approximation is not valid only when we are considering *extremely small* characteristic times.

It is also assumed that the macroscopic electric neutrality is maintained with a high degree of accuracy and, therefore, the electric charge density, ρ_c, is set equal to zero.

A questionable approximation in the set of MHD equations is the generalized Ohm's law, in the form given in Eq. (5.36). In this form, the terms containing the time derivatives and pressure gradient (or divergence of the pressure dyad) are omitted, even though these terms are considered in other equations of the set. This approximation is not therefore justifiable in a direct manner. It is common to simply assume that all time derivatives are negligibly small and that the plasma is *almost* a cold plasma, so that the generalized Ohm's law reduces to the form given in (5.36).

For convenience, we collect here the following set of simplified magnetohydrodynamic equations

$$\partial\rho \ / \ \partial t + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0 \tag{6.5}$$

$$\rho \ D\underset{\sim}{u} \ / \ Dt = \underset{\sim}{J} \times \underset{\sim}{B} - \underset{\sim}{\nabla} p \tag{6.6}$$

$$\underset{\sim}{\nabla} p = V_S^2 \ \underset{\sim}{\nabla}\rho \tag{6.7}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \partial\underset{\sim}{B} \ / \ \partial t \tag{6.8}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o \ \underset{\sim}{J} \tag{6.9}$$

$$\underset{\sim}{J} = \sigma_o \ (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) - (\sigma_o \ /ne) \ \underset{\sim}{J} \times \underset{\sim}{B} \tag{6.10}$$

In this set of equations, viscosity and thermal conductivity are neglected. The pressure dyad reduces, therefore, to a scalar pressure.

Note that Eq. (6.9) implies in

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{J} = 0 \tag{6.11}$$

which is the equation of conservation of electric charge in the absence of changes in the total macroscopic charge density, ρ_c. It is for this reason that the equation of conservation of electric charge is not explicitly considered in the set of MHD equations (6.5) to (6.10). Except in some special circumstances, it is also common to neglect the Hall effect term $(\sigma_o /en)\, \underset{\sim}{J} \times \underset{\sim}{B}$ in Eq. (6.10).

PROBLEMS

9.1 - Show that the total kinetic energy density of all species in a fluid can be written as the sum of the thermal energy density of the whole fluid, plus the kinetic energy of the mass motion, that is

$$\sum_\alpha \frac{1}{2} \rho_\alpha < v^2 >_\alpha = \frac{3}{2} p + \sum_\alpha \frac{1}{2} \rho_\alpha u_\alpha^2$$

where

$$\frac{3}{2} p = \sum_\alpha \frac{1}{2} \rho_\alpha < c_{\alpha o} >^2$$

$$= \sum_\alpha \frac{1}{2} \rho_\alpha < c_\alpha^2 > + \sum_\alpha \frac{1}{2} \rho_\alpha w_\alpha^2$$

9.2 - Show that when there is no heat flow ($\underset{\sim}{q}=0$), no Joule heating ($\underset{\sim}{J}' \cdot \underset{\sim}{E}' = 0$) and when the pressure tensor is isotropic given by $\underset{\approx}{p} = p \underset{\approx}{1}$, the energy equation (4.14) reduces to the following adiabatic equation

$$p \rho^{-5/3} = \text{constant}$$

9.3 - From the momentum conservation equation with the MHD approximation [see Eq. (6.6)], and the generalized Ohm's law in the simplified form (6.10), but without considering the Hall effect term, derive the following equation:

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \sigma_0 (\underset{\sim}{E} \times \underset{\sim}{B}) + \sigma_0 (\underset{\sim}{u} \times \underset{\sim}{B}) \times \underset{\sim}{B} - \underset{\sim}{\nabla} p$$

Solve this equation, considering that $\underset{\sim}{E} = 0$ and p = constant, to show that the fluid velocity perpendicular to $\underset{\sim}{B}$ is given by

$$u_\perp (t) = u_\perp (0) \exp (- t/\tau)$$

where τ is a characteristic time for diffusion of the fluid across the field lines, given by

$$\tau = \frac{\rho}{\sigma_0 B^2}$$

9.4 - In Eqs. (1.5) and (1.6), explain the reason why the mass flux $\underset{\sim}{J}_m$ is given by $\rho\underset{\sim}{u}$, whereas the electric charge flux $\underset{\sim}{J}$ is *not* given by $\rho_c\underset{\sim}{u}$.

9.5 - Obtain an expression for the heat flux triad $\underset{\approx}{Q}$, for the plasma as a whole, defined as

$$\underset{\approx}{Q} = \sum_\alpha \rho_\alpha < \underset{\sim}{c}_{\alpha 0} \; \underset{\sim}{c}_{\alpha 0} \; \underset{\sim}{c}_{\alpha 0} >$$

where $\underset{\sim}{c}_{\alpha o} = \underset{\sim}{c}_{\alpha} + \underset{\sim}{w}_{\alpha}$, in terms of a summation over the heat flux triad for each species $\underset{\approx}{Q}_{\alpha}$, and of terms involving the diffusion velocity $\underset{\sim}{w}_{\alpha}$. Then, simplify this expression for the isotropic case.

9.6 - Derive an energy equation [of higher order than Eq. (4.14)] involving the total time rate of change of the total pressure dyad, that is, $D\underset{\approx}{p}/Dt$.

9.7 - For a perfectly conducting fluid characterized by a scalar pressure, under steady state conditions, use the equation of motion (6.6) and the generalized Ohm's law (6.10), to derive the following equation for the fluid velocity component perpendicular to $\underset{\sim}{B}$,

$$\underset{\sim}{u}_{\perp} = - \frac{\underset{\sim}{B}}{B^2} \times \left(\underset{\sim}{E} + \frac{\underset{\sim}{\nabla} p}{ne} \right)$$