1. Classification *INPE-COM.4/RPE* *C.D.U.: 533.9*	2. Period	4. Distribution Criterion internal [ ] external [X]
3. Key Words (selected by the author) *WAVE PROPAGATION* *HOT PLASMAS*		
5. Report Nº *INPE-1563-RPE/068*	6. Date *September, 1979*	7. Revised by *René A. Medrano-B.*
8. Title and Sub-title *WAVES IN HOT ISOTROPIC PLASMAS*		9. Authorized by *Nelson de Jesus Parada* *Director*
10. Sector *DCE/DGA/GIO* Code		11. Nº of Copies *10*
12. Authorship *J.A. Bittencourt*		14. Nº of Pages *49*
13. Signature of the responsible *Bittencourt*		15. Price

16. Summary/Notes

This chapter analyzes the problem of wave propagation in a hot isotropic plasma, in the absence of an externally applied magnetic field, from the point of view of kinetic theory, using the Vlasov equation. A major point in this chapter is to emphasize those aspects of wave propagation which arise from the kinetic theory treatment, and which were missing when the problem was previously analyzed using the cold and warm plasma models. A detailed treatment is presented for the characteristics of the longitudinal plasma mode and the transverse electromagnetic mode. The problem of wave-particle interaction is also analyzed, and a description is given for the phenomenon of collisionless Landau damping and for the two-stream instability.

17. Remarks *Revised October 1984.*

INDEX

CHAPTER 18

WAVES IN HOT ISOTROPIC PLASMAS

LIST OF FIGURES

CHAPTER 18

WAVES IN HOT ISOTROPIC PLASMAS

1. INTRODUCTION

We consider in this chapter the propagation of small amplitude waves in unbounded hot plasmas which are close to equilibrium conditions, from the kinetic theory point of view. The problem is examined using the Vlasov equation and only electron motion is considered. The ions, in view of their greater inertia, are assumed to stay immobile. A major point of this chapter will be to emphasize those effects which arise when the Vlasov equation is used, and which were missing when the problem was treated using the cold and warm plasma models (Chapters 16 and 17).

The treatment presented in this chapter is restricted to *isotropic* plasmas, in the absence of an externally applied magnetic field. It is shown that the plasma waves can be separated into three groups, the first group being the *longitudinal plasma wave*, and the second and third groups being the two different polarizations of the *transverse electromagnetic wave*. The chapter ends with a brief discussion of plasma instabilities which arise from the interaction of the plasma particles with the wave electric field. To illustrate the wave-particle interaction phenomenon we describe just one important example, the two-stream instability.

2. BASIC EQUATIONS

The relevant equations for the kinetic theory treatment of small amplitude waves in an electron gas of infinite extent are the Vlasov and Maxwell equations. The Vlasov equation, satisfied by the electron distribution function $f(\underset{\sim}{r},\underset{\sim}{v},t)$, can be written as

$$\frac{\partial f(\underset{\sim}{r},\underset{\sim}{v},t)}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f(\underset{\sim}{r},\underset{\sim}{v},t) + \left\{ - \frac{e}{m_e} [\underset{\sim}{E}(\underset{\sim}{r},t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)] + \frac{\underset{\sim}{F}_{ext}}{m_e} \right\} \cdot$$

$$\cdot \underset{\sim}{\nabla}_v f(\underset{\sim}{r},\underset{\sim}{v},t) = 0 \tag{2.1}$$

where $\underset{\sim}{F}_{ext}$ denotes any force *externally* applied to the plasma, and $\underset{\sim}{E}(\underset{\sim}{r},t)$ and $\underset{\sim}{B}(\underset{\sim}{r},t)$ are the *internal* smoothed, self-consistent, macroscopic electric and magnetic induction fields associated with the distributions of charge density and charge current density inside the plasma. The fields $\underset{\sim}{E}(\underset{\sim}{r},t)$ and $\underset{\sim}{B}(\underset{\sim}{r},t)$ satisfy Maxwell equations

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E}(\underset{\sim}{r},t) = \rho_c(\underset{\sim}{r},t)/\varepsilon_0 \tag{2.2}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B}(\underset{\sim}{r},t) = 0 \tag{2.3}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E}(\underset{\sim}{r},t) = - \frac{\partial \underset{\sim}{B}(\underset{\sim}{r},t)}{\partial t} \tag{2.4}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B}(\underset{\sim}{r},t) = \mu_0 \underset{\sim}{J}(\underset{\sim}{r},t) + \frac{1}{c^2} \frac{\partial \underset{\sim}{E}(\underset{\sim}{r},t)}{\partial t} \tag{2.5}$$

where the charge and current densities are given, respectively, by

$$\rho_c(\underset{\sim}{r},t) = \sum_\alpha q_\alpha n_\alpha(\underset{\sim}{r},t) = \sum_\alpha q_\alpha \int_v f_\alpha(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \tag{2.6}$$

$$\underset{\sim}{J}(\underset{\sim}{r},t) = \sum_{\alpha} q_{\alpha} n_{\alpha}(\underset{\sim}{r},t)\underset{\sim}{u}_{\alpha}(\underset{\sim}{r},t) = \sum_{\alpha} q_{\alpha} \int_{v} \underset{\sim}{v}\, f_{\alpha}(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \qquad (2.7)$$

Eqs. (2.1) to (2.7) form a complete self-consistent set of equations, which were first introduced in section 7 of Chapter 5. It is worth noting that even though there is no explicit collision term in the Vlasov equation (2.1), an important contribution to the charged particle interactions is included through the internal self-consistent electromagnetic fields.

3. GENERAL RESULTS FOR A PLANE PLASMA WAVE IN A HOT ISOTROPIC PLASMA

Consider an unbounded uniform electron plasma with a fixed neutralizing ion background and without any external field present. This is obviously an equilibrium arrangement. Suppose that some electrons are slightly displaced from their equilibrium position. As a result of this small space-dependent perturbation in the electron gas, some sort of oscillatory or wave phenomenon can be expected to arise as a consequence of the electric fields produced by charge separation. The ions, because of their much larger mass, can be assumed to remain nearly stationary during the process, since the frequencies of interest are sufficiently high. Since we are dealing with small deviations from equilibrium, the equations can be linearized, that is, the products of two nonequilibrium quantities, which are considered to be of second order, can be neglected.

3.1 - Perturbation charge density and current density

To describe small deviations from equilibrium we express the electron distribution function in the form

$$f(\underset{\sim}{r},\underset{\sim}{v},t) = f_0(v) + f_1(\underset{\sim}{r},\underset{\sim}{v},t) \qquad (|f_1| \ll f_0) \tag{3.1}$$

where $f_0(v)$ is the equilibrium distribution function, considered to be homogeneous and isotropic, and $f_1(\underset{\sim}{r},\underset{\sim}{v},t)$ is a perturbation in the distribution function, always small compared to $f_0(v)$. Before the application of the perturbation the plasma is in equilibrium, so that the macroscopic self-consistent electric and magnetic fields as well as the charge and current densities vanish throughout the plasma. The equilibrium number density of the electrons is everywhere the same as that of the ions, and is given by

$$n_0 = \int_V f_0(v)\, d^3v \tag{3.2}$$

Since $f_1(\underset{\sim}{r},\underset{\sim}{v},t)$ is a first order quantity, the internal electric and magnetic fields that arise due to the perturbation are also first order quantities. From (2.6) the perturbation charge density is given by

$$\rho_c(\underset{\sim}{r},t) = en_0 - e\int_V f(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \tag{3.3}$$

Using (3.1) and (3.2), we obtain

$$\rho_c(\underset{\sim}{r},t) = -e\int_V f_1(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \tag{3.4}$$

The perturbation current density is obtained from (2.7), noting that the ions are assumed to stay immobile,

$$\underset{\sim}{J}(\underset{\sim}{r},t) = -e\int_V \underset{\sim}{v} f(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \tag{3.5}$$

Substituting (3.1) into (3.5), and considering that the current density in the equilibrium state vanishes, that is,

$$- e \int_{v} \underset{\sim}{v} f_0(v)\, d^3v = 0 \tag{3.6}$$

we find

$$\underset{\sim}{J}(\underset{\sim}{r},t) = - e \int_{v} \underset{\sim}{v} f_1(\underset{\sim}{r},\underset{\sim}{v},t)\, d^3v \tag{3.7}$$

3.2 - Solution of the linearized Vlasov equation

Substituting (3.1) into the Vlasov equation (2.1), without any external fields present, we obtain

$$\frac{\partial f_1(\underset{\sim}{r},\underset{\sim}{v},t)}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_1(\underset{\sim}{r},\underset{\sim}{v},t) - \frac{e}{m_e} [\underset{\sim}{E}(\underset{\sim}{r},t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)] \cdot \underset{\sim}{\nabla}_v f_0(v) -$$

$$- \frac{e}{m_e} [\underset{\sim}{E}(\underset{\sim}{r},t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)] \cdot \underset{\sim}{\nabla}_v f_1(\underset{\sim}{r},\underset{\sim}{v},t) = 0 \tag{3.8}$$

Since $\underset{\sim}{E}(\underset{\sim}{r},t)$, $\underset{\sim}{B}(\underset{\sim}{r},t)$ and $f_1(\underset{\sim}{r},\underset{\sim}{v},t)$ are first order quantities, the last term in the left-hand side of (3.8) involves the product of two first order quantities and therefore it is of second order and can be neglected as compared to the remaining terms. Thus, the *linearized* Vlasov equation becomes

$$\frac{\partial f_1(\underset{\sim}{r},\underset{\sim}{v},t)}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f_1(\underset{\sim}{r},\underset{\sim}{v},t) - \frac{e}{m_e} [\underset{\sim}{E}(\underset{\sim}{r},t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)] \cdot \underset{\sim}{\nabla}_v f_0(v) = 0 \tag{3.9}$$

A convenient way to solve this equation is to use the method of integral transforms. For an initial-value problem the equation is simplified by taking its Laplace transform in the time domain and the Fourier transform with respect to the space variables. This method reduces the differential equation to an algebraic equation which can then be solved for the desired transform variables. Next, in order to regain the original variables, we have to invert the Laplace and Fourier transforms of the dependent variables. This mathematical treatment, however, involves the calculation of some complicated contour integrals in the complex plane, which is out of the scope of this text. Therefore, in order to simplify the mathematical analysis of the problem, without losing the essentials of the plasma behavior under consideration, we shall look for periodic harmonic solutions of $f_1(\underset{\sim}{r},\underset{\sim}{v},t)$ in the space and time variables, according to

$$f_1(\underset{\sim}{r},\underset{\sim}{v},t) = f_1(\underset{\sim}{v}) \exp(i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t) \tag{3.10}$$

where the vectors involved are referred to a Cartesian coordinate system. With this special choice for $f_1(\underset{\sim}{r},\underset{\sim}{v},t)$, (3.4) and (3.7) become

$$\rho_c(\underset{\sim}{r},t) = \rho_c \exp(i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t) \tag{3.11}$$

$$\underset{\sim}{J}(\underset{\sim}{r},t) = \underset{\sim}{J} \exp(i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t) \tag{3.12}$$

where

$$\rho_c = - e \int_v f_1(\underset{\sim}{v})\, d^3v \tag{3.13}$$

$$\underset{\sim}{J} = - e \int_v \underset{\sim}{v}\, f_1(\underset{\sim}{v})\, d^3v \tag{3.14}$$

Consequently, the macroscopic self-consistent electric and magnetic fields have the same harmonic time and space dependence,

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \underset{\sim}{E} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \tag{3.15}$$

$$\underset{\sim}{B}(\underset{\sim}{r},t) = \underset{\sim}{B} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t) \tag{3.16}$$

Furthermore, sice we are assuming that the equilibrium distribution function, $f_0(v)$, is a function of only the magnitude of $\underset{\sim}{v}$, we have the very useful identity

$$\underset{\sim}{\nabla}_v f_0(v) = \frac{\underset{\sim}{v}}{v}\frac{df_0(v)}{dv} \tag{3.17}$$

so that, for the term involving the magnetic force in (3.9), we have

$$[\underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)]\cdot\underset{\sim}{\nabla}_v f_0(v) = [\underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r},t)]\cdot\frac{\underset{\sim}{v}}{v}\frac{df_0(v)}{dv} = 0 \tag{3.18}$$

Substituting (3.10), (3.15), (3.16) and (3.18) into the linearized Vlasov equation (3.9), we get

$$- i\omega f_1(\underset{\sim}{v}) + i\underset{\sim}{k}\cdot\underset{\sim}{v} f_1(\underset{\sim}{v}) - \frac{e}{m_e}\underset{\sim}{E}\cdot\underset{\sim}{\nabla}_v f_0(v) = 0 \tag{3.19}$$

whose solution is

$$f_1(\underset{\sim}{v}) = \frac{ie}{m_e}\frac{\underset{\sim}{E}\cdot\underset{\sim}{\nabla}_v f_0(v)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})} \tag{3.20}$$

For definiteness we shall consider the direction of propagation of the plane waves as being the x direction, that is, $\underset{\sim}{k} = k\,\hat{\underset{\sim}{x}}$. Therefore, $\underset{\sim}{k}\cdot\underset{\sim}{v} = kv_x$ and (3.20) becomes

$$f_1(\underset{\sim}{v}) = \frac{ie}{m_e}\frac{\underset{\sim}{E}\cdot\underset{\sim}{\nabla}_v f_0(v)}{(\omega - kv_x)} \tag{3.21}$$

With this orientation chosen for the coordinate system, the longitudinal component of the wave electric field is $\underset{\sim}{E}_\ell = E_x\hat{\underset{\sim}{x}}$, whereas the transverse component is $\underset{\sim}{E}_t = E_y \hat{\underset{\sim}{y}} + E_z \hat{\underset{\sim}{z}}$, as illustrated in Fig. 1.

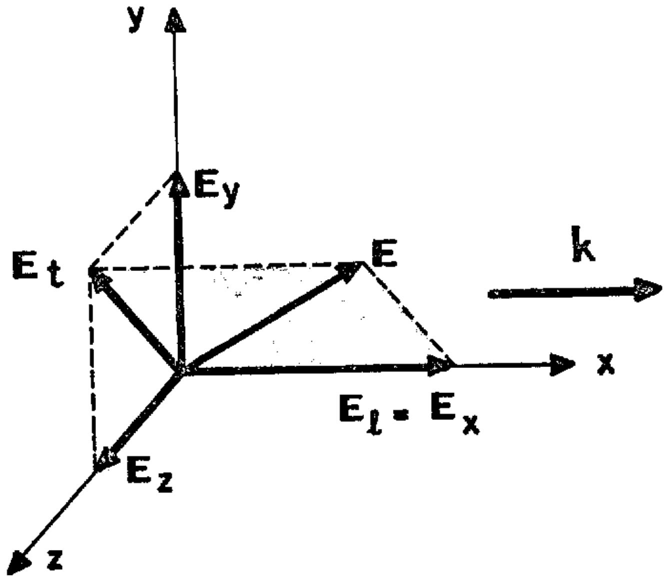

Fig. 1 - Illustrating the relative orientations of the wave propagation vector $\underset{\sim}{k}$ and the wave electric field $\underset{\sim}{E}$ in a Cartesian coordinate system.

3.3 - Expression for the current density

Next we derive expressions for the Cartesian components of the charge current density, $\underset{\sim}{J}$. Substituting (3.21) into (3.14), we obtain

$$\underset{\sim}{J} = - \frac{ie^2}{m_e} \int_v \frac{\underset{\sim}{v}\,[\underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v f_0(v)]}{(\omega - kv_x)} \, d^3v \tag{3.22}$$

Note that the x component of this equation is given by

$$J_x = - \frac{ie^2}{m_e} \int_v \frac{v_x\,[\underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v f_0(v)]}{(\omega - kv_x)} \, d^3v \tag{3.23}$$

where the triple integral with respect to the three variables v_x, v_y and v_z range from $-\infty$ to $+\infty$. Using the identity (3.17), we note that

$$\int_v \frac{v_x}{(\omega - kv_x)} \frac{E_j v_j}{v} \frac{df_0(v)}{dv} d^3v = 0 \quad \text{(for j=y,z)} \tag{3.24}$$

since the integrand is an odd function of v_j, for $j = y,z$. Consequently, the only contribution from $\underset{\sim}{E} \cdot \underset{\sim}{\nabla}_v f_0(v)$ to the x-component of $\underset{\sim}{J}$ comes from the term $E_x \partial f_0(v)/\partial v_x$, so that (3.23) can be written as

$$J_x = - \frac{ie^2}{m_e} E_x \int_v \frac{v_x}{(\omega - kv_x)} \frac{\partial f_0(v)}{\partial v_x} d^3v \tag{3.25}$$

Similarly, the y and z components of (3.22) are found to be given by

$$J_y = - \frac{ie^2}{m_e} E_y \int_v \frac{v_y}{(\omega - kv_x)} \frac{\partial f_0(v)}{\partial v_y} d^3v \tag{3.26}$$

$$J_z = - \frac{ie^2}{m_e} E_z \int_v \frac{v_z}{(\omega - kv_x)} \frac{\partial f_0(v)}{\partial v_z} d^3v \tag{3.27}$$

Note that J_x, J_y and J_z are linearly related to E_x, E_y and E_z, respectively, a feature which is a consequence of the plasma isotropy, as expected in the absence of an external magnetic field.

3.4 - Separation into the various modes

To complete the specification of the problem we use the two Maxwell curl equations (2.4) and (2.5), which for the fields given by (3.15) and (3.16) reduce to

$$i k \hat{\underset{\sim}{x}} \times \underset{\sim}{E} = i\omega \underset{\sim}{B} \tag{3.28}$$

$$i k \hat{\underset{\sim}{x}} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} - \frac{i\omega}{c^2} \underset{\sim}{E} \tag{3.29}$$

In Cartesian coordinates, $\hat{\underset{\sim}{x}} \times \underset{\sim}{E} = E_y \hat{\underset{\sim}{z}} - E_z \hat{\underset{\sim}{y}}$, so that the components of the vector equations (3.28) and (3.29) become, respectively,

$$\omega B_x = 0 \tag{3.30}$$

$$\omega B_y = - kE_z \tag{3.31}$$

$$\omega B_z = kE_y \tag{3.32}$$

and

$$0 = \mu_0 J_x - \frac{i\omega}{c^2} E_x \tag{3.33}$$

$$- i k B_z = \mu_0 J_y - \frac{i\omega}{c^2} E_y \tag{3.34}$$

$$i k B_y = \mu_0 J_z - \frac{i\omega}{c^2} E_z \tag{3.35}$$

where the components of $\underset{\sim}{J}$ are given by (3.25) to (3.27).

An examination of these equations shows that the electromagnetic fields can be separated into *four independent groups*, each one of them involving the following variables:

(a) J_x, E_x [Eq. (3.33)]

(b) B_x [Eq. (3.30)]

(c) J_y, E_y, B_z [Eqs. (3.32) and (3.34)]

(d) J_z, E_z, B_y [Eqs. (3.31) and (3.35)]

The first group contains an electric field and a current density in the direction of the propagation coefficient $\underset{\sim}{k}$, that is, parallel to the wave normal of the initial plane wave disturbance produced in the plasma, but contains no magnetic field. This group gives the *longitudinal plasma wave* mode, since the average particle velocity is also in the direction of $\underset{\sim}{k}$. The second group does not constitute a natural wave mode, since it has no current associated with it and therefore is not influenced by the collective electron motion. It only indicates that there is no magnetic field associated with the longitudinal plasma wave so that these waves are electrostatic in character. The third and fourth groups involve electric and magnetic fields which are perpendicular to $\underset{\sim}{k}$. The electric current density and therefore the average particle velocity are also perpendicular to the wave normal direction. Note that $\underset{\sim}{E}$, $\underset{\sim}{B}$ and $\underset{\sim}{k}$ form a mutually perpendicular triad. These two groups constitute the two different polarizations of the *transverse electromagnetic wave* mode. In the next section we discuss the characteristics of the longitudinal plasma wave. The characteristics of the transverse electromagnetic wave are discussed in section 5.

4. ELECTROSTATIC LONGITUDINAL WAVE IN A HOT ISOTROPIC PLASMA

4.1 - Development of the dispersion relation

The intrinsic behavior of the longitudinal plasma wave is contained in the dispersion relation. This equation, which relates the variables k and ω, determines the natural wave modes of the system. To obtain the dispersion relation for the longitudinal plasma wave we use (3.33) with J_x as given by (3.25),

$$E_x = \frac{\omega_{pe}^2}{n_0\omega} E_x \int_v \frac{v_x}{(kv_x - \omega)} \frac{\partial f_0(v)}{\partial v_x} d^3v \tag{4.1}$$

Dividing this equation by $E_x \neq 0$, yields the *dispersion relation* for the *longitudinal plasma wave*

$$1 = \frac{\omega_{pe}^2}{n_0\omega} \int_v \frac{v_x}{(kv_x - \omega)} \frac{\partial f_0(v)}{\partial v_x} d^3v \tag{4.2}$$

It is convenient to simplify Eq. (4.2) by noting that

$$\int_v \frac{v_x}{(kv_x-\omega)} \frac{\partial f_0(v)}{\partial v_x} d^3v = \frac{1}{k} \int_v \frac{\partial f_0(v)}{\partial v_x} \left(\frac{\omega}{kv_x-\omega} +1 \right) d^3v = \frac{\omega}{k} \int_v \frac{\partial f_0(v)/\partial v_x}{(kv_x-\omega)} d^3v \tag{4.3}$$

since

$$\int_v \frac{\partial f_0(v)}{\partial v_x} d^3v = \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z [f_0(v)] \Bigg|_{v_x = -\infty}^{v_x = +\infty} = 0 \tag{4.4}$$

because $f_0(v)$ vanishes at both limits. Therefore, the dispersion relation (4.2) becomes

$$1 = \frac{\omega_{pe}^2}{n_0 k^2} \int_v \frac{\partial f_0(v)/\partial v_x}{(v_x - \omega/k)} d^3v \tag{4.5}$$

A useful alternative form of this dispersion relation can be obtained by an integration by parts in the v_x variable. Thus, using the relation

$$\int_a^b U\,dV = UV\Big|_a^b - \int_a^b V\,dU \tag{4.6}$$

for the integration with respect to v_x in Eq. (4.5), where

$$\begin{aligned} U &= (v_x - \omega/k)^{-1} & dU &= -(v_x - \omega/k)^{-2}\,dv_x \\ V &= f_0(v) & dV &= \frac{\partial f_0(v)}{\partial v_x}\,dv_x \end{aligned} \tag{4.7}$$

the triple integral in (4.5) becomes

$$\iiint_{-\infty}^{+\infty} \frac{\partial f_0(v)/\partial v_x}{(v_x - \omega/k)}\,dv_x\,dv_y\,dv_z = \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \left[\frac{f_0(v)}{(v_x - \omega/k)}\Bigg|_{v_x=-\infty}^{v_x=+\infty} + \int_{-\infty}^{+\infty} \frac{f_0(v)}{(v_x - \omega/k)^2}\,dv_x \right]$$

$$= \int_v \frac{f_0(v)}{(v_x - \omega/k)^2}\,d^3v \tag{4.8}$$

Therefore, the dispersion relation (4.5) can also be written as

$$\begin{aligned} 1 &= \frac{\omega_{pe}^2}{n_0 k^2} \int_v \frac{f_0(v)}{(v_x - \omega/k)^2}\,d^3v \\ &= \frac{\omega_{pe}^2}{k^2} \left\langle (v_x - \omega/k)^{-2} \right\rangle_0 \end{aligned} \tag{4.9}$$

where the average value with the subscript 0 is calculated using the equilibrium distribution function f_0.

4.2 - Limiting case of a cold plasma

Before proceeding further with the analysis of the dispersion relation (4.9), it is instructive to examine the results for the limiting case of a cold plasma, for which the electron velocity distribution, under equilibrium conditions and at rest, is given by

$$f_0(v) = n_0 \, \delta(v_x) \, \delta(v_y) \, \delta(v_z) \tag{4.10}$$

where $\delta(x)$ is the Dirac delta function, defined by

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0; \qquad \int_{-\infty}^{+\infty} \delta(x) \, dx = 1 \tag{4.11}$$

Substituting (4.10) into the dispersion relation (4.9) and using the following property of the Dirac delta function

$$\int_{-\infty}^{+\infty} f(x) \, \delta(x - x_0) \, dx = f(x_0) \tag{4.12}$$

we obtain

$$1 = \frac{\omega_{pe}^2}{k^2} \int_v \frac{\delta(v_x) \, \delta(v_y) \, \delta(v_z)}{(v_x - \omega/k)^2} \, d^3v \tag{4.13}$$

or

$$\omega^2 = \omega_{pe}^2 \tag{4.14}$$

in agreement with the cold plasma result (section 4 of Chapter 16).

4.3 - High phase velocity limit

Another important result can be immediately obtained from the dispersion relation (4.9), for the limiting case in which the wave phase velocity, ω/k, is very large compared to the velocity of almost all of the electrons. In this high phase velocity limit it is reasonable to expand $(1 - kv_x/\omega)^{-2}$ into a binomial series and retain only the first few terms, since $kv_x/\omega \ll 1$. Thus, recalling that for any $|\varepsilon| < 1$ we have

$$\frac{1}{(1-\varepsilon)^2} = 1 + 2\varepsilon + 3\varepsilon^2 + 4\varepsilon^3 + \dots \qquad (4.15)$$

the dispersion relation (4.9) becomes (for $|v_x| \ll |\omega/k|$),

$$1 = \frac{\omega_{pe}^2}{\omega^2} \left\langle \left(1 - \frac{kv_x}{\omega}\right)^{-2} \right\rangle_0$$

$$= \frac{\omega_{pe}^2}{\omega^2} \left\{ 1 + 2\frac{k}{\omega} \langle v_x \rangle_0 + 3\frac{k^2}{\omega^2} \langle v_x^2 \rangle_0 + \dots \right\} \qquad (4.16)$$

Since the plasma is considered to be stationary, we have $\langle v_x \rangle_0 = u_x = 0$, so that the second term in the right-hand side of (4.16) vanishes. To a first degree of aproximation we obtain $\omega^2 = \omega_{pe}^2$, which is again the cold plasma result given in (4.14). For a small correction to the cold plasma result, we consider the next non-zero term in the expansion (4.16). Assuming that the equilibrium distribution function is isotropic and using the definition of absolute temperature,

$$\langle v_x^2 \rangle_0 = \langle c_x^2 \rangle_0 = \frac{1}{3} \langle c^2 \rangle_0 = \frac{k_B T_e}{m_e} \qquad (4.17)$$

where T_e is the temperature of the electron gas at equilibrium and k_B is Boltzmann's constant, the dispersion relation (4.16) becomes

$$\omega^2 = \omega_{pe}^2 \left(1 + 3 \frac{k_B T_e k^2}{m_e \omega^2} + \ldots \right) \tag{4.18}$$

Since the second term in the right-hand side of (4.18) is very small in the high phase velocity limit, we can replace ω, in just this small term, by ω_{pe} (which is the value of ω when this term is zero) and write (4.18) as

$$\omega^2 = \omega_{pe}^2 + 3 \left(\frac{k_B T_e}{m_e} \right) k^2 \tag{4.19}$$

This results is known as the *Bohm-Gross dispersion relation*. Note that it is identical to the result obtained using the warm plasma model when collisions are neglected and when the ratio of specific heats, γ, is taken equal to 3. Since γ is related to the number of degrees of freedom, N, by the relation

$$\gamma = (2 + N)/N \tag{4.20}$$

we see that $\gamma = 3$ corresponds to the case when the electron gas has one degree of freedom (N = 1), so that the electrons move only in the direction of wave propagation.

If additional terms are retained in the binomial series expansion (4.16), additional aproximations can be obtained for the dispersion relation $\omega(k)$. In all these approximations we find that ω remains real, so that the longitudinal plasma wave has a constant amplitude in time. There is neither temporal growth nor decay. It is usual to terminate the approximations to $\omega(k)$ at the stage given by (4.19). Using the definition of the Debye length, λ_D, the Bohm-Gross dispersion relation can be rewritten as

$$\omega^2 = \omega_{pe}^2(1 + 3\,k^2\lambda_D^2) \tag{4.21}$$

4.4 - Dispersion relation for Maxwellian distribution function

The longitudinal wave dispersion relation (4.5) is now evaluated for the important case when $f_0(v)$ is the Maxwellian distribution function for a stationary equilibrium plasma ($\underset{\sim}{u} = 0$),

$$f_0(v) = n_0 \left(\frac{m_e}{2\,k_B T_e} \right)^{3/2} \exp\left(- \frac{m_e v^2}{2k_B T_e} \right) \tag{4.22}$$

In this case, a careful analysis of (4.5) shows that ω has a negative imaginary part, causing a temporal damping of the electron plasma wave. This temporal damping, which arises in the *absence* of collisions, is known as *Landau damping* and will be discussed in the next sub-section.

For the moment, we evaluate the dispersion relation for the longitudinal electron wave using the Maxwell-Boltzmann equilibrium distribution function. Substituting (4.22) into (4.5) yields

$$1 = \frac{\omega_{pe}^2}{n_0 k^2} \int_V \frac{\partial f_0(v)/\partial v_x}{(v_x - \omega/k)}\, d^3v$$

$$= - \frac{\omega_{pe}^2}{n_0 k^2} \int_V \frac{(m_e/k_B T_e)\, v_x\, f_0(v)}{(v_x - \omega/k)}\, d^3v$$

$$= - \frac{\omega_{pe}^2}{k^2} \left(\frac{m_e}{k_B T_e} \right) \left(\frac{m_e}{2\pi k_B T_e} \right)^{3/2} \int_{-\infty}^{+\infty} \frac{v_x}{(v_x - \omega/k)} \exp \left(- \frac{m_e v_x^2}{2k_B T_e} \right) dv_x \;\cdot$$

$$\cdot \int_{-\infty}^{+\infty} \exp \left(- \frac{m_e v_y^2}{2k_B T_e} \right) dv_y \int_{-\infty}^{+\infty} \exp \left(- \frac{m_e v_z^2}{2k_B T_e} \right) dv_z \qquad (4.23)$$

The second and third integrals are each equal to $(2\pi k_B T_e/m_e)^{1/2}$. It is convenient to introduce the following dimensionless parameters

$$C = \frac{(\omega/k)}{(2k_B T_e/m_e)^{1/2}} \qquad (4.24)$$

$$q = \frac{v_x}{(2k_B T_e/m_e)^{1/2}} \qquad (4.25)$$

so that the dispersion relation (4.23) reduces to

$$1 = - \frac{\omega_{pe}^2}{k^2} \left(\frac{m_e}{k_B T_e} \right) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{q \exp(- q^2)}{(q - C)} \, dq \qquad (4.26)$$

Using the notation

$$I(C) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{q \exp(-q^2)}{(q - C)} \, dq \qquad (4.27)$$

and substituting $(k_B T_e/m_e)/\omega_{pe}^2$ by λ_D^2, (4.26) becomes

$$k^2 \lambda_D^2 + I(C) = 0 \qquad (4.28)$$

The evaluation of the integral $I(C)$ is not straightforward because of the singularity at $q = C$, since for real $\omega(k)$ the denominator vanishes on the real v_x axis. For complex $\omega(k)$, which corresponds to damped [$Im(\omega) < 0$] or unstable [$Im(\omega) > 0$] waves, the singularity lies off the path of integration along the real v_x axis. However, this simplified derivation of the dispersion relation gives no indication of the proper integration contour to be chosen in the complex v_x plane. Possible contours of integration are shown in Fig. 2 for the cases: (a) unstable wave, with $Im(\omega) > 0$; (b) real $\omega(k)$; and (c) damped wave, with $Im(\omega) < 0$. Landau was the first to treat this problem properly as an *initial-value* problem. If we are interested in the evalution of the plasma after an initial perturbation, then the causality principle demands that there should be no fields before the starting of the source. According to the well known theorem of residues in complex variables, the value of an integral in the complex domain with a closed contour of integration, such as in Fig. 2, is equal to $2\pi i$ times the sum of the residues within the closed path. The integral vanishes if there are no singularities enclosed by the integration path. Thus, the nature of the singularities of the integrand determines the behavior of the fields after the initial pertubation. The correct contour prescribed by Landau is along the real v_x axis, indented such as to pass below the singularities, and closed by an infinite semicircular path in the upper half of the complex v_x plane, as shown in Fig. 2.

This technique of integration around a contour closed by an infinite semicircle in the upper half plane works if the contribution to the integral from the semicircular path vanishes as its radius goes to infinity. The integral I(C) given in (4.27), the way it stands, cannot be handled by the usual method of residues, since the integrand diverges for $q = \pm i \infty$. To put this integral in a form suitable for evaluation by the method of residues, or by any other method, note first that we can write

$$\frac{q}{(q - C)} = 1 + \frac{C}{(q - C)} \tag{4.29}$$

so that we have

$$I(C) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left(1 + \frac{C}{q - C} \right) \exp(-q^2) \, dq \tag{4.30}$$

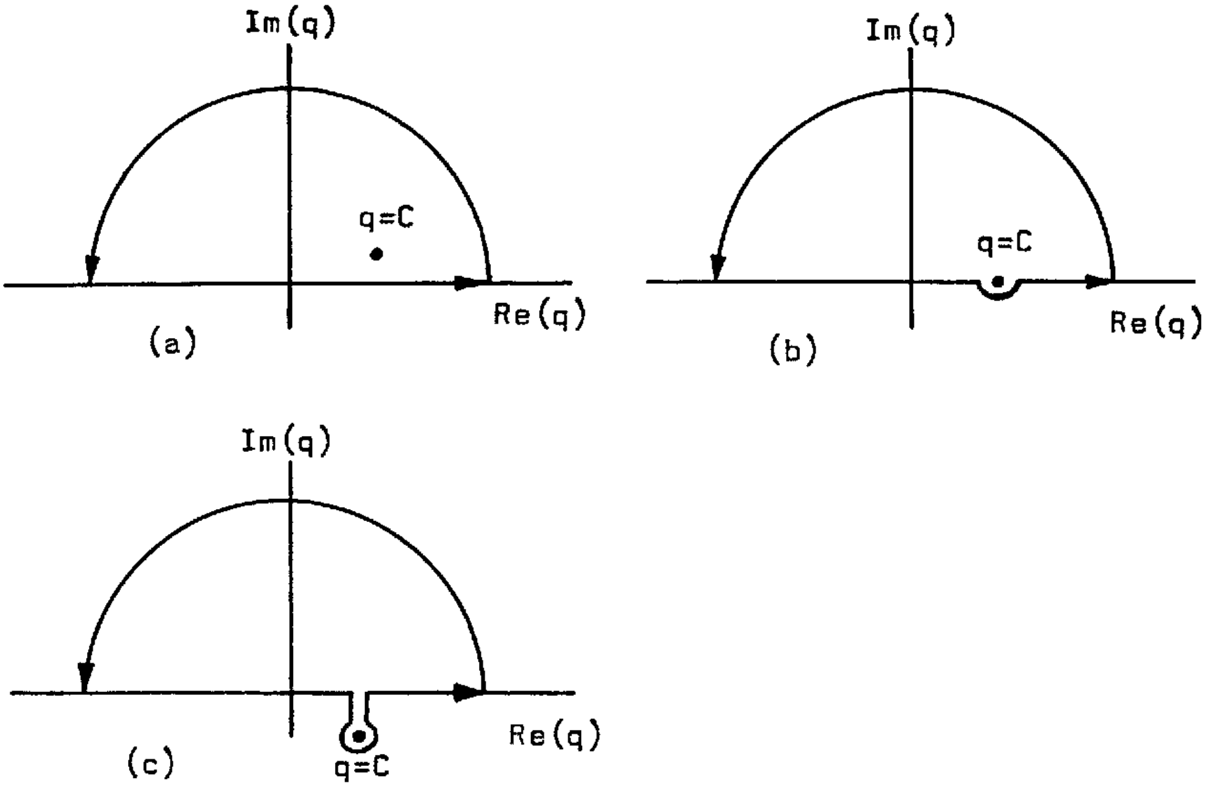

Fig. 2 - Contour of integration in the complex v_x plane for (a) $\mathrm{Im}(\omega) > 0$, (b) $\mathrm{Im}(\omega) = 0$ and (c) $\mathrm{Im}(\omega) < 0$.

The first integral in the right-hand side of this equation is equal to unity. Therefore,

$$I(C) = 1 + \frac{C}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-q^2)}{(q - C)} dq \tag{4.31}$$

For purposes of integration it is convenient to introduce a parameter s in the integral of Eq. (4.31), by defining

$$G(C,s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-s\,q^2)}{(q - C)} dq \tag{4.32}$$

Hence, we identify the integral I(C) as

$$I(C) = 1 + C\,G(C,1) \tag{4.33}$$

so that the dispersion relation (4.28) becomes

$$k^2\,\lambda_D^2 + 1 + C\,G(C,1) = 0 \tag{4.34}$$

The purpose of defining G(C,s), as in (4.32), is that this relation allows us to evaluate G(C,1) through a transformation of the integral into a differential equation. Initially, note that the integral in (4.32) can also be written as

$$G(C,s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{(q + C)}{(q^2 - C^2)} \exp(-s\,q^2)\, dq \tag{4.35}$$

The first integral in the right-hand side of this equation vanishes, since the integrand is an odd function of q. Therefore, an alternative expression for G(C,s) is

$$G(C,s) = \frac{C}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-s\,q^2)}{(q^2 - C^2)}\, dq \tag{4.36}$$

Taking the derivate of (4.36) with respect to s, yields

$$\frac{d\,G(C,s)}{ds} = - \frac{C}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{q^2 \exp(-s\,q^2)}{(q^2 - C^2)}\, dq$$

$$= - \frac{C}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left[1 + \frac{C^2}{(q^2 - C^2)} \right] \exp(-s\,q^2)\, dq \tag{4.37}$$

Evaluating the first integral we obtain $-C/\sqrt{s}$ so that

$$\frac{d\,G(C,s)}{ds} = - \frac{C}{\sqrt{s}} - C^2\, G(C,s) \tag{4.38}$$

Next, multiply this differential equation by $\exp(s\,C^2)$ and note that

$$\frac{d}{ds} \left[G(C,s) \exp(s\,C^2) \right] = \exp(s\,C^2) \left[\frac{d\,G(C,s)}{ds} + C^2\, G(C,s) \right] \tag{4.39}$$

Thus, it is possible to write Eq. (4.38) in the form

$$\frac{d}{ds} \left[G(C,s) \exp(s\,C^2) \right] = - \frac{C}{\sqrt{s}} \exp(s\,C^2) \tag{4.40}$$

Upon integrating both sides of this equation from s = 0 to s = 1, gives

$$G(C,1) \exp(C^2) - G(C,0) = - C \int_0^1 \frac{\exp(s\,C^2)}{\sqrt{s}}\, ds \tag{4.41}$$

or, rearranging,

$$G(C,1) = G(C,0)\,\exp(-C^2) - C\,\exp(-C^2)\int_0^1 \frac{\exp(s\,C^2)}{\sqrt{s}}\,ds \qquad (4.42)$$

The integral G(C,0) is easily evaluated for the case of weak damping (large phase velocity). In this case, the pole at $v_x = \omega/k$ lies near the real v_x axis, and G(C,0) can be evaluated as an improper integral as follows:

$$G(C,0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \frac{dq}{(q - C)}$$

$$= \lim_{X \to \infty}\left[\frac{1}{\sqrt{\pi}}\int_{-X}^{+X} \frac{dq}{(q - C)}\right]$$

$$= \lim_{X \to \infty}\left[\frac{1}{\sqrt{\pi}}\,\ln\left(-\frac{X - C}{X + C}\right)\right]$$

$$= \frac{1}{\sqrt{\pi}}\,\ln(-1) = \frac{1}{\sqrt{\pi}}\,\ln(e^{i\pi})$$

$$= i\,\sqrt{\pi} \qquad (4.43)$$

The integral C(C,0) can also be evaluated by the method of residues, using an appropriate contour of integration in the complex q-plane [as shown in Fig.2(b)], which gives the same result (4.43) for the *Cauchy principal value* of the integral. Therefore, (4.42) becomes

$$G(C,1) = i\,\sqrt{\pi}\,\exp(-C^2) - C\,\exp(-C^2)\int_0^1 \frac{\exp(s\,C^2)}{\sqrt{s}}\,ds \qquad (4.44)$$

The remaining integral in the right-hand side of (4.44) can be rewritten in a different form by changing the variable s to W^2/C^2. Consequently, $ds/\sqrt{s} = 2\ dW/C$ and

$$G(C,1) = i\ \sqrt{\pi}\ \exp(-\ C^2) - 2 \int_0^C \exp(W^2 - C^2)\ dW \qquad (4.45)$$

Although this integral cannot be evaluated explicitly, it is now in a more convenient form for numerical calculation.

Substituting (4.45) into (4.34), results in the following expression for the *dispersion relation*

$$-\ k^2\ \lambda_D^2 = 1 + i\ \sqrt{\pi}\ C \exp(-C^2)\ - 2C \int_0^C \exp(W^2 - C^2)\ dW \qquad (4.46)$$

The integral remaining here can be evaluated numerically and its values have been extensively tabulated*, while the imaginary term is known as the *Landau damping* term. The formal procedure to evaluate k as a function of ω (or vice versa) from this dispersion relation, consists in choosing a given value of C i.e. of $(\omega/k)/(2\ k_B\ T_e/m_e)^{1/2}$ and find

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* *Remark. See, for example B.D. Fried and S.D. Conte, "The Plasma Dispersion Function", Acad. Press, N.Y., 1961. The function G(C,1), defined here, is the same as the plasma dispersion function defined by Fried & Conte as*

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-\ x^2)}{(x - \zeta)}\ dx$$

with $\zeta \equiv C$ *and* $x \equiv q$.

the (tabulated) corresponding value of the dispersion function from the tables. Eq. (4.46) can then be used to evaluate the propagation coefficient k.

4.5 - Landau damping

In order to show that (4.46) predicts the temporal damping of the longitudinal plasma wave, it is convenient to perform an approximate evaluation of the dispersion relation. The special case of high phase velocity and weak damping can be obtained in a straightforward way and, at the same time, provides a partial check on the accuracy of the Bohm-Gross dispersion relation obtained earlier. Furthermore, an explicit expression is obtained for the imaginary part of ω. Thus, for the limiting case of $C \gg 1$, lets us find an approximate expression for the dispersion function integral

$$I_1 = 2C \int_0^C \exp(W^2 - C^2)\, dW \tag{4.47}$$

As the first step, Eq. (4.47) can be rewritten by transforming the variable of integration to $\xi = C^2 - W^2$, which gives

$$I_1 = \int_0^{C^2} \left(1 - \frac{\xi}{C^2} \right)^{-1/2} \exp(-\xi)\, d\xi \tag{4.48}$$

Since ξ is less than C^2 over the entire range of integration, we can expand $(1 - \xi/C^2)^{-1/2}$ in a binomial series,

$$\left(1 - \frac{\xi}{C^2} \right)^{-1/2} = 1 + \frac{\xi}{2C^2} + \frac{3\xi^2}{8C^4} + \ldots +$$

$$+ \frac{1 \times 3 \times \ldots \times (2n - 1)}{2^n \; n!} \left(\frac{\xi}{C^2} \right)^n + \ldots \qquad (4.49)$$

If this expansion is substituted into Eq. (4.48), and each term is integrated by noting that

$$\int_0^{C^2} \left(\frac{\xi}{C^2} \right)^n \exp(-\xi) \, d\xi =$$

$$= \frac{n!}{(C^2)^n} - \exp(-C^2) \left[1 + \frac{n}{C^2} + \frac{n(n-1)}{C^4} + \ldots + \frac{n!}{(C^2)^n} \right] \qquad (4.50)$$

we find

$$I_1 = 1 + \frac{1}{2C^2} + \frac{3}{4C^4} + \ldots + \frac{1 \times 3 \times \ldots \times (2n-1)}{(2C^2)^n} + \ldots + O[\exp(-C^2)] \qquad (4.51)$$

where $O[\exp(-C^2)]$ denotes terms of order $\exp(-C^2)$. Although this is an asymptotic expansion, and actually diverges when $n \to \infty$, a good estimate of I_1 can be obtained by retaining only the first few terms, provided C is large. Therefore, on retaining only the first three terms of (4.51), the dispersion relation (4.46) in the high phase velocity limit becomes

$$k^2 \lambda_D^2 = \frac{1}{2C^2} + \frac{3}{4C^4} - i \sqrt{\pi} \; C \exp(-C^2) \qquad (4.52)$$

With the help of Eq. (4.24), which defines C, and the definition of the Debye length λ_D, Eq. (4.52) can be written as

$$\frac{\omega^2}{\omega_{pe}^2} = 1 + 3k^2 \lambda_D^2 \left(\frac{\omega_{pe}}{\omega} \right)^2 -$$

$$- i \frac{\sqrt{\pi/2}}{k^3 \lambda_D^3} \left(\frac{\omega}{\omega_{pe}} \right)^3 \exp \left[- \frac{1}{2k^2 \lambda_D^2} \left(\frac{\omega}{\omega_{pe}} \right)^2 \right] \quad (4.53)$$

In the high phase velocity limit the second term in the right-hand side of (4.53) is small as compared to the first one, and the third term is exponentially small as compared to the first one, so that in this limit the plasma oscillates very close to the plasma frequency ω_{pe}. Note that this limit corresponds also to a long-wavelength limit. Thus, (4.53) can be further approximated as

$$\omega^2 = \omega_{pe}^2 + 3k^2 \left(\frac{k_B T_e}{m_e} \right) -$$

$$- \frac{i\sqrt{\pi/2}\ \omega_{pe}^5}{k^3(k_B T_e/m_e)^{3/2}} \exp \left[- \frac{\omega_{pe}^2}{2k^2 (k_B T_e/m_e)} - \frac{3}{2} \right] \quad (4.54)$$

where in the right-hand side of (4.53) we have replaced ω by ω_{pe}, except in the exponetial term where ω^2 has been replaced by the Bohm-Gross result (4.19). Note that the first two terms in (4.54) correspond to the Bohm-Gross result, whereas the imaginary term is new. Separating ω in its real and imaginary parts, according to $\omega = \omega_r + i\omega_i$, and noting that $\omega_i = (\omega^2)_i/(2\omega_r)$, we obtain (taking $\omega_r \simeq \omega_{pe}$)

$$\omega_i = - \frac{\sqrt{\pi/8}\ \omega_{pe}^4}{k^3(k_B T_e/m_e)^{3/2}} \exp \left[- \frac{\omega_{pe}^2}{2k^2(k_B T_e/m_e)} - \frac{3}{2} \right] \quad (4.55)$$

This *negative* imaginary term in ω leads to *temporal decay*, since for a standing wave problem (where k is real) the waves are proportional to

$$\exp(ikx - i\omega t) = \exp(ikx - i\omega_r t)\exp(\omega_i t) \qquad (4.56)$$

This damping of the longitudinal plasma wave with time was first pointed out by L.D. Landau and, for this reason, the expression (4.55) is usually called the *Landau damping factor*.

This temporal decay of the longitudinal plasma wave amplitude arises in the absence of dissipative mechanisms, such as collisions of the electrons with heavy particles. The physical mechanism responsible for collisionless Landau damping is the *wave - particle interaction* i.e. the interaction of the electrons with the electric field $E\hat{\underset{\sim}{x}}\cos(kx - \omega t)$ of the wave. The electrons that initially have velocities quite close to the phase velocity of the wave are trapped inside the moving potential wells of the wave and this trapping results in a net interchange of energy between the electrons and the wave. For the Maxwell - Boltzmann velocity distribution function we find that, for small k, the phase velocity lies far out on the tail and the damping is negligible, but for values of k close to $1/\lambda_D$ the phase velocity lies within the tail, as shown in Fig. 3, so that there is a velocity band, Δv, around $v = \omega/k$, where there are more electrons in Δv moving initially slower than ω/k, than moving faster than ω/k. Consequently, the trapping of the electrons in the potential troughs of the wave will cause a net increase in the electron energy at the expense of the wave energy. This happens in the region where $\partial f_0/\partial v_x$ is negative, like the one shown in Fig. 3. In some cases, the initial velocity distribution of the electrons may be appropriately chosen in such a way that ω_i becomes positive. This would indicate an unstable situation, with the wave amplitude growing with time. This happens when $\partial f_0/\partial v_x$ is positive at $v_x = \omega/k$.

It is important to note that the Landau damping factor, ω_i, is essentially due to the pole of the integrand in (4.31), which occurs at the value of the electron velocity component v_x(parallel to $\underset{\sim}{k}$) equal to the phase velocity of the wave (ω/k). This property is a

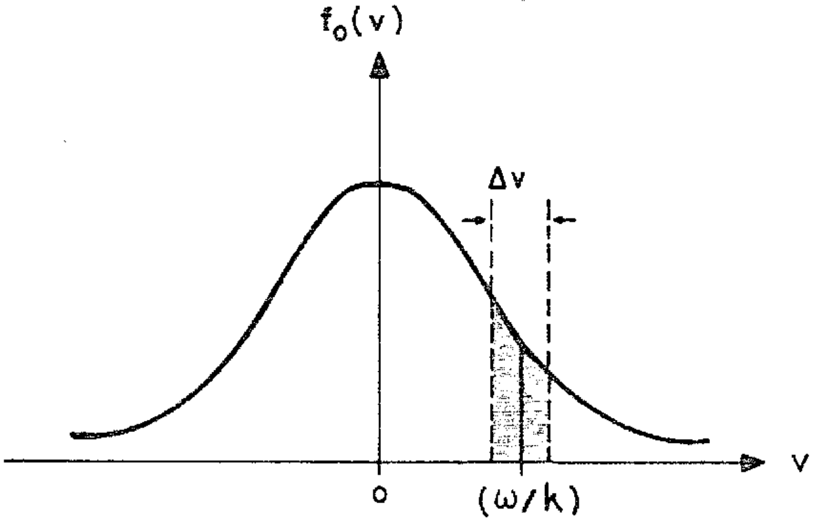

Fig. 3 - Equilibrium distribution function of the electrons showing a velocity band, Δv, around the phase velocity (ω/k), in which there are more electrons moving slower than (ω/k), than moving faster than (ω/k).

mathematical manifestation of the fact that the wave-particle interaction is effective only when the velocity of the electrons are very close to the phase velocity of the wave.

5. TRANSVERSE WAVE IN A HOT ISOTROPIC PLASMA

5.1 - Development of the dispersion relation

The third and fourth independent groups of fields, consisting of J_y, E_y, B_z and J_z, E_z, B_y, respectively, constitute the two different polarizations of the transverse wave mode. In order to deduce the dispersion relation for the transverse electromagnetic wave, let us consider initially Eqs. (3.26), (3.32) and (3.34). Substituting B_z form (3.32) into (3.34), yields

$$E_y = \frac{i\omega}{\varepsilon_0\,(k^2c^2 - \omega^2)}\; J_y \tag{5.1}$$

Combining this equation with (3.26), to eliminate J_y, we obtain

$$E_y = \frac{\omega_{pe}^2\,\omega\,E_y}{n_0(\omega^2 - k^2c^2)} \int_v \frac{v_y}{(kv_x - \omega)}\,\frac{\partial f_0(v)}{\partial v_y}\, d^3v \tag{5.2}$$

In a similar way, combining (3.27), (3.31) and (3.35) we find that the equation for E_z is identical to (5.2). The integral with respect to v_y in (5.2) can be simplified by an integration by parts

$$\int_{-\infty}^{+\infty} v_y \frac{\partial f_0(v)}{\partial v_y}\, dv_y = v_y f_0(v)\Big|_{v_y=-\infty}^{v_y=+\infty} - \int_{-\infty}^{+\infty} f_0(v)\, dv_y \tag{5.3}$$

The first term in the right-hand side of this equation vanishes, since $f_0(v)$ vanishes at $v_y = \pm\infty$. Thus, we obtain from (5.2) the following dispersion relation for the transverse electromagnetic wave

$$k^2c^2 - \omega^2 = \frac{\omega_{pe}^2\,\omega}{n_0 k} \int_v \frac{f_0(v)}{(v_x - \omega/k)}\, d^3v \tag{5.4}$$

5.2 - Cold plasma result

Again, we examine first the limiting case of a cold plasma characterized by the distribution function (4.10). Substituting

(4.10) into (5.4) and using the property (4.12) of the Dirac delta function, we find

$$k^2c^2 = \omega^2 - \omega_{pe}^2 \tag{5.5}$$

This result is identical to the one obtained in Chapter 16 using the cold plasma approximation [see Eq. (16.4.12)].

5.3 - Dispersion relation for Maxwellian distribution function

Considering $f_0(v)$, in (5.4), as the Maxwell-Boltzmann distribution function, we find, after integrating over v_y and v_z,

$$k^2c^2 - \omega^2 = \frac{\omega_{pe}^2 C}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-q^2)\, dq}{(q - C)} \tag{5.6}$$

where, as before, we have introduced the dimensionless parameters C and q, defined in (4.24) and (4.25), respectively. The integral appearing in (5.6) is the same as the integral G(C,s), for s = 1, defined in (4.32), so that we can write the dispersion relation (5.6) as

$$k^2c^2 - \omega^2 = \omega_{pe}^2\, C\, G(C, 1) \tag{5.7}$$

For weak damping we can use (4.45), obtaining

$$k^2c^2 - \omega^2 = \omega_{pe}^2 \left[i\sqrt{\pi}\, C \exp(-C^2) - 2C \int_0^C \exp(W^2 - C^2)\, dW\right] \tag{5.8}$$

5.4 - Landau damping of the transverse wave

In contrast with the Landau damping of the longitudinal plasma wave, the Landau damping of the transverse electromagnetic wave, which is due to the small negative imaginary part of ω in (5.8), is negligibly small. For the purpose of establishing this result, it is convenient to evaluate approximately the dispersion relation (5.8) in the high phase velocity limit. In the limit when C is very large we can use (4.51). To obtain a first approximation to the real part of ω, it is sufficient to retain only the first term in (4.51), so that in the high phase velocity limit (5.8) reduces to

$$k^2c^2 = \omega^2 - \omega_{pe}^2 + i\sqrt{\pi}\ \omega_{pe}^2\ C\exp(-C^2) \tag{5.9}$$

This result is identical to the dispersion relation obtained using the cold plasma model without collisions, except for the Landau damping term.

In the high phase velocity limit ($C \gg 1$) the Landau damping factor is very small and can be omitted in a first approximation, with the result that (5.12) reduces to the cold plasma result (5.5). From (5.5) we see that for $\omega > \omega_{pe}$ the phase velocity ω/k is greater than c (the velocity of electromagnetic waves in free space). Thus, C is of the order of $c/(2k_B\, T_e/m_e)^{1/2}$, and is therefore a very large number. Since C is very large, the Landau damping of the transverse electromagnetic wave is negligible. As a matter of fact it can be argued that, for this case, the Landau damping term is really zero, since the integration over v_x should really extend only from -c to +c, while the phase velocity is always gerater than c. This implies that the pole at $v_x = (\omega/k)$, or equivalently at $q = C$, lies outside the path of integration along the real axis. Therefore, the conditions for efficient wave-particle interaction are not met for the transverse electromagnetic wave throughout the frequency range of propagation (since ω/k is greater than c), resulting in no wave damping. On the other hand, for the longitudinal plasma wave there are frequencies for which the wave phase velocity is of the order of the electron thermal velocities, so that wave-particle interaction can take place efficiently, with the result that the Landau damping factor becomes important for the lower phase velocity longitudinal waves.

6. THE TWO-STREAM INSTABILITY

As an example of a situation in which wave-particle interaction leads to a *growing wave amplitude*, at the expense of the kinetic energy of the plasma particles, we consider in this section the *two-stream instability*. Although the instability arises under a wide range of beam conditions, we shall consider only the simple case of two contrastreaming uniform beams of electrons with the same number density $n_0/2$. The first stream travels in the x-direction with drift velocity $\underset{\sim}{v}_D = v_D\hat{\underset{\sim}{x}}$, and the second stream in the opposite direction with drift velocity $\underset{\sim}{v}_D = -v_D\hat{\underset{\sim}{x}}$. We shall assume that each particle, in each stream, has exactly the stream velocity i.e. the particles are assumed to be cold, so that the electron distribution function can be written in terms of the Dirac delta function as

$$f_0(v) = \frac{1}{2} n_0 [\,\delta(v_x - v_D) + \delta(v_x + v_D)\,]\ \delta(v_y)\ \delta(v_z) \tag{6.1}$$

This distribution function is illustrated in Fig. 4 for the v_x component only.

For longitudinal plasma waves propagating in the x direction ($\underset{\sim}{k} = k\hat{\underset{\sim}{x}}$) in an electron gas, described by the Vlasov equation, the dispersion relation is, from (4.9),

$$1 = \frac{\omega_{pe}^2}{n_0\ k^2} \int_V \frac{f_0(v)}{(v_x - \omega/k)^2}\ d^3v \tag{6.2}$$

Substituting (6.1) into (6.2), yields

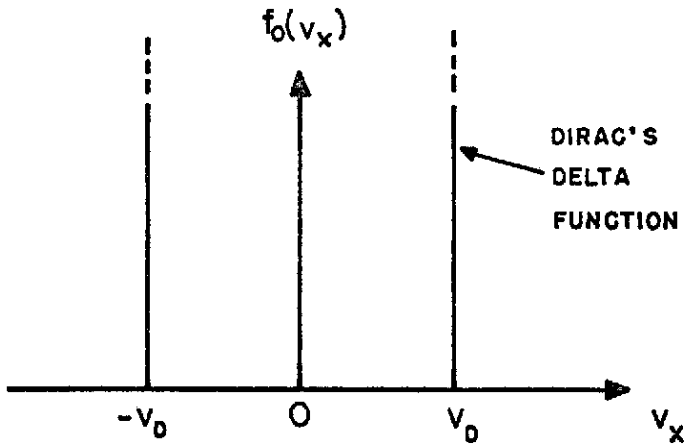

Fig. 4 - Illustrating the v_x component of the distribution function (6.1).

$$1 = \frac{\omega_{pe}^2}{2} \int_{-\infty}^{+\infty} \frac{[\,\delta(v_x - v_D) + \delta(v_x + v_D)\,]}{(kv_x - \omega)^2}\, dv_x \int_{-\infty}^{+\infty} \delta(v_y)\ dv_y \int_{-\infty}^{+\infty} \delta(v_z)\, dv_z \tag{6.3}$$

and integrating over each of the δ functions, we obtain

$$1 = \frac{\omega_{pe}^2}{2} \left[\frac{1}{(kv_D - \omega)^2} + \frac{1}{(kv_D + \omega)^2} \right] \tag{6.4}$$

This is the dispersion relation for longitudinal waves (with the wave normal in the direction of the first electron stream) in a constrastreaming electron plasma characterized by the distribution function (6.1). We assume that the propagation coefficient, k, of the longitudinal plasma wave is real (standing waves), and investigate the existence of temporal growth or damping of the wave amplitude.

Eq. (6.4) can be rearranged in the following polynomial form

$$\omega^4 - B\omega^2 + C = 0 \tag{6.5}$$

where

$$B \equiv \omega_{pe}^2 + 2k^2 v_D^2 \tag{6.6}$$

$$C \equiv k^2 \, v_D^2 \, (k^2 v_D^2 - \omega_{pe}^2) \tag{6.7}$$

Note that B is always positive, whereas C can be either positive or negative, depending on whether $k^2 v_D^2 > \omega_{pe}^2$ or $k^2 v_D^2 < \omega_{pe}^2$, respectively. The polynomial equation (6.5) has two roots for ω^2, which are

$$\omega_1^2 = \frac{B}{2} + \left[\left(\frac{B}{2} \right)^2 - C \right]^{1/2} \tag{6.8}$$

$$\omega_2^2 = \frac{B}{2} - \left[\left(\frac{B}{2} \right)^2 - C \right]^{1/2} \tag{6.9}$$

In what follows it will be shown that an instability can arise only when $k^2 v_D^2 < \omega_{pe}^2$. First we note that for $k^2 v_D^2 > \omega_{pe}^2$ we have $C > 0$, so that both ω_1^2 and ω_2^2 are positive real quantities and therefore there can be no temporal growth or damping of the wave amplitude. On the other hand, for $k^2 v_D^2 < \omega_{pe}^2$ we have $C < 0$, so that ω_1^2 is still a positive real quantity, whereas ω_2^2 is a negative real quantity. Therefore, ω_2 has two imaginary values (one positive and one negative). The positive imaginary value of ω_2 corresponds to an unstable mode, since for $\omega_2 = i\omega_{2i}$ (with ω_{2i} real, positive) we have $\exp(-i\omega t) = \exp(\omega_{2i} t)$. Hence, the growth rate is given by

$$\omega_{2i} = \left\{ -\frac{B}{2} + \left[\left(\frac{B}{2} \right)^2 - C \right]^{1/2} \right\}^{1/2} ; C < 0 \tag{6.10}$$

or, using (6.6) and (6.7)

$$\omega_{2i} = \left\{ -\left(\frac{\omega_{pe}^2}{2} + k^2 v_D^2 \right) + \left[\left(\frac{\omega_{pe}^2}{2} + k^2 v_D^2 \right)^2 - k^2 v_D^2 \left(k^2 v_D^2 - \omega_{pe}^2 \right) \right]^{1/2} \right\}^{1/2} ; k^2 v_D^2 < \omega_{pe}^2 \tag{6.11}$$

The maximum value of the growth rate (6.11) corresponds to the minimum value of ω_2^2 in (6.9), since $\omega_{2i}^2 = -\omega_2^2$. Examining the derivate of ω_2^2 with respect to k, we find that the minimum value of ω_2^2 occurs when $k^2 v_D^2 = (3/8)\,\omega_{pe}^2$, and the corresponding value of ω_2^2 is $-\omega_{pe}^2/8$. Consequently, the maximum value of the growth rate is

$$(\omega_{2i})_{max} = \frac{1}{\sqrt{8}}\,\omega_{pe} \tag{6.12}$$

7. SUMMARY

7.1 - Longitudinal mode

The *dispersion relation* is (for $\underset{\sim}{k} = k\underset{\sim}{\hat{x}}$)

$$1 = \frac{\omega_{pe}^2}{n_0\ \omega} \int_V \frac{v_x}{(kv_x - \omega)} \frac{\partial f_0(v)}{\partial v_x} d^3v \tag{4.2}$$

Alternative forms for this dispersion relation are

$$1 = \frac{\omega_{pe}^2}{n_0\ k^2} \int_V \frac{\partial f_0(v)/\partial v_x}{(v_x - \omega/k)} d^3v \tag{4.5}$$

$$1 = \frac{\omega_{pe}^2}{n_0\ k^2} \int_V \frac{f_0(v)}{(v_x - \omega/k)^2} d^3v \tag{4.9}$$

For $f_0(v)$ as the Maxwell-Boltzmann distribution function,

$$- k^2\lambda_D^2 = 1 + i\ \sqrt{\pi}\ C \exp(-C^2) - 2C \int_0^C \exp(W^2 - C^2)\ dW \tag{4.46}$$

The *cold plasma* limit gives stationary electrostatic oscillations at the plasma frequency,

$$\omega^2 = \omega_{pe}^2 \tag{4.14}$$

The high phase velocity limit gives the *warm plasma* model result (Bohm-Gross dispersion relation) for the electron plasma wave,

$$\omega^2 = \omega_{pe}^2 + 3 \left(\frac{k_B\ T_e}{m_e} \right) k^2 \tag{4.19}$$

The *Landau* (temporal) *damping factor* is (with $\omega = \omega_r + i\omega_i$)

$$\omega_i = - \frac{\sqrt{\pi/8}\ \omega_{pe}^4}{k^3(k_B\ T_e/m_e)^{3/2}} \exp\left[- \frac{\omega_{pe}^2}{2k^2(k_B\ T_e/m_e)} - \frac{3}{2} \right] \tag{4.55}$$

7.2 - Transverse mode

The *dispersion relation* is (for $\underset{\sim}{k} = k\underset{\sim}{\hat{x}}$)

$$k^2c^2 - \omega^2 = \frac{\omega_{pe}^2 \, \omega}{n_0 \, k} \int_v \frac{f_0(v)}{(v_x - \omega/k)} \, d^3v \qquad (5.4)$$

For $f_0(v)$ as the Maxwell-Boltzmann distribution function,

$$k^2c^2 - \omega^2 = \omega_{pe}^2 \left[i \sqrt{\pi} \, C \exp(-C^2) - 2C \int_0^C \exp(W^2 - C^2) dW \right] \qquad (5.8)$$

The *cold* and *warm plasma* limits give

$$k^2c^2 = \omega^2 - \omega_{pe}^2 \qquad (5.5)$$

The high phase velocity limit gives

$$k^2c^2 - \omega^2 = \omega_{pe}^2 + i \sqrt{\pi} \; \omega_{pe}^2 \; C \exp(-C^2) \qquad (5.9)$$

The *Landau damping term* is negligible, since $v_{ph} \geqq c$.

PROBLEMS

18.1 - Since the longitudinal plasma wave is an electrostatic oscillation, it is possible to derive its dispersion relation using Poisson equation, satisfied by the electrostatic potential $\phi(\underset{\sim}{r}, t)$, instead of Maxwell equations. Consider the problem of small amplitude longitudinal waves propagating in the x direction in an electron gas (only electrons move in a background of stationary ions), in the absence of a magnetic field. Assume that

$$f(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_0(\underset{\sim}{v}) + f_1(\underset{\sim}{v}) \exp(ikx - i\omega t)$$

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \hat{\underset{\sim}{x}}\, E \exp(ikx - i\omega t)$$

where $|f_1| << f_0$, with $f_0(\underset{\sim}{v})$ the nonperturbed equilibrium distribution function, and where $\underset{\sim}{E}(\underset{\sim}{r}, t)$ is the internal electric field due to the small amplitude perturbation in the electron gas. Using the linearized Vlasov equation (neglecting second-order terms) determine the expression for $f_1(\underset{\sim}{v})$ in terms of $\underset{\sim}{E} = -\underset{\sim}{\nabla}\phi$ and $\underset{\sim}{\nabla}_v f_0$. Using this result in Poisson equation, obtain the following *dispersion relation* for longitudinal waves propagating in the x-direction:

$$1 = \frac{\omega_{pe}^2}{n_0 k^2} \int_v \frac{(\partial f_0/\partial v_x)}{(v_x - \omega/k)}\, d^3v$$

18.2 - Show that

$$2C \int_0^C \exp(W^2 - C^2)\, dW = 2C^2 \sum_{n=0}^{\infty} (-1)^n \frac{2^n C^{2n}}{1.\ 3.\ 5\ \ldots\ (2n-1)\ (2n+1)}$$

by making a series expansion of the integrand. For $C \ll 1$, that is, for $(\omega/k) \ll (2k_B T_e/m_e)^{1/2}$, show that the dipersion relation for the longitudinal plasma wave reduces to

$$k^2 \lambda_D^2 = -1$$

or

$$k^2 (k_B T_e/m_e)^{1/2} = -\omega_{pe}^2$$

This result is the low-frequency limit of the result obtained from the macroscopic warm plasma model, using the isothermal sound speed of the electron gas $V_{se} = (k_B T_e/m_e)^{1/2}$.

18.3 - (a) Show that the dispersion relation for the longitudinal plasma wave (with $\underset{\sim}{k} = k\, \hat{\underset{\sim}{x}}$), for the case of an unbounded homogeneous plasma in which the motion of the electrons and the ions is taken into account, can be written as

$$1 = \frac{\omega_{pe}^2}{n_0 k} \int_v \frac{\partial f_{0e}(v)/\partial v_x}{(v_x - \omega/k)} \, d^3v + \frac{\omega_{pi}^2}{n_0 k^2} \int_v \frac{\partial f_{0i}(v)/\partial v_x}{(v_x - \omega/k)} \, d^3v$$

Show that this dispersion relation can be recast into the form

$$1 = \frac{1}{k^2} \left[\omega_{pe}^2 < (v_x - \omega/k)^{-2} >_{0e} + \omega_{pi}^2 < (v_x - \omega/k)^{-2} >_{0i} \right]$$

where (with $\alpha = e,\ i$)

$$<(v_x - \omega/k)^{-2}>_{0\alpha} = \frac{1}{n_0} \int_v \frac{f_{0\alpha}(v)}{(v_x - \omega/k)^2} \, d^3v$$

(b) For the cold plasma model, for which

$$f_{0\alpha}(v) = n_0\ \delta(v_x)\ \delta(v_y)\ \delta(v_z)$$

show that the dispersion relation reduces to

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 = \frac{n_0\ e^2}{\varepsilon_0 \mu}$$

where $\mu = m_e m_i/(m_e + m_i)$ is the reduced mass of an electron and an ion.

(c) In the high-phase velocity limit, show, by making a binomial expansion, that the dispersion relation becomes

$$1 = \frac{\omega_{pe}^2}{\omega^2} \left(1 + 3 \frac{k^2}{\omega^2} \frac{k_B\ T_e}{m_e} + \ldots \right)$$

$$+ \frac{\omega_{pi}^2}{\omega^2} \left(1 + 3 \frac{k^2}{\omega^2} \frac{k_B\ T_i}{m_i} + \ldots \right)$$

Show that this equation can be written as

$$1 = \frac{(\omega_{pe}^2 + \omega_{pi}^2)}{\omega^2} \left(1 + 3 \frac{k^2}{\omega^2} \frac{k_B\ T_h}{\mu} + \ldots \right)$$

where T_h is a "hibrid" temperature given by

$$T_h = \frac{m_i^2\ T_e + m_e^2\ T_i}{(m_e + m_i)^2}$$

Under what conditions does this relation reduce to the Bohm--Gross dispersion relation for a warm electron plasma?

(d) Show that the dispersion relation of part (a) can be expressed as

$$1 = -\frac{1}{k^2\ \lambda_{De}^2}\left[1 + i\sqrt{\pi}\ C_e \exp(-C_e^2) - 2C_e \int_0^{C_e} \exp(W^2 - C_e^2)dW\right] -$$

$$-\frac{1}{k^2\ \lambda_{Di}^2}\left[1 + i\sqrt{\pi}\ C_i\ \exp(-C_i^2) - 2C_i \int_0^{C_i} \exp(W^2 - C_i^2)dW\right]$$

where (with α = e, i)

$$\lambda_{D\alpha} = \left(\frac{\varepsilon_0\ kT_\alpha}{n_0\ e^2}\right)^{1/2} \quad ; \quad C_\alpha = \frac{(\omega/k)}{(2k_B\ T_\alpha/m_\alpha)^{1/2}}$$

For weakly damped oscillations ($\omega_i << \omega_r$) and in the low frequency and low phase velociy range specified by the condition

$$C_i >> 1 >> C_e$$

show that the dispersion relation reduces to

$$1 = -\frac{1}{k^2\ \lambda_{De}^2}\left(1 + i\sqrt{\pi}\ C_e - \frac{m_e}{2m_i\ C_e^2}\right)$$

Consequently, verify that the frequency of oscillation and the Landau damping constant are given by

$$\omega_r = \left(\frac{k_B\, T_e}{m_i} \right)^{1/2} k(1 + k^2\, \lambda_{De}^2)^{-1/2}$$

$$\omega_i = - \left(\frac{\pi}{8}\, \frac{m_e}{m_i} \right)^{1/2} \left(\frac{k_B\, T_e}{m_i} \right)^{1/2} k(1 + k^2\, \lambda_{De}^2)^{-2}$$

Note that the condition $C_i >> 1 >> C_e$ is fulfilled only if $T_e/T_i >> (1 + k^2\, \lambda_{De}^2)$, which implies in a strongly nonisothermal plasma, with hot electrons and cold ions. Show that in the long-wave range we find

$$\omega_r = k \left(\frac{k_B\, T_e}{m_i} \right)^{1/2}$$

which are essentially the same as the low frequency ion acoustic waves that propagate at a sound speed determined by the ion mass and the electron temperature.

18.4 - A longitudinal plasma wave is set up propagating in the x direction ($\underset{\sim}{k} = k\, \hat{\underset{\sim}{x}}$) in a plasma whose equilibrium state is characterized by the following so-called *resonance distribution* of velocities in the direction of the wave normal of the longitudinal plasma wave

$$f_0(\underset{\sim}{v}) = n_0\, \frac{A}{\pi}\, \frac{1}{(v_x^2 + A^2)}\; \delta(v_y)\, \delta(v_z)$$

where A is a constant.

(a) Using this expression for $f_0(\underset{\sim}{v})$ in the dispersion relation for the longitudinal plasma wave [Eq. (4.9)], obtain the result

$$1 = \frac{\omega_{pe}^2}{k^2} \frac{A}{\pi} \int_{-\infty}^{+\infty} \frac{dv_x}{(V_x - \omega/k)^2 \; (v_x^2 + A^2)}$$

(b) Evaluate the integral of part (a) by closing the contour in the upper half plane (note that there is a double pole at $v_x = \omega/k$ and a simple pole at $v_x = i\,A$), to obtain the dispersion relation

$$1 = \frac{\omega_{pe}^2}{k^2} \frac{1}{(\omega/k + i\,A^2)}$$

(c) Analyse this dispersion relation ($\omega = \omega_r + i\omega_i$) to show that the longitudinal wave in this plasma is not unstable and determine the frequency of oscillation (ω_r) and the Landau damping constant (ω_i). Compare this Landau damping constant with the corresponding value for a Maxwellian distribution of velocities, for the cases when $k\,\lambda_D \ll 1$ and $k\,\lambda_D \geqq 1$.

18.5 - Solve the linearized Vlasov equation (3.9) by the method of integral transforms, taking its Laplace transform in the time domain and the Fourier transform with respect to the space variables. Then, determine the dispersion relation for the modes of wave propagation in a hot isotropic plasma.

18.6 - Evaluate the integral G(C, 0), defined in Eq (4.32) with s = 0, by the method of residues using the contours of integration in the complex plane shown in Fig. 2.

18.7 - Consider a longitudinal wave propagating along the x direction in a plasma, whose electric field is given by

$$E_x(x, t) = E_0 \sin (kx - \omega t)$$

(a) Show that, for small displacements, the electrons which are moving with a velocity approximately equal to the phase velocity of the wave will oscillate with a frequency given by

$$\omega' = \left(\frac{e E_0 k}{m_e} \right)^{1/2}$$

(b) Estabilish the necessary conditions for trapping of the electrons by the wave.

18.8 - Consider the two-stream problem using the macroscopic cold plasma equations for two beams of electrons having number densities given by

$$n_{1,2} = \frac{1}{2} n_0 + n_{1,2} \exp [\, i(kx - \omega t)]$$

and average velocities given by

$$u_{1,2} = \pm u_0 + u_{1,2} \exp [\, i(kx - \omega t)]$$

Consider that the electric field is given by

$$E_x = E_0 \exp [\, i(kx - \omega t)]$$

Determine the dispersion relation for this two-stream problem and verify if the oscillations with real k are stable or unstable.