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16. Summary/Notes

This is the twelfth chapter, in a series of twenty two, written as an introduction to the fundamentals of plasma physics. Initially, it is presented a derivation of the Parker modified momentum equation, and of the CGL double adiabatic energy equations. Some special cases of the double adiabatic equations are analysed. The concepts of magnetic viscosity and magnetic Reynolds number are introduced, and an analysis is given for the phenomena of diffusion of the magnetic field lines through a plasma, and of freezing of the magnetic field lines to the plasma. The concept of magnetic pressure is also introduced and the subject of plasma confinement in a magnetic field is investigated.

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SIMPLE APPLICATIONS OF MAGNETOHYDRODYNAMICS

1. FUNDAMENTAL EQUATIONS OF MAGNETOHYDRODYNAMICS

The basic equations governing the behavior of a conducting fluid have been presented and discussed in Chapter 9. For convenience, we reproduce here the simplified form of the magnetohydrodynamic equations. They include the equation of continuity for the whole conducting fluid

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0 \tag{1.1}$$

the equation of motion in the form

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \underset{\sim}{J} \times \underset{\sim}{B} - \underset{\sim}{\nabla} p \tag{1.2}$$

and the adiabatic equation of conservation of energy

$$\underset{\sim}{\nabla} p = V_s^2 \underset{\sim}{\nabla} \rho \tag{1.3}$$

where ρ denotes the total mass density, $\underset{\sim}{u}$ is the average fluid velocity, $\underset{\sim}{J}$ is the electric current density, $\underset{\sim}{B}$ is the magnetic flux density, p is the total scalar pressure, and V_S is the adiabatic sound speed, given by $(\gamma p/\rho)^{1/2}$, where γ is the ratio of the specific heats at constant pressure and at constant volume. To these equations we must add Maxwell curl equations, in the following reduced form,

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o \underset{\sim}{J} \tag{1.4}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{\partial \underset{\sim}{B}}{\partial t} \tag{1.5}$$

and the generalized Ohm's law, in the simplified form,

$$\underset{\sim}{J} = \sigma_o (\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) \tag{1.6}$$

where σ_o denotes the electric conductivity of the fluid, and $\underset{\sim}{E}$ is the electric field.

In this set of simplified MHD equations, it has been assumed that macroscopic electrical neutrality is maintained to a high degree of approximation, so that the electric charge, and the force due to the electric field, are negligible. The neglect of the term $\partial \underset{\sim}{E}/\partial t$, in Maxwell equation(1.4), is justified for *very low frequency* phenomena and *highly conducting fluids*, as discussed in section 6, of Chapter 9. As far as the generalized Ohm's law (1.6) is concerned, it is assumed that the time derivatives and pressure gradients are negligible, even though these terms are retained in the other MHD equations. Also, viscosity and thermal conductivity are neglected and the pressure dyad degrades to a scalar pressure.

The advantage of this approximate set of equations is that they reduce substantially the complexity of the more general equations for a conducting fluid and, therefore, facilitate the understanding of the physical processes that take place in a highly conducting fluid at very low frequencies.

1.1 - Parker modified momentum equation

In the presence of a strong magnetic field the pressure tensor of an inviscid conducting fluid is *anisotropic*. When the cyclotron frequency is much larger than the collision frequency, a charged particle gyrates many times around a line of magnetic force during the time between collisions, implying that there is equipartition between the kinetic energies of the particles in the two independent directions in the plane perpendicular to $\underset{\sim}{B}$ but not, in general, in the direction along $\underset{\sim}{B}$. If we denote by $p_{\perp}$ and $p_{\parallel}$ the scalar pressures in the plane normal to $\underset{\sim}{B}$ and along

$\underset{\sim}{B}$, respectively, and consider a *local* coordinate system in which the third axis is in the direction of $\underset{\sim}{B}$, we can write the pressure tensor of an inviscid fluid as

$$\underset{\approx}{p} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix} \tag{1.7}$$

When the magnetic field is not constant, the orientation of the axes of the local coordinate system changes from point to point and this change in direction must be taken into account in evaluating the divergence of the pressure tensor. Thus, we can express $\underset{\approx}{p}$, in Eq. (1.7), as the sum of a hydrostatic scalar pressure $p_\perp$ and another tensor referred to the local coordinate system, as

$$\underset{\approx}{p} = p_\perp \underset{\approx}{1} + (p_\parallel - p_\perp) \hat{\underset{\sim}{B}} \hat{\underset{\sim}{B}} \tag{1.8}$$

where $\underset{\approx}{1}$ is the unit dyad

$$\underset{\approx}{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.9}$$

and $\hat{\underset{\sim}{B}} \hat{\underset{\sim}{B}} = \underset{\sim}{B} \underset{\sim}{B} / B^2$ is the dyad formed from the unit vector $\hat{\underset{\sim}{B}}$, parallel to $\underset{\sim}{B}$,

$$\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$

The momentum equation (1.2) must be modified to include the anisotropy of the pressure dyad. Thus, we write

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \underset{\sim}{J} \times \underset{\sim}{B} - \underset{\sim}{\nabla} \cdot \underset{\approx}{p} \tag{1.11}$$

To evaluate $\underset{\sim}{\nabla} \cdot \underset{\approx}{p}$, with $\underset{\approx}{p}$ as given by Eq. (1.8), we note that

$$\underset{\sim}{\nabla} \cdot (p_{\perp} \underset{\approx}{1}) = \underset{\sim}{\nabla}\, p_{\perp} \tag{1.12}$$

and using the following identity

$$\underset{\sim}{\nabla} \cdot \left[\frac{(p_{\parallel} - p_{\perp})}{B^2} \underset{\sim\sim}{BB} \right] = (\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \left[\frac{(p_{\parallel} - p_{\perp})}{B^2} \underset{\sim}{B} \right] + \frac{(p_{\parallel} - p_{\perp})}{B^2} \underset{\sim}{B} (\underset{\sim}{\nabla} \cdot \underset{\sim}{B}) \tag{1.13}$$

where the second term in the right-hand side vanishes, in virtue of $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$, we obtain

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{p} = \underset{\sim}{\nabla}\, p_{\perp} + (\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \left[\frac{(p_{\parallel} - p_{\perp})}{B^2}\, \underset{\sim}{B} \right] \tag{1.14}$$

Furthermore, using Maxwell equation (1.4) we can write the magnetic force per unit volume as

$$\underset{\sim}{J} \times \underset{\sim}{B} = \frac{1}{\mu_o} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} \tag{1.15}$$

The term in the right-hand side can be expanded, using a vector identity, with the result that

$$\underset{\sim}{J} \times \underset{\sim}{B} = \frac{1}{\mu_o} \left[(\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\underset{\sim}{B} - \frac{1}{2} \underset{\sim}{\nabla} (B^2) \right] \tag{1.16}$$

Substituting expressions (1.14) and (1.16) into the momentum equation (1.11), we obtain, finally,

$$\rho \frac{D\underset{\sim}{u}}{Dt} = - \underset{\sim}{\nabla} \left(p_{\perp} + \frac{B^2}{2\mu_o} \right) + (\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \left\{ \left[\frac{1}{\mu_o} - \frac{(p_{\parallel} - p_{\perp})}{B^2} \right] \underset{\approx}{B} \right\} \tag{1.17}$$

This equation differs from the usual momentum equation (1.2), for a highly conducting inviscid fluid, only through the term $(p_{\parallel}-p_{\perp})/B^2$. It was derived, although in a quite different way, by E.N. Parker in 1957 and, for this reason, it is usually referred to as Parker modified momentum equation.

1.2 - The double adiabatic equations of Chew, Goldberger and Low (CGL)

To complement the momentum equation(1.17), we need equations for the time rate of change of $p_{\parallel}$ and $p_{\perp}$. These equations will take the place of the familiar adiabatic energy equation (1.3) which applies for the isotropic case. From the general energy equation (9.4.14) for a conducting fluid, if we do not take into account heat conduction, and Joule heating, we have

$$\frac{D}{Dt}\left(\frac{3}{2}p\right) + \frac{3}{2}\, p\, \underset{\sim}{\nabla}.\underset{\sim}{u} + (\underset{\approx}{p}\cdot\underset{\sim}{\nabla})\cdot\underset{\sim}{u} = 0 \qquad (1.18)$$

with the pressure dyad $\underset{\approx}{p}$ as given by Eq. (1.8), and where the scalar pressure p is one third the trace of $\underset{\approx}{p}$, that is,

$$p = \frac{1}{3}\left(2\,p_{\perp} + p_{\parallel}\right) \qquad (1.19)$$

Note that 3p/2 is the total thermal energy density. By direct expansion, using Eq. (1.8) for $\underset{\approx}{p}$, we find that

$$(\underset{\approx}{p}\cdot\underset{\sim}{\nabla})\cdot\underset{\sim}{u} = \left[p_{\perp}\underset{\sim}{\nabla} + (p_{\parallel} - p_{\perp})(\underset{\sim}{\hat{B}}\underset{\sim}{\hat{B}}\cdot\underset{\sim}{\nabla})\right]\cdot\underset{\sim}{u} \qquad (1.20)$$

and taking this expression, together with (1.19), into Eq. (1.18), we obtain

$$\frac{D}{Dt}(2\, p_\perp + p_\parallel) + (p_\parallel + 4\, p_\perp)\, \underset{\sim}{\nabla} \cdot \underset{\sim}{u} + 2(p_\parallel - p_\perp)(\hat{\underset{\sim}{B}}\, \hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}) \cdot \underset{\sim}{u} = 0 \tag{1.21}$$

A strong magnetic field constrains the motion of the charged particles only in the direction transverse to $\underset{\sim}{B}$, but they are still free to move large distances along $\underset{\sim}{B}$. Thus, it is reasonable to suppose that the contribution to the total thermal energy, arising from the motion of the particles parallel to the field, also satisfy an equation of conservation of energy similar to (1.18). This leads to the following equation for the part of the total thermal energy due to the random motions of the particles along the magnetic field,

$$\frac{Dp_\parallel}{Dt} + p_\parallel\, \underset{\sim}{\nabla} \cdot \underset{\sim}{u} + 2\, p_\parallel (\hat{\underset{\sim}{B}}\, \hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}) \cdot \underset{\sim}{u} = 0 \tag{1.22}$$

Eqs. (1.21) and (1.22) can also be obtained from an energy equation of higher order than (9.4.14), involving the total time rate of change of the pressure dyad $\underset{\approx}{p}$. When this equation, involving $D\, \underset{\approx}{p} / D\, t$, is contracted with the unit dyad $\underset{\approx}{1}$ we obtain Eq. (1.21), and when contracted with the dyad $\hat{\underset{\sim}{B}}\, \hat{\underset{\sim}{B}}$ results in Eq. (1.22). From these two equations, we obtain

$$\frac{Dp_\perp}{Dt} + 2\,p_\perp\,\underset{\sim}{\nabla}\,.\,\underset{\sim}{u} - p_\perp\,(\,\underset{\sim}{\hat{B}}\,\underset{\sim}{\hat{B}}\,.\,\underset{\sim}{\nabla}\,).\,\underset{\sim}{u} = 0 \tag{1.23}$$

Eqs. (1.22) and (1.23) enable $p_\parallel$ and $p_\perp$ to be calculated. They can be written in a more succinct form, as follows. First we note that,using Maxwell curl equation

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \tag{1.24}$$

and considering a perfectly conducting fluid for which

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = 0 \tag{1.25}$$

we have,

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla} \times (\,\underset{\sim}{u} \times \underset{\sim}{B}\,) \tag{1.26}$$

Expanding the right-hand side using the vector identity $\underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) = (\underset{\sim}{B}\,.\,\underset{\sim}{\nabla})\,\underset{\sim}{u} - \underset{\sim}{B}(\,\underset{\sim}{\nabla}\,.\,\underset{\sim}{u}\,) - (\,\underset{\sim}{u}\,.\,\underset{\sim}{\nabla}\,)\underset{\sim}{B} + \underset{\sim}{u}(\underset{\sim}{\nabla}\,.\,\underset{\sim}{B})$, and noting that $\underset{\sim}{\nabla}\,.\,\underset{\sim}{B} = 0$, we obtain

$$\frac{D\underset{\sim}{B}}{Dt} = (\underset{\sim}{B}\,.\,\underset{\sim}{\nabla})\,\underset{\sim}{u} - \underset{\sim}{B}\,(\underset{\sim}{\nabla}\,.\,\underset{\sim}{u}) \tag{1.27}$$

If we now take the scalar product of Eq (1.27) with $\underset{\sim}{B}/B^2$, we obtain

$$\frac{1}{2B^2}\frac{D(B^2)}{Dt} = \hat{\underset{\sim}{B}} \cdot (\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla})\,\underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \tag{1.28}$$

which may be written as

$$\frac{1}{B}\frac{DB}{Dt} = (\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}).\,\underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \tag{1.29}$$

Furthermore, from the equation of continuity (1.1), we get

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{1.30}$$

and using Eqs. (1.29) and (1.30), to eliminate $(\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}).\underset{\sim}{u}$ and $\underset{\sim}{\nabla} \cdot \underset{\sim}{u}$ in Eqs.(1.22) and (1.23), we obtain

$$\frac{1}{p_{\parallel}}\frac{Dp_{\parallel}}{Dt} - \frac{3}{\rho}\frac{D\rho}{Dt} + \frac{2}{B}\frac{DB}{Dt} = 0 \tag{1.31}$$

$$\frac{1}{p_{\perp}}\frac{Dp_{\perp}}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} - \frac{1}{B}\frac{DB}{Dt} = 0 \tag{1.32}$$

These two equations can be written in compact form as

$$\frac{D}{Dt}\left(\frac{p_{\parallel}\,B^2}{\rho^3}\right) = 0 \tag{1.33}$$

$$\frac{D}{Dt}\left(\frac{p_\perp}{\rho B}\right) = 0 \qquad (1.34)$$

They are known as the *double adiabatic equations* for a conducting fluid in a strong magnetic field, and are due to G.F. Chew, M.L. Goldberger and F.E. Low (1956). They are also known as the CGL equations. They take the place of the adiabatic energy equation

$$\frac{D}{Dt}\left(p\, \rho^{-\gamma} \right) = 0 \qquad (1.35)$$

1.3 - Special cases of the double adiabatic equations

As a simple application of the double adiabatic equations, consider the case in which the only variations are *parallel* to the magnetic field as, for example, in sound waves travelling along the field lines. This situation is usually referred to as *linear compression* parallel to the $\underset{\sim}{B}$ field or one-dimensional compression. The magnetic field is assumed to be straight and uniform, and directed along the z axis. Thus, $B_x = B_y = 0$ and $\underset{\sim}{B} = B_z \hat{\underset{\sim}{z}}$, as well as $\partial/\partial x = \partial/\partial y = 0$. In this case, we find

$$(\hat{\underset{\sim}{B}}\, \hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}) \cdot \underset{\sim}{u} = \frac{\partial u_z}{\partial z} = \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \qquad (1.36)$$

and from Eq.(1.29), we see that B stays constant. Eqs.(1.31) and (1.32), with $DB/Dt = 0$, then yields

$$\frac{D}{Dt} \left(\frac{p_{\parallel}}{\rho^3} \right) = 0 \tag{1.37}$$

$$\frac{D}{Dt} \left(\frac{p_{\perp}}{\rho} \right) = 0 \tag{1.38}$$

If we compare these results with Eq. (1.35), we find that γ may be assigned the value 3 along the field lines (one-dimensional compression), and the value 1 across the field lines.

It is useful to introduce a parallel and a perpendicular temperature through the relations

$$p_{\parallel} = n\ k\ T_{\parallel} \tag{1.39}$$

$$p_{\perp} = n\ k\ T_{\perp} \tag{1.40}$$

Therefore, for the case of *one - dimensional compression* parallel to the magnetic field,

$$T_{\parallel} \propto n^2 \tag{1.41}$$

$$T_{\perp} = \text{constant} \tag{1.42}$$

which shows that this type of compression is isothermal with respect to the perpendicular temperature $T_\perp$. The changes in $p_\perp$ are, therefore, entirely due to the changes in the number density n, whereas those of $p_\parallel$ are due to changes in both n and $T_\parallel$.

Another special case of interest is the *two-dimensional compression perpendicular* to the $\underset{\sim}{B}$ field, in which all motion is transverse to the field lines. This situation can be pictured as the motion of magnetic flux tubes, identified by the particles contained in them. Assuming straight field lines along the z axis ($B_x = B_y = 0$, $\underset{\sim}{B} = B_z \; \hat{\underset{\sim}{z}}$) and variations only in the transverse direction ($\partial / \partial z=0$) we find that

$$(\hat{\underset{\sim}{B}} \, \hat{\underset{\sim}{B}} \, . \, \underset{\sim}{\nabla}) . \, \underset{\sim}{u} = (\hat{\underset{\sim}{z}} \, \frac{\partial}{\partial z}) \, . \, \underset{\sim}{u} = 0 \tag{1.43}$$

and Eqs. (1.22) and (1.23) yield

$$\frac{Dp_\parallel}{Dt} - \frac{p_\parallel}{\rho} \frac{D\rho}{Dt} = 0 \tag{1.44}$$

$$\frac{Dp_\perp}{Dt} - 2 \frac{p_\perp}{\rho} \frac{D\rho}{Dt} = 0 \tag{1.45}$$

Therefore, in the the case of cylindrical compression perpendicular to $\underset{\sim}{B}$ the adiabatic equations reduce to

$$\frac{D}{Dt} \left(\frac{p_\parallel}{\rho} \right) = 0 \tag{1.46}$$

$$\frac{D}{Dt}\left(\frac{p_{\perp}}{\rho^2}\right) = 0 \tag{1.47}$$

Comparing with (1.35), γ takes the value 1 parallel to the magnetic field, and 2 transverse to it. Using Eqs.(1.39) and (1.40) it is seen that for a two-dimensional (cylindrically symmetric) compression perpendicular to the magnetic field,

$$T_{\parallel} = \text{constant} \tag{1.48}$$

$$T_{\perp} \propto n \tag{1.49}$$

so that this type of compression is isothermal with respect to the parallel temperature. The changes in $p_{\parallel}$ are due entirely to variations in the number density n, whereas those of $p_{\perp}$ result from variations in n as well as in $T_{\perp}$.

In the case of *three - dimensional spherically symmetric* compression, we have

$$p_{\perp} = p_{\parallel} = p \tag{1.50}$$

and Eq. (1.21) reduces to

$$\frac{D}{Dt}(3p) + 5p\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = 0 \tag{1.51}$$

Thus, we obtain

$$\frac{D}{Dt}\left(\frac{p}{\rho^{5/3}}\right) = 0 \qquad (1.52)$$

which is the familiar adiabatic equation (1.35) of gas dynamics, with $\gamma = 5/3$. In any of the cases of adiabatic compression, the fluid has to be subjected to a certain system of forces in order to achieve the desired type of adiabatic compression. The required system of forces has to be determined from the momentum equation in conjunction with the conditions appropriate to the particular problem under analysis.

1.4 - Energy integral

As a final consideration in this section, we will show that the system of hydromagnetic equations (1.1) to (1.6) possesses an energy integral. Using Maxwell equation (1.4), to substitute $\underset{\sim}{J}$ in the equation of motion (1.2), yields

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \frac{1}{\mu_o} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} - \underset{\sim}{\nabla} p \qquad (1.53)$$

Now, take the dot product of this equation with $\underset{\sim}{u}$,

$$\rho\, \underset{\sim}{u} . \frac{D\underset{\sim}{u}}{Dt} = \frac{1}{\mu_o} \underset{\sim}{u} . (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} - \underset{\sim}{u} . \underset{\sim}{\nabla} p \qquad (1.54)$$

The term on the left-hand side can be expanded as

$$\rho \underset{\sim}{u} \cdot \frac{D\underset{\sim}{u}}{Dt} = \rho \underset{\sim}{u} \cdot \left[\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} \right]$$

$$= \frac{1}{2} \rho \left[\frac{\partial u^2}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) u^2 \right]$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) - \frac{u^2}{2} \frac{\partial \rho}{\partial t} + \frac{1}{2} \rho (\underset{\sim}{u} . \underset{\sim}{\nabla}) u^2 \qquad (1.55)$$

Using the continuity equation (1.1), to eliminate $\partial\rho/\partial t$ in Eq.(1.55), yields

$$\rho \underset{\sim}{u} \cdot \frac{D\underset{\sim}{u}}{Dt} = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) + \frac{u^2}{2} \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) + \frac{1}{2} \rho (\underset{\sim}{u} . \underset{\sim}{\nabla}) u^2$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) + \underset{\sim}{\nabla} . \left(\frac{1}{2} \rho u^2 \underset{\sim}{u} \right) \qquad (1.56)$$

In order to transform the term $\underset{\sim}{u} . \underset{\sim}{\nabla} p$, we write the adiabatic energy equation (1.35) as

$$\rho^{-\gamma} \frac{Dp}{Dt} - \gamma p \rho^{-(\gamma + 1)} \frac{D\rho}{Dt} = 0 \qquad (1.57)$$

and use the continuity equation in the form

$$\frac{D\rho}{Dt} = - \rho (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) \qquad (1.58)$$

Combining these two equations, we obtain

$$\frac{\partial p}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} p + \gamma p \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0 \tag{1.59}$$

which may also be written as

$$\frac{\partial p}{\partial t} + (1 - \gamma) \underset{\sim}{u} \cdot \underset{\sim}{\nabla} p + \gamma \underset{\sim}{\nabla} \cdot (p \underset{\sim}{u}) = 0 \tag{1.60}$$

from which we get

$$\underset{\sim}{u} \cdot \underset{\sim}{\nabla} p = \frac{1}{(\gamma - 1)} \frac{\partial p}{\partial t} + \frac{1}{(\gamma - 1)} \underset{\sim}{\nabla} (p \underset{\sim}{u}) \tag{1.61}$$

Finally, for the $\underset{\sim}{u} . (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}$ term in Eq (1.54), considering a perfectly conducting fluid for which $\underset{\sim}{E} = - \underset{\sim}{u} \times \underset{\sim}{B}$, and using a vector identity, we can write

$$\begin{aligned} \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} &= - (\underset{\sim}{u} \times \underset{\sim}{B}) \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \\ &= \underset{\sim}{E} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \\ &= \underset{\sim}{B} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{E}) - \underset{\sim}{\nabla} \cdot (\underset{\sim}{E} \times \underset{\sim}{B}) \end{aligned} \tag{1.62}$$

Using Maxwell equation (1.5) we arrive at

$$\frac{1}{\mu_o} \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} = - \frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_o} \right) - \frac{1}{\mu_o} \underset{\sim}{\nabla} \cdot (\underset{\sim}{E} \times \underset{\sim}{B}) \tag{1.63}$$

Substituting Eqs. (1.56), (1.61) and (1.63), into Eq. (1.54), yields the following energy conservation equation

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \rho u^2 + \frac{p}{(\gamma - 1)} + \frac{B^2}{2\mu_o} \right] + \underset{\sim}{\nabla} \cdot \left[\frac{1}{2} \rho u^2 \underset{\sim}{u} + \frac{\gamma}{(\gamma - 1)} p \underset{\sim}{u} + \underset{\sim}{E} \times \underset{\sim}{H} \right] = 0 \tag{1.64}$$

The first three terms of this equation represent the kinetic energy density associated with the macroscopic motion of the fluid, the thermal energy density, and the energy density stored in the magnetic field, respectively, whereas the last three terms denote the flux of macroscopic kinetic energy, the flux of thermal energy trasported at the macroscopic mean velocity $\underset{\sim}{u}$, and the flux of electromagnetic energy (Poynting vector $\underset{\sim}{E} \times \underset{\sim}{H}$), respectively.

If we integrate Eq. (1.64) over the entire fluid-plus-vacuum volume and, use Gauss' divergence theorem to transform the divergence term into a surface integral, we find that the first two terms in the surface integral vanish, since ρ, p and $\underset{\sim}{u}$ are zero outside the fluid. The remaining surface term is the surface integral of the Poynting vector which, for an isolated system, also vanishes. Therefore, we obtain the energy conservation integral

$$\int_V \left[\frac{1}{2} \rho u^2 + \frac{p}{(\gamma - 1)} + \frac{B^2}{2\mu_o} \right] dV = \text{constant} \qquad (1.65)$$

The first integral represents the macroscopic kinetic energy of the fluid, the second term is the thermal free energy, and the last one represents the total energy of the magnetic field. It is usually useful to separate Eq.(1.65) into a kinetic energy part

$$K = \int_V \frac{1}{2} \rho u^2 \, dV \qquad (1.66)$$

and a potential energy part

$$U = \int_V \left[\frac{p}{(\gamma - 1)} + \frac{B^2}{2\mu_o} \right] dV \qquad (1.67)$$

with the energy conservation law K + U = constant. In these equations the integration extends over the entire fluid - plus - vacuum volume.

2. MAGNETIC VISCOSITY AND REYNOLDS NUMBER

The behavior of the magnetic field is of great importance in many MHD problems. To obtain a simple equation for $\underset{\sim}{B}$, we start by taking the curl of the generalized Ohm's law (1.6),

$$\underset{\sim}{\nabla} \times \underset{\sim}{J} = \sigma_o \left[\underset{\sim}{\nabla} \times \underset{\sim}{E} + \underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) \right] \quad (2.1)$$

Replacing $\underset{\sim}{J}$ and $\underset{\sim}{\nabla} \times \underset{\sim}{E}$, using Maxwell curl equations (1.4) and (1.5), we obtain

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) = \mu_o \, \sigma_o \left[- \frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) \right] \quad (2.2)$$

Making use of the following identity (with $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$)

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) = - \nabla^2 \underset{\sim}{B} \quad (2.3)$$

equation (2.2) reduces to

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) + \eta_m \nabla^2 \underset{\sim}{B} \quad (2.4)$$

where

$$\eta_m = \frac{1}{\mu_o \sigma_o} \quad (2.5)$$

is called the *magnetic viscosity.*

The first term in the right-hand side of Eq.(2.4) is called the *flow term*, while the second term is called the *diffusion term*. To compare the relative magnitude of these two terms, we can use dimensional analysis and take, approximately,

$$|\underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B})| \simeq \frac{uB}{L} \tag{2.6}$$

$$\eta_m \, |\nabla^2 \underset{\sim}{B}| \simeq \frac{\eta_m B}{L^2} \tag{2.7}$$

where L denotes some characteristic length for the variation of the parameters. The ratio of the flow term to the diffusion term is called the *magnetic Reynolds number* and is, therefore, given by

$$R_m = \frac{u\,L}{\eta_m} \tag{2.8}$$

In most MHD problems one or the other of these two terms is of predominant importance and R_m is either very large, or very small compared to unity.

It is instructive to compare the magnetic viscosity, η_m, and the magnetic Reynolds number, R_m, with the ordinary hydrodynamic viscosity, η_k, and Reynolds number, R. For this purpose, consider the Navier-Stokes equation of hydrodynamics

$$\frac{D\underset{\sim}{u}}{Dt} = \underset{\sim}{f} - \frac{1}{\rho} \underset{\sim}{\nabla} p + \eta_k \left[\nabla^2 \underset{\sim}{u} + \frac{1}{3} \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) \right] \tag{2.9}$$

where $\underset{\sim}{f}$ denotes the average force per unit mass of the fluid, and η_k is the kinematic viscosity (viscosity divided by density). Comparing this equation with Eq.(2.4), we see that the role played by η_m , in the rate of change of $\underset{\sim}{B}$, is completely analogous to the role played by η_k, in the rate of change of the mean fluid velocity $\underset{\sim}{u}$. The ordinary Reynolds number is defined as the ratio of the inertia term $(\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{u}$ to the viscosity term $\eta_k \nabla^2 \underset{\sim}{u}$. Using dimensional analysis, we have

$$|(\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{u}| \simeq \frac{u^2}{L} \tag{2.10}$$

$$\eta_k\, |\nabla^2 \underset{\sim}{u}| \simeq \eta_k \frac{u}{L^2} \tag{2.11}$$

which gives the following expression, completely analogous to R_m , for the ordinary Reynolds number

$$R = \frac{uL}{\eta_k} \tag{2.12}$$

3 . DIFFUSION OF MAGNETIC FIELD LINES

When $R_m << 1$, that is, when the diffusion term dominates, Eq. (2.4) becomes approximately,

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \eta_m \nabla^2 \underset{\sim}{B} \qquad (R_m << 1) \tag{3.1}$$

This is the equation of diffusion of a magnetic field in a stationary conductor, resulting in the decay of the magnetic field. It is analogous to the particle diffusion equation studied in Chapter 10. The characteristic time of decay of the magnetic field can be obtained by dimensional analysis, taking

$$\left| \frac{\partial \underset{\sim}{B}}{\partial t} \right| \simeq \frac{B}{\tau_D} \tag{3.2}$$

$$\left| \eta_m \nabla^2 \underset{\sim}{B} \right| \simeq \frac{\eta_m B}{L^2} \tag{3.3}$$

where τ_D represents a characteristic time for variation of the plasma parameters. Thus, according to Eq. (3.1), the magnetic field diffuses away with a characteristic time of decay of the order of

$$\tau_D = \frac{L^2}{\eta_m} = L^2 \mu_o \sigma_o \tag{3.4}$$

For ordinary conductors the time of decay is very small. If we consider, for example, a copper sphere of radius 1 meter, we find that τ_D is less than 10 seconds. For a celestial body, however, because of the large dimensions, the time of decay can be very large. For the Earth's core, considering it to be molten iron, the time of free decay is approximately 10^4 years, while for the general magnetic field of the Sun it is found to be of the order of 10^{10} years.

4. FREEZING OF THE MAGNETIC FIELD LINES TO THE PLASMA

A completely different type of behavior appears when $R_m >> 1$. In this case the flow term dominates over the diffusion term and Eq. (2.4) reduces to

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla} \times (\underset{\sim}{u} \times \underset{\sim}{B}) \qquad (R_m >> 1) \tag{4.1}$$

This equation implies that, in a highly conducting fluid, the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. Alfvēn has expressed this type of behavior by saying that the magnetic field lines are "frozen" in the conducting fluid. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid.

In order to show this implication of Eq (4.1), it is convenient to consider, initially, the concept of magnetic tubes of force, which are used to visually describe the direction and magnitude of $\underset{\sim}{B}$ at various points in space. One can think of the space pervaded by a magnetic field as divided into a large number of elementary magnetic tubes of force, all of them enclosing the same magnetic flux $\Delta\ \Phi_B$. If ΔS is the local cross sectional area of an elementary magnetic tube of force (Fig.1), then the magnitude of $\underset{\sim}{B}$, at the local point P, is equal to $\Delta\ \Phi_B/\Delta S$. According to this definition, the magnitude of $\underset{\sim}{B}$ is everywhere inversely proportional to the cross sectional area of the elementary tubes of force.

Let us now consider a closed line whose points move with velocity $\underset{\sim}{u}$ in a space pervaded by a magnetic field. Assume, for the moment, that $\underset{\sim}{u}$ is an arbitrary function of position and time (not necessarily equal to the fluid velocity), with the result that the closed curve may change in shape, as well as undergo translational and rotational motion. Let C_1 denote the closed curve at time t, bounding

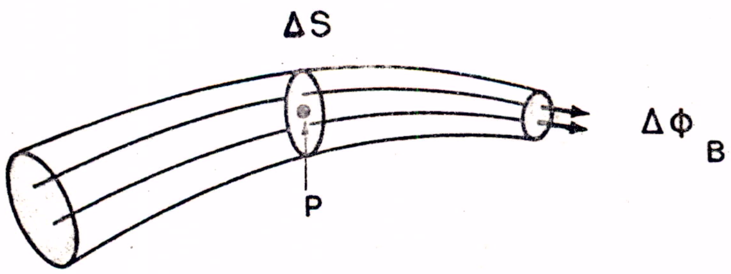


Fig. 1 - Elementary magnetic tube of force. The magnitude of $\underset{\sim}{B}$, at the point P, is equal to $\Delta\ \Phi_B\ /\ \Delta S$.

the open surface $\underset{\sim}{S}(t) = \underset{\sim}{S}_1$. At a time Δt later, let C_2 and $\underset{\sim}{S}(t+\Delta t)=\underset{\sim}{S}_2$ denote the corresponding closed curve and open surface (Fig. 2). The flux of the magnetic field through an open surface $\underset{\sim}{S}$, at time t,

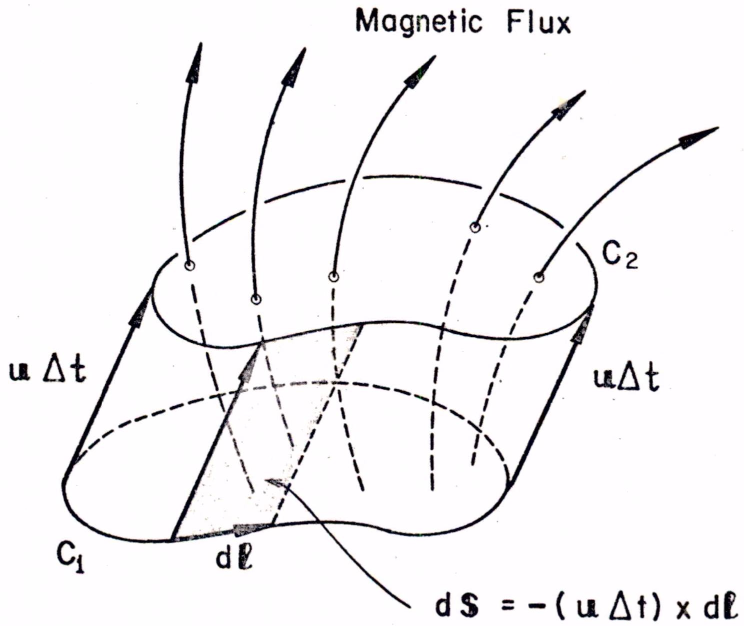


Fig.2 - A closed line bounding an open surface moving in a magnetic field with velocity $\underset{\sim}{u}$ ($\underset{\sim}{r}$, t), viewed at the instants of time t and t + Δt. The shaded area is the part of the cylindrical surface described by an element d $\underset{\sim}{\ell}$ of the contour curve.

is given by

$$\Phi_B(t) = \int_S \underset{\sim}{B}\,(\underset{\sim}{r},t).\ d\,\underset{\sim}{S} \qquad (4.2)$$

The rate of change of the magnetic flux through an open surface $\underset{\sim}{S}$ can be written as

$$\frac{d}{dt}\left[\int_S \underset{\sim}{B}(\underset{\sim}{r},t).\ d\,\underset{\sim}{S}\right] = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \underset{\sim}{B}(\underset{\sim}{r},t+\Delta t)\ .\ d\,\underset{\sim}{S}\ - \right.$$

$$\left. - \int_{S_1} \underset{\sim}{B}(\underset{\sim}{r},t).\ d\,\underset{\sim}{S}\right] \qquad (4.3)$$

If we expand $\underset{\sim}{B}(\underset{\sim}{r},\ t+\Delta t)$ in a Taylor series about $\underset{\sim}{B}\ (\underset{\sim}{r},\ t)$, we have

$$\underset{\sim}{B}(\underset{\sim}{r},t+\Delta t) = \underset{\sim}{B}(\underset{\sim}{r},\ t) + \frac{\partial \underset{\sim}{B}(\underset{\sim}{r},t)}{\partial t}\ \Delta t + \ldots \qquad (4.4)$$

so that, in the limit as $\Delta t \to 0$, Eq. (4.3) reduces to

$$\frac{d}{dt}\left[\int_S \underset{\sim}{B}(\underset{\sim}{r},t)\ .\ d\,\underset{\sim}{S}\right] = \lim_{\Delta t \to 0} \left\{ \int_{S_2} \frac{\partial \underset{\sim}{B}(\underset{\sim}{r},t)}{\partial t}\ .\ d\,\underset{\sim}{S}\ + \right.$$

$$+ \frac{1}{\Delta t}\left[\int_{S_2} \underset{\sim}{B}(\underset{\sim}{r},\ t).\ d\,\underset{\sim}{S}\ - \right.$$

$$- \int_{S_1} \underset{\sim}{B} (\underset{\sim}{r} , t) \, . \, d \underset{\sim}{S} \Big] \Big\} \qquad (4.5)$$

To evaluate the term within brackets in the right-hand side of this equation, we can use the fact that for any *closed surface* at time t we have, from Gauss' divergence theorem,

$$\oint \underset{\sim}{B} \, . \, d \underset{\sim}{S} = \int_V \underset{\sim}{\nabla} \, . \, \underset{\sim}{B} \, d \, V = 0 \qquad (4.6)$$

since $\underset{\sim}{\nabla} \, . \, \underset{\sim}{B} = 0$. Thus, if we apply this result to the closed surface consisting of $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$ and the sides of the cylindrical surface of length $\underset{\sim}{u} \, \Delta \, t$ shown in Fig. 2, we obtain

$$- \int_{S_1} \underset{\sim}{B}(\underset{\sim}{r},t).d \underset{\sim}{S} + \int_{S_2} \underset{\sim}{B}(\underset{\sim}{r},t).d \underset{\sim}{S} - \oint_{C_1} \underset{\sim}{B}(\underset{\sim}{r},t).\left[(\underset{\sim}{u}\Delta t) \times d\underset{\sim}{\ell}\right] = 0 \qquad (4.7)$$

where the minus sign in the first term on the left-hand side is due to the fact that the *outwardly drawn* unit normal to the surface S_1 is in a direction opposite to that of the surface S_2, and $- (\underset{\sim}{u}\Delta t) \times d\underset{\sim}{\ell}$ is the element of area(pointing outwards) covered by the vector element $d\underset{\sim}{\ell}$ of the closed curve in the time interval Δt. If Eq.(4.7) is substituted into Eq.(4.5) and the limit $\Delta t \to 0$ is evaluated, noting that in this limit $S_2 = S_1 = S(t)$, we obtain

$$\frac{d}{dt}\left[\int_S \underset{\sim}{B}\,(\underset{\sim}{r},t)\;.\;d\,\underset{\sim}{S}\right] = \int_S \frac{\partial \underset{\sim}{B}(\underset{\sim}{r},t)}{\partial t}\;.\;d\,\underset{\sim}{S}\;+$$

$$+\oint_C \underset{\sim}{B}\,(\underset{\sim}{r},\underset{\sim}{t})\;.\;(\underset{\sim}{u}\times d\,\underset{\sim}{\ell}) \tag{4.8}$$

The last term in the right-hand side of this equation can be transformed using the vector identity

$$\underset{\sim}{B}(\underset{\sim}{r},t)\;.\;(\underset{\sim}{u}\times d\,\underset{\sim}{\ell}) = -\left[\underset{\sim}{u}\times \underset{\sim}{B}(\underset{\sim}{r},\,t)\right]\;.\;d\,\underset{\sim}{\ell} \tag{4.9}$$

and from Stokes' theorem we can write

$$\oint_C \left[\underset{\sim}{u}\times \underset{\sim}{B}(\underset{\sim}{r},t)\right].\,d\underset{\sim}{\ell} = \int_S \underset{\sim}{\nabla}\times\left[\underset{\sim}{u}\times \underset{\sim}{B}(\underset{\sim}{r},t)\right]\;.\;d\,\underset{\sim}{S} \tag{4.10}$$

Thus, using this expression in Eq.(4.8), we obtain

$$\frac{d}{dt}\left[\int_S \underset{\sim}{B}(\underset{\sim}{r},\,t).d\underset{\sim}{S}\right] = \int_S\left\{\frac{\partial \underset{\sim}{B}(\underset{\sim}{r},t)}{\partial t} - \underset{\sim}{\nabla}\times\left[\underset{\sim}{u}\times \underset{\sim}{B}(\underset{\sim}{r},\,t)\right]\right\}.\;d\,\underset{\sim}{S} \tag{4.11}$$

This result is quite general.

Suppose now that the space is filled with a highly conducting fluid, so that Eq. (4.1), valid for $R_m \gg 1$, applies.

If the velocity $\underset{\sim}{u}$ in Eq. (4.11) is taken as the fluid velocity, we conclude, from Eqs.(4.1) and (4.11), that

$$\frac{d}{dt}\left[\int_S \underset{\sim}{B}\,(\underset{\sim}{r},t).d\,\underset{\sim}{S}\right] = 0 \qquad (4.12)$$

which is a mathematical statement of the fact that the magnetic flux linked by a closed line (bounding the open surface S) moving with the fluid velocity $\underset{\sim}{u}$ is constant. Note that this conclusion requires that only the component of the velocity of the closed line perpendicular to $\underset{\sim}{B}$ be the same as the component of the fluid velocity perpendicular to $\underset{\sim}{B}$, since the velocity component parallel to $\underset{\sim}{B}$ gives no contribution to the term $\underset{\sim}{u} \times \underset{\sim}{B}$. Thus, Eq.(4.1) implies that the lines of magnetic flux are frozen into the highly conducting fluid and are carried by any motion of the fluid *perpendicular* to the magnetic field lines. There is no restriction, however, on the motion along the field lines and, therefore, the conducting fluid can flow freely in the direction *parallel* to $\underset{\sim}{B}$.

This result is expected on physical grounds since, as the conducting fluid moves across the magnetic field, it induces an electric field which is proportional to the component of the fluid velocity perpendicular to $\underset{\sim}{B}$. However, if the conductivity of the fluid is infinite, this perpendicular component of velocity must be infinitesimally small if the flow of electric current is to remain finite.

In a fluid of *finite* conductivity the result (4.12) is no longer true. Using Eq. (2.4) in the general result (4.11), yields

$$\frac{d\Phi_B}{dt} = \frac{1}{\mu_o \sigma_o} \int_S \nabla^2 \underset{\sim}{B} \cdot d\underset{\sim}{S} \tag{4.13}$$

where the right-hand side of this equation gives rise to a slipping of magnetic flux through a closed material line.

5. MAGNETIC PRESSURE

5.1 - Concept of magnetic pressure

The concept of *magnetic pressure* is very useful in the study of the confinement of high temperature plasmas. Under steady state conditions the MHD equations reduce to the following closed set of *magnetohydrostatic* equations

$$\underset{\sim}{\nabla} p = \underset{\sim}{J} \times \underset{\sim}{B} \tag{5.1}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_o \underset{\sim}{J} \tag{5.2}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \tag{5.3}$$

If we eliminate $\underset{\sim}{J}$ from these equations, we obtain the equivalent set of magnetohydrostatic equations involving only p and $\underset{\sim}{B}$,

$$\underset{\sim}{\nabla} p = \frac{1}{\mu_o} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} \tag{5.4}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \tag{5.5}$$

The term in the right-hand side of Eq. (5.4) can be written as the divergence of the magnetic part of the electromagnetic stress dyad. Using the vector identity

$$\begin{aligned} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B} &= (\underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B} - \frac{1}{2} \underset{\sim}{\nabla} (B^2) \\ &= \underset{\sim}{\nabla} \cdot (\underset{\sim}{B} \underset{\sim}{B}) - \frac{1}{2} \underset{\sim}{\nabla} \cdot (\underset{\approx}{1} B^2) \end{aligned} \tag{5.6}$$

where $\underset{\approx}{1}$ is the unit dyad, and using the following definition of the magnetic stress dyad

$$\underset{\approx}{T}^{(m)} = \frac{1}{\mu_o} (\underset{\sim}{B} \underset{\sim}{B} - \underset{\approx}{1} \frac{B^2}{2}) \tag{5.7}$$

which written out in matrix form (in a Cartesian coordinate system) is

$$\underset{\approx}{T}^{(m)} = \frac{1}{\mu_o} \begin{pmatrix} (B_x^2 - B^2/2) & B_x \; B_y & B_x \; B_z \\ B_y \; B_x & (B_y^2 - B^2/2) & B_y \; B_z \\ B_z \; B_x & B_z \; B_y & (B_z^2 - B^2/2) \end{pmatrix} \tag{5.8}$$

we can write Eq. (5.4) as

$$\underset{\sim}{\nabla} p = \underset{\sim}{\nabla} \cdot \underset{\approx}{T}^{(m)} \tag{5.9}$$

or, equivalently,

$$\underset{\sim}{\nabla} \cdot \left[\underset{\approx}{1} p - \underset{\approx}{T}^{(m)} \right] = 0 \tag{5.10}$$

The stress is considered to be positive if it is tensile, and negative if it is compressive. Thus, we see that $- \underset{\approx}{T}^{(m)}$ may be defined as the *magnetic pressure* dyad, playing the same role as the fluid pressure dyad.

It is instructive to consider a *local* magnetic coordinate system in which the third axis points along the local direction of $\underset{\sim}{B}$, as shown in Fig. 3. For this local coordinate system, the

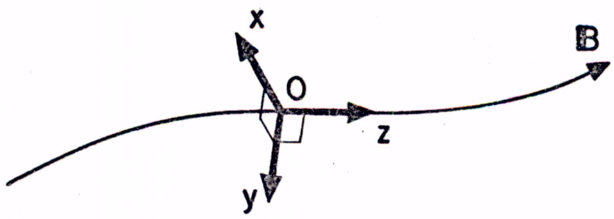


Fig. 3 - Local magnetic coordinate system with the z axis pointing along the local direction of $\underset{\sim}{B}$.

off-diagonal elements of the magnetic stress dyad vanish, since $\underset{\sim}{B} = (0,0,B)$, so that

$$\underset{\approx}{T}^{(m)} = \begin{pmatrix} -B^2/2\mu_o & 0 & 0 \\ 0 & -B^2/2\mu_o & 0 \\ 0 & 0 & B^2/2\mu_o \end{pmatrix} \qquad (5.11)$$

Therefore, the principal stresses are equivalent to a *tension* $B^2/2\mu_o$ along the magnetic field lines, and a *pressure* $B^2/2\mu_o$ perpendicular to the magnetic field lines, similar to a mutual repulsion of the field lines. Alternatively, we can express Eq. (5.11) in the form

$$\underset{\approx}{T}^{(m)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B^2/\mu_o \end{pmatrix} + \begin{pmatrix} -B^2/2\mu_o & 0 & 0 \\ 0 & -B^2/2\mu_o & 0 \\ 0 & 0 & -B^2/2\mu_o \end{pmatrix} \qquad (5.12)$$

so that the stress caused by the magnetic flux can also be thought of as an *isotropic magnetic pressure* $B^2/2\mu_o$ and a *tension* B^2/μ_o along the magnetic flux lines as if they were elastic cords (Fig. 4).

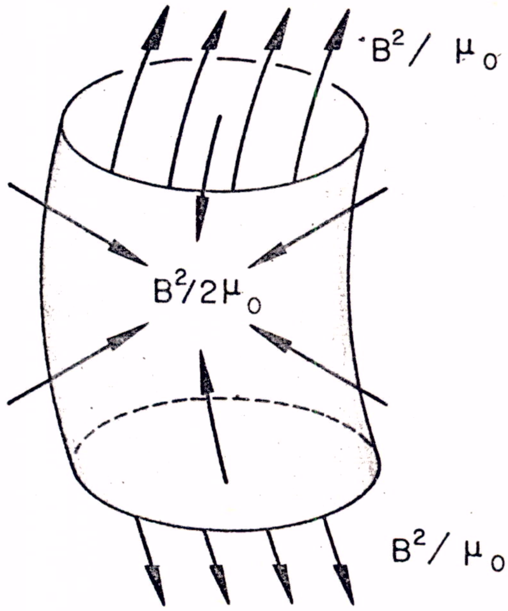


Fig. 4 - The stress caused by the magnetic flux can be decomposed into an isotropic magnetic pressure $B^2/2\mu_o$ and a magnetic tension B^2/μ_o along the field lines.

This latter representation is very useful, since the isotropic pressure $B^2/2\mu_o$ can always be incorporated with the fluid pressure, resulting in a decrease in the pressure exerted by the fluid, while the tension B^2/μ_o along the magnetic flux lines gives the effect of the magnetic forces.

5.2 - Isobaric surfaces

It is convenient to consider in the plasma hypothetical surfaces over which the kinetic pressure is constant, called *isobaric surfaces*. At any point, the vector ∇p is normal to the isobaric surface passing through the point considered. From Eq. (5.1), we see that ∇p is normal to the plane containing $\mathbf{J}$ and $\mathbf{B}$, that is

$$\underset{\sim}{J} \cdot \underset{\sim}{\nabla} p = 0 \tag{5.13}$$

$$\underset{\sim}{B} \cdot \underset{\sim}{\nabla} p = 0 \tag{5.14}$$

Therefore, both $\underset{\sim}{J}$ and $\underset{\sim}{B}$ lie on isobaric surfaces. To illustrate this fact, consider the particular case in which the isobaric surfaces are closed concentric cylindrical surfaces, with the kinetic pressure increasing in the direction towards the central axis of the concentric cylindrical surfaces. Thus, $\underset{\sim}{\nabla} p$ is along a radial line directed toward the axis. From Eqs. (5.13) and (5.14) we see that neither $\underset{\sim}{B}$, nor $\underset{\sim}{J}$, pass through the isobaric surfaces and, therefore, it follows that the cylindrical isobaric surfaces are formed by a network of magnetic field lines and electric currents. Further, in view of Eq.(5.1), the magnetic field lines and electric currents, lying on the isobaric surfaces, must cross each other in such a way that $\underset{\sim}{J} \times \underset{\sim}{B}$ is equal to $\underset{\sim}{\nabla} p$. This situation is shown in Fig. 5. The maximum kinetic pressure occurs along the central axis, which also coincides with a magnetic field line. For this reason, this axis is usually called the *magnetic axis* of the magnetoplasma configuration.

6. PLASMA CONFINEMENT IN A MAGNETIC FIELD

The subject of plasma confinement in a magnetic field is of considerable interest in the theory of controlled thermonuclear fusion. Consider, for simplicity, the special case in which the magnetic field is along the z-axis, that is $\underset{\sim}{B} = B\,\underset{\sim}{\hat{z}}$.

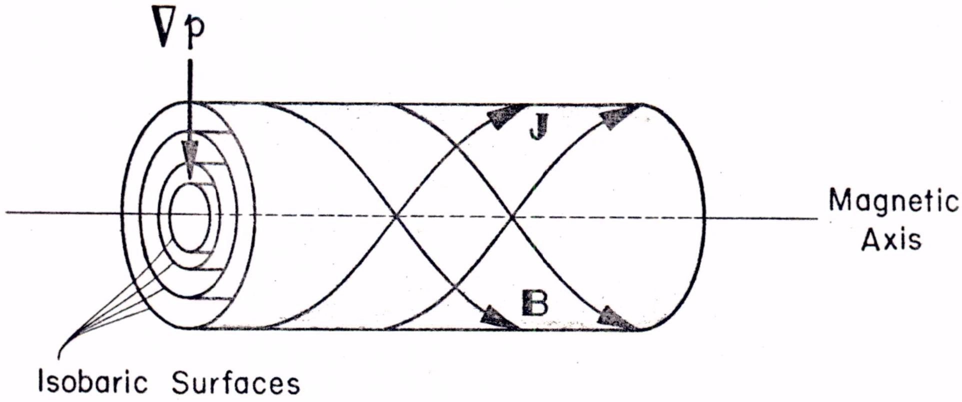


Fig.5 - Isobaric concentric cylindrical surfaces, with $\underset{\sim}{\nabla}$ p along a radial directed towards the magnetic axis. The lines of $\underset{\sim}{B}$ and $\underset{\sim}{J}$ lie on the isobaric surfaces and cross each other in such a manner that $\underset{\sim}{J} \times \underset{\sim}{B}$ is equal to $\underset{\sim}{\nabla}$ p.

In this case, Eq. (5.10) simplifies to

$$\underset{\sim}{\nabla} \cdot \begin{pmatrix} (p + B^2/2\mu_o) & 0 & 0 \\ 0 & (p + B^2/2\mu_o) & 0 \\ 0 & 0 & (p - B^2/2\mu_o \end{pmatrix} = 0 \qquad (6.1)$$

from which we obtain

$$\frac{\partial}{\partial_x} \left(p + \frac{B^2}{2\mu_o} \right) = 0 \qquad (6.2)$$

$$\frac{\partial}{\partial y}\left(p + \frac{B^2}{2\mu_o} \right) = 0 \qquad (6.3)$$

$$\frac{\partial}{\partial z}\left(p - \frac{B^2}{2\mu_o} \right) = 0 \qquad (6.4)$$

Also, from $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$, we have

$$\frac{\partial B}{\partial z} = 0 \qquad (6.5)$$

since, in the local coordinate system , $\underset{\sim}{B} = (0,0,B)$. This last equation, together with (6.4), implies that both p and B do not vary in the direction of $\underset{\sim}{B}$. The solutions of Eqs. (6.2) and (6.3), combined with this result, give

$$\left(p + \frac{B^2}{2\mu_o} \right) = \text{constant} \qquad (6.6)$$

Therefore, in the presence of an externally applied magnetic field, if the plasma is bounded, the kinetic pressure of the plasma decreases from the axis radially outwards, whereas the magnetic pressure increases in the same direction in such a manner that their sum is constant, according to Eq. (6.6). The plasma kinetic pressure can be forced to vanish on an outer surface if the applied magnetic field is sufficiently strong, with the result that the plasma is confined within this outer surface by the magnetic field.

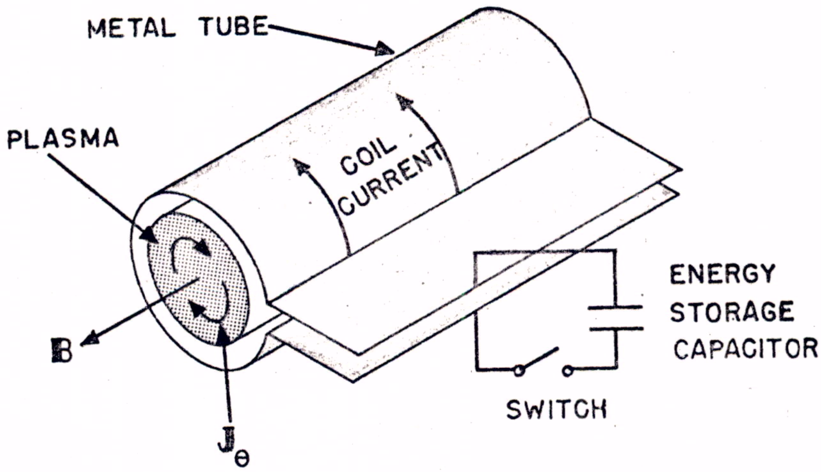
Let $\underset{\sim}{B}_o$ be the value of the magnetic induction external to the plasma (which is the value at the boundary of the plasma). Since the kinetic pressure at the plasma boundary is zero (ideally), we can evaluate the constant in Eq.(6.6) by calculating it at the boundary of the plasma. Therefore,

$$p + \frac{B^2}{2\mu_o} = \frac{B_o^2}{2\mu_o} \tag{6.7}$$

The maximum fluid pressure that can be confined for a given applied field $\underset{\sim}{B}_o$ is, consequently,

$$p_{max} = \frac{B_o^2}{2\mu_o} \tag{6.8}$$

A device that can be used to confine a magnetoplasma by straight parallel field lines is shown in Fig. 6, called a *theta* (θ) - *pinch*, since the effect responsible for the confinement is due to electric currents flowing in the plasma in the azimuthal θ-direction. The plasma is initially confined inside a hollow cylindrical metal tube, whose side is split in the longitudinal direction in such a way as to form a capacitor. When a high voltage is discharged through the capacitor, the large azimuthal (θ) current produced in the metal tube produces a magnetic field in the longitudinal (axial) direction inside the plasma. The electric current, *induced* in the plasma, is also in the azimuthal direction, but in a sense opposite to that on the metal tube.



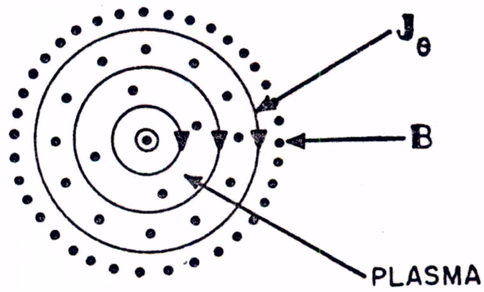


Fig. 6. Magnetoplasma confined by straight parallel field lines in a theta-pinch device.

The resulting $\underset{\sim}{J} \times \underset{\sim}{B}$ force acting on the plasma pushes it inwards, towards the axis, until a balance is reached between the kinetic pressure due to the random thermal motions of the plasma particles, and the magnetic pressure which acts to constrict or *pinch* the plasma.

A parameter β, defined as the ratio of the kinetic pressure at a point inside the plasma, to the confining magnetic pressure at the plasma boundary, is usually introduced as a measure of the relative magnitudes of the kinetic and the magnetic pressures. It is given by

$$\beta = \frac{p}{B_o^2/2\mu_o} \tag{6.9}$$

Note that β ranges between 0 and 1, since the field inside the plasma is less than B_o . From Eq. (6.7), we can also express the parameter β as

$$\beta = 1 - (B/B_o)^2 \tag{6.10}$$

Two special cases of plasma confinement schemes are the so called *low* β and *high* β devices. In the low β devices, the kinetic pressure is small in comparison with the magnetic pressure at the plasma boundary , whereas in the high β devices they are of an equal order of magnitude ($\beta \simeq 1$) .

An important property of a plasma is its *diamagnetic* character. Eq (6.7) implies that the magnetic field inside the plasma is less than its value at the plasma boundary. As the kinetic pressure increases inside the plasma, the magnetic field decreases. Under the action of the externally aplied $\underset{\sim}{B}$ field,the motions of the plasma particles give rise to internal electric currents which produce a magnetic field in a direction opposite to the externally aplied field. Consequently, the *resultant* magnetic field inside the plasma is reduced to a value less than the plasma boundary value. The electric current, induced in the plasma, depends on the number density of the charged particles and their velocity. Therefore, as the plasma kinetic pressure increases, the induced electric current and the induced magnetic field also increase, thus enhancing the diamagnetic effect.

PROBLEMS

12.1 - Consider the energy equation involving the time rate of change of the total pressure dyad $\underset{\approx}{p}$, derived in Problem 9.6. Show that, when this equation is contracted with the unit dyad $\underset{\approx}{1}$ results in Eq. (1.21), whereas when contracted with the dyad $\hat{\underset{\sim}{B}}\,\hat{\underset{\sim}{B}}$ yields Eq. (1.22).

12.2 - Derive an energy conservation equation, similar to Eq. (1.64), but considering the Parker modified momentum equation and the CGL energy equations, instead of Eqs. (1.2) and (1.3).

12.3 - Calculate the *minimum* intensity of the magnetic induction ($\underset{\sim}{B}_o$) necessary to confine a plasma at:

(a) an internal pressure of 100 atm.

(b) a temperature of 10 keV and density of 8×10^{15} cm^{-3}.

12.4 - A plasma is confined by a unidirectional magnetic induction $\underset{\sim}{B}$ of magnitude 5 Weber/m^2. Considering that the plasma temperature is 10 keV and $\beta = 0.4$, calculate the number density of the particles. If the temperature increases to 50 keV, what is the value of the $\underset{\sim}{B}$ field necessary to confine the plasma, assuming that β stays the same?

12.5 - Calculate the diffusion time, τ_D, and the magnetic Reynolds number, R_m, for a typical MHD generator, with L = 0.1 m, $u = 10^3$ m/sec and σ_o = 100 mho/m. Verify that, in this case, τ_D is very short, so that inhomogeneities in the magnetic field are smoothed out rapidly.

12.6 - Consider a plasma in the form of a straight circular cylinder with a helical magnetic field given by

$$\underset{\sim}{B} = B_\theta(r)\ \underset{\sim}{\hat{\theta}} + B_z(r)\ \underset{\sim}{\hat{z}}$$

Show that the force per unit volume, associated with the inward magnetic pressure for this configuration, is

$$-\underset{\sim}{\nabla}_\perp \left(\frac{B^2}{2\mu_o} \right) = -\underset{\sim}{\hat{r}} \frac{\partial}{\partial r} \left(\frac{B^2(r)}{2\mu_o} \right)$$

and the force per unit volume, associated with the magnetic tension due to the curvature of magnetic field lines, is

$$\frac{B^2}{\mu_o} \underset{\sim}{\hat{B}} \cdot \underset{\sim}{\nabla} \underset{\sim}{\hat{B}} = -\underset{\sim}{\hat{r}} \frac{B_\theta^2(r)}{\mu_o r}$$

12.7 - Use Eq. (4.1), for a perfectly conducting fluid, and the nonlinear equation of continuity (1.1), to show that the change of $\underset{\sim}{B}$ with time in a fluid element is related to changes of density according to

$$\frac{D}{Dt}\left(\frac{\underset{\sim}{B}}{\rho}\right) = \left(\frac{\underset{\sim}{B}}{\rho}\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{u}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}$$

Use this relation to establish that, in a perfectly conducting fluid, the fluid elements which lie initially on a magnetic flux line, continue to lie on a flux line.

12.8 - The boundary of the Earth's magnetosphere, in the direction of the Earth-Sun line, occurs at a distance where the kinetic pressure of the solar wind particles is equal to the (modified) Earth's magnetic field pressure. Show that the distance of the magnetopause from the center of the Earth, along the Earth-Sun line, is given approximately by

$$R_M = \left(\frac{2B_o^2}{\mu_o \rho_s v_s^2}\right)^{1/6} R_E$$

where R_E is the Earth's radius, ρ_s is the mass density of the solar wind, v_s is its undisturbed speed, and B_0 is the surface value of the undisturbed Earth's magnetic field.

12.9 - Consider a cylindrically symmetric plasma column ($\partial/\partial z = 0$; $\partial/\partial\theta = 0$), under equilibrium conditions, confined by a magnetic field. Verify that, in cylindrical coordinates, the radial component of Eq. (5.1) becomes

$$\frac{dp(r)}{dr} = J_\theta(r)\, B_z(r) - J_z(r)\, B_\theta(r)$$

Using Maxwell equation (5.2), show that

$$J_\theta = - \frac{1}{\mu_o} \frac{dB_z}{dr}$$

$$J_z = \frac{1}{\mu_o} \frac{1}{r} \frac{d}{dr} (r\, B_\theta)$$

Therefore, obtain the following basic equation for the equilibrium of a plasma column with cylindrical symmetry

$$\frac{d}{dr} \left(p + \frac{B_z^2}{2\mu_o} + \frac{B_\theta^2}{2\mu_o} \right) = - \frac{1}{\mu_o} \frac{B_\theta^2}{r}$$

Give a physical interpretation for the various terms in this equation.

$$\mathbb{J}_p = \frac{1}{\Delta V}\sum_i q_i \mathbb{v}_{pi} = \left(\frac{1}{\Delta V}\sum_i m_i\right)\frac{\partial \mathbb{E}_\perp/\partial t}{B^2}$$

$$= \rho_m \frac{\partial \mathbb{E}_\perp/\partial t}{B^2} \tag{2.12}$$

where the summation is over all positive and negative charges contained in the volume element ΔV, and ρ_m is the mass density of the plasma.

2.2 Plasma dielectric constant

The polarization effect in a plasma is a consequence of the time variation of the electric field. The application of a steady $\mathbb{E}$-field does not result in a polarization field, since the ions and electrons can move around to preserve quasineutrality. Since the plasma behaves like an ordinary dielectric, the polarization current density $\mathbb{J}_p$ can be taken into account through the introduction of the dielectric constant of the plasma. For this purpose, we can separate the total current density $\mathbb{J}$ into the polarization current density $\mathbb{J}_p$ and the current density $\mathbb{J}_0$ due to other sources,

$$\mathbb{J} = \mathbb{J}_p + \mathbb{J}_0 \tag{2.13}$$

Thus, combining $\mathbb{J}_p$ with the term $\epsilon_0 \partial \mathbb{E}_\perp/\partial t$ which appears in the right hand side of Maxwell's $\nabla \times \mathbb{B}$ equation, we obtain

$$\epsilon_0 \frac{\partial \mathbb{E}_\perp}{\partial t} + \frac{\rho_m}{B^2}\frac{\partial \mathbb{E}_\perp}{\partial t} = \epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right)\frac{\partial \mathbb{E}_\perp}{\partial t}$$

$$= \epsilon \frac{\partial \mathbb{E}_\perp}{\partial t} \tag{2.14}$$

where

$$\epsilon = \epsilon_0 \epsilon_r = \epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right) \tag{2.15}$$

is the effective electric permittivity perpendicular to the magnetic field. For low frequencies, the relative permittivity ϵ_r of a plasma is very high; for a number density of 10^{15} particles/cm^3