INSTITUTO DE PESQUISAS ESPACIAIS

AUTORIZAÇÃO PARA PUBLICAÇÃO
AUTHORIZATION FOR PUBLICATION

PALAVRAS CHAVES/KEY WORDS

Surrogate and Lagrangean Relaxations
0-1 Multiknapsack

AUTORIZADA POR/AUTHORIZED BY

L. A. V. Dias
Luiz A. Vieira Dias
Chairman of LAC

AUTOR RESPONSAVEL / RESPONSIBLE AUTHOR

Luiz A.N. Lorena

DISTRIBUIÇÃO/DISTRIBUTION

[ ] INTERNA / INTERNAL
[X] EXTERNA / EXTERNAL
[ ] RESTRITA / RESTRICTED

REVISADA POR / REVISED BY

CDU/UDC

519.87

DATA / DATE

February 1990

PUBLICAÇÃO Nº / PUBLICATION NO

INPE-5036-PRE/1571

ORIGEM / ORIGIN

LAC

PROJETO / PROJECT

OTIS

Nº DE PAG. / NO OF PAGES

29

ULTIMA PAG. / LAST PAGE

26

VERSÃO / VERSION

Nº DE MAPAS / NO OF MAPS

TÍTULO/TITLE

THE 0-1 MULTIKNAPSACK PROBLEM: MONOTONE SURROGATE AND LAGRANGEAN ALGORITHMS

AUTORES/AUTHORSHIP

Luiz Antonio Nogueira Lorena
G. Plateau*

RESUMO-NOTAS/ABSTRACT-NOTES

In this work an approach for generating monotone decreasing algorithms to the 0-1 Multiknapsack surrogate and Lagrangean relaxations in showed. The work is centralized in controling the step size of a subgradient type algorithm. A repetition of optimal intermediate values for the surrogate algorithm is avoided at least at the third consecutive iteration, and for the Lagrangean case a monotone decreasing sequence of values is assured. A lot of computational tests with problems of the literature are presented.

OBSERVAÇÕES/REMARKS

This work will be submitted to the "International Conference on Operations Research", Vienna, Austria, 28-31 August, 1990.

*LIPN-Université Paris Nord, CSP, França.

MINISTÉRIO DA CIÊNCIA E TECNOLOGIA
INSTITUTO DE PESQUISAS ESPACIAIS

PROPOSTA PARA PUBLICAÇÃO

☐ DISSERTAÇÃO
☐ TESE
☒ RELATÓRIO
☐ OUTROS

IDENTIFICAÇÃO

TÍTULO

The 0-1 multiknapsack problem: monotone surrogate and Lagrangean algorithms.

AUTOR(ES)

Luiz Antonio Nogueira Lorena
G. Plateau*

*LIPN-Université de Paris-Nord, CSP, França.

ORIENTADOR

CO-ORIENTADOR

DISS. OU TESE

LIMITE __/__/__ | DEFESA __/__/__ | CURSO | ORGÃO

DIVULGAÇÃO

☒ EXTERNA ☐ INTERNA ☐ RESTRITA

EVENTO/MEIO

☒ CONGRESSO ☐ REVISTA ☐ OUTROS

REV. TÉCNICA

NOME DO REVISOR

RECEBIDO __/__/__ | DEVOLVIDO __/__/__ | ASSINATURA

APROVAÇÃO

NOME DO RESPONSÁVEL

Luiz Alberto Vieira Dias

APROVADO ☐ SIM ☐ NÃO | DATA 6/2/90 | ASSINATURA

Luiz Alberto Vieira Dias
Chefe do Laboratório
Associado de Computação
e Matemática Aplicada

REV. LINGUAGEM

Nº | PRIOR. | RECEBIDO __/__/__ | NOME DO REVISOR

PÁG. | DEVOLVIDO __/__/__ | ASSINATURA

DATILOGRAFIA

OS AUTORES DEVEM MENCIONAR NO VERSO INSTRUÇÕES ESPECÍFICAS, ANEXANDO NORMAS, SE HOUVER

RECEBIDO | DEVOLVIDO __/__/__ | NOME DA DATILÓGRAFA

Nº DA PUBLICAÇÃO: PÁG.:
CÓPIAS: Nº DISCO: LOCAL:

DIRETOR

AUTORIZO A PUBLICAÇÃO

☐ SIM
☐ NÃO __/__/__

OBSERVAÇÕES E NOTAS

Este trabalho será submetido ao "international Conference on Operations Research, Vienna, Austria, de 28 a 31/08/90.

APÓS A PUBLICAÇÃO, ESTA FICHA SERÁ ARQUIVADA JUNTAMENTE COM A VERSÃO ORIGINAL

INPE 106

INSTITUTO DE PESQUISAS ESPACIAIS
MINISTÉRIO DA CIÊNCIA E TECNOLOGIA

AUTORIZAÇÃO PARA PUBLICAÇÃO
AUTHORIZATION FOR PUBLICATION

PALAVRAS CHAVES/KEY WORDS

AUTORES AUTHORS

Surrogate and Lagrangean Relaxations
0-1 Multiknapsack

AUTORIZADA POR/AUTHORIZED BY

L. A. V. D.
Luiz A. Vieira Dias
Chairman of LAC

AUTOR RESPONSAVEL
RESPONSIBLE AUTHOR

Luiz A.N. Lorena

DISTRIBUIÇÃO/DISTRIBUTION

☐ INTERNA / INTERNAL
☒ EXTERNA / EXTERNAL
☐ RESTRITA / RESTRICTED

REVISADA POR / REVISED BY

CDU/UDC

519.87

DATA / DATE

February 1990

PUBLICAÇÃO Nº
PUBLICATION NO

INPE-5036-PRE/1571

ORIGEM
ORIGIN

LAC

PROJETO
PROJECT

OTIS

TÍTULO/TITLE

THE 0-1 MULTIKNAPSACK PROBLEM: MONOTONE SURROGATE AND LAGRANGEAN ALGORITHMS

Nº DE PAG
NO OF PAGES

29

ULTIMA PAG
LAST PAGE

26

VERSÃO
VERSION

Nº DE MAPAS
NO OF MAPS

AUTORES/AUTHORSHIP

Luiz Antonio Nogueira Lorena
G. Plateau*

RESUMO - NOTAS / ABSTRACT - NOTES

In this work an approach for generating monotone decreasing algorithms to the 0-1 Multiknapsack surrogate and Lagrangean relaxations in showed. The work is centralized in controling the step size of a subgradient type algorithm. A repetition of optimal intermediate values for the surrogate algorithm is avoided at least at the third consecutive iteration, and for the Lagrangean case a monotone decreasing sequence of values is assured. A lot of computational tests with problems of the literature are presented.

OBSERVAÇÕES / REMARKS

This work will be submitted to the "International Conference on Operations Research", Vienna, Austria, 28-31 August, 1990.

*LIPN-Université Paris Nord, CSP, França.

Resumo:

Apresenta-se neste trabalho uma abordagem para gerar algoritmos monotonicamente decrescentes para as relaxações "surrogate" e Lagrangeana do problema multidimensional da mochila em variáveis 0-1. O trabalho mostra a importância do controle do tamanho do passo em um algoritmo tipo subgradientes. Não é permitida a repetição de valores intermediários no algoritmo "surrogate" em duas iterações seguidas. No algoritmo Lagrangeano obtém-se uma seqüência monótona decrescente de valores ótimos. Apresenta-se um grande número de testes computacionais com problemas da literatura.

THE 0-1 MULTIKNAPSACK PROBLEM: MONOTONE SURROGATE AND LAGRANGEAN ALGORITHMS

L.A.N.LORENA(*) and G.PLATEAU(**)

Abstract:

In this work an approach for generating monotone decreasing algorithms to the 0-1 Multiknapsack surrogate and Lagrangean relaxations is showed. The work is centralized in controling the step size of a subgradient type algorithm. A repetition of optimal intermediate values for the surrogate algorithm is avoided at least at the third consecutive iteration, and for the Lagrangean case a monotone decreasing sequence of values is assured. A lot of computational tests with problems of the literature are presented.

Key words : 0-1 Multiknapsack, Surrogate and Lagrangean relaxations

(*) Instituto de Pesquisas Espaciais - INPE, Secretaria Especial de Ciência e Tecnologia, Caixa Postal 515, 12201 São José dos Campos-SP, Brazil.
(**) LIPN - Université de PARIS-NORD, CSP, Département de Mathématiques et Informatique, Av.J.B.Clément, 93430 Villetaneuse, France.

1. INTRODUCTION:

In this work we study monotone decreasing algorithms for the 0-1 Multiknapsack surrogate and Lagrangean relaxations.

The 0-1 Multiknapsack problem can be defined as

$$(P) \qquad \begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \leq b \\ & x \in \{0,1\}^n, \end{array}$$

where, $c \in N^n$, $b \in N^m$, A is a mxn dense non--negative integer matrix and $\{0,1\}^n = \{x \in R^n : x_j=0 \text{ or } x_j=1; j=1,...,n\}$.

Other surrogate algorithms were proposed by Karwan and Rardin [10], Dyer [2], Gavish and Pirkul [6], and Sarin, Karwan and Rardin [14], [15]. The research in surrogate duality were developed mainly by Glover [7], Greenberg [8], Karwan and Rardin [10] and Dyer [2].

In section 2 we define the Lagrangean and Surrogate relaxations, and show one condition for the equality between the optimal values of the Surrogate Continuous and Lagrangean relaxations. A general subgradient algorithm is also presented. The surrogate version of the algorithm is analysed in section 3, and the Lagragean version in section 4. In section 5 we present a reduction phase for the Lagrangean version and computational tests with both algorithms using 13 problems of the literature.

2. LAGRANGEAN AND SURROGATE RELAXATIONS

In this section, we recall the surrogate and Lagrangean relaxation definition and some properties. One condition for the equality between the optimal values of the Lagrangean and the Surrogate Continuous relaxations of (P) is showed. A general subgradient type algorithm is also given.

The Surrogate relaxation of (P) can be defined as

$$(SR_w) \qquad \begin{aligned} &\max\ cx \\ &\text{s.t. } wAx \leq wb \\ &\qquad x \in \{0,1\}^n \end{aligned}$$

where $w \in R_+^m$,

and for the Continuous version (SRC_w), $x \in \{0,1\}^n$ is substituted by $x \in [0,1]^n$ that means $0 \leq x_j \leq 1; j=1,\dots,n$.

The Lagrangean relaxation of (P) is

$$(LR_w) \quad = \quad \max \{cx - w(Ax - b)\}, \quad \text{s.t} \quad x \in \{0,1\}^n$$

where $w \in R_+^m$.

Let us define the functions

$l: R_+^m \to R, \ w \to l(w) = v(LR_w)$;

$s: R_+^m \to R, \ w \to s(w) = v(SR_w)$; and

$sc: R_+^m \to R, \ w \to sc(w) = v(SRC_w)$;

where v(.) is the optimal value of problem (.). It's well known that:

- l is a function continuous, convex and linear by parts in R_+^m(Rockafellar [13]);

- s is a quasi-convex function in R_+^m and upper semicontinuous in the compact $B = \{w \geq 0: \|w\|_1 = 1\}$, and

- sc is a quasi-convex function in R_+^m and continuous in B (Greenberg H. and Pierskalla W.P. [9]).

These characteristics contribute for successful application of subgradient type algorithms in the Lagrangean case and for a source of problems in the surrogate case.

For a given w:

(i) the solution of (LR_w) is trivial;

(ii) the solution of (SRC_w) is obtained as follows (Dantzig [1]):

- sort the ratios c_j/wA^j in decreasing order,

- fix variables at 1 according this order until the infeasibility of the constraint.

Let $i*$ be the variable index that makes the infeasibility of the constraint; $\overline{x}_{i*}$ is the basic variable, and let $J_w = \{j \in \{1,2,\ldots,n\}: \overline{x}_j = 1$ in the solution of $(SRC_w)\}$. Then,

$$v(SRC_w) = \sum_{j \in J_w} c_j + c_{i*} x_{i*},$$

$$\text{where } \overline{x}_{i*} = w(b - \sum_{j \in J_w} A^j)/wA^{i*}.$$

$$v(SRC_w) = \sum_{j \in J_w} c_j - c_{i*} w g_w / wA^{i*}, \text{ where } g_w = \sum_{j \in J_w} A^j - b,$$

$$wg_w \leq 0 \text{ and } |wg_w| < wA^{i*}; \text{ and}$$

(iii) the solution of (SR_w) can be obtained by a real constrained version of the algorithm **FPK** for 0-1 Knapsack problems (Fayard D. and Plateau G. [3]). The optimal value $v(SR_w)$ is integer but the constraint is real.

Let $\lambda_w = c_{i*}/wA^{i*}$, that is, λ_w is the optimal solution of the (SRC_w) dual. In the following we show a sufficient condition for the equality between the optimal values of the Surrogate Continuous and the Lagrangean relaxation of (P).

PROPOSITION 1: If $w_C = \lambda_W . w$ then $v(SRC_W) = v(LRw_C)$.

Proof:

$$v(LRw_C) = \max_{x\in\{0,1\}^n} \{cx - w_C(Ax - b)\} =$$

$$= \max_{x\in\{0,1\}^n} \{cx - \lambda_W . w(Ax - b)\} =$$

$$= \max_{x\in\{0,1\}^n} \{cx - (c_{i*}/wA^{i*})w(Ax - b)\} =$$

$$= \max_{x\in\{0,1\}^n} \{\sum_{j=1}^{n} [c_j - (c_{i*}/wA^{i*})wA^j]x_j + (c_{i*}/wA^{i*})wb\} =$$

$$= \sum_{j\in J_W} c_j - c_{i*} wg_W/wA^{i*} = v(SRC_W).|||$$

This result shows that for an optimal solution for problem (SRC_W), there is a Lagrangean Relaxation (LRw_C) with the same optimal value. The optimal solution of (LRw_C) will be:

$$\begin{cases} \bar{x}_j = 1, & \text{for all } j\in J_W , \\ \bar{x}_j = 0, & \text{otherwise.} \end{cases}$$

This will be used in section 4 in a Lagrangean version of the general subgradient type algorithm given in the following.

Algorithm G

{ For a given relaxation of (P), (REL_w), let

$$g_w = \sum_{j \in J_w} A^j - b, \qquad J_w = \{j : x_j = 1 \text{ in the optimal solution of } (REL_w)\} \quad \}$$

Compute $w \in R_+^m$; solve (REL_w); $\theta \leftarrow +\infty$;

while $v(REL_w) \leq \theta$ **do**

    $\theta \leftarrow v(REL_w)$;

    compute $t \in R_+$;

    $w \leftarrow w + t\, g_w / \| g_w \|^2$;

    for each i in $\{1, \ldots, m\}$ such that $w_i < 0$ **do**

        $w_i \leftarrow 0$;

    endfor;

    solve (REL_w);

endwhile

The step size t will be defined at the appropriated section depending on the relaxation (REL_w) used to (P), and their control will make possible a monotone non-increasing sequence of $v(REL_w)$.

3. SURROGATE VERSION:

In this section a surrogate version of algorithm G is presented, with some theoretical results and suggestion for controling the step size t at implementation phase.

The **Algorithm** S is derived of algorithm G by making $(REL_w) = (SR_w)$. Suppose that $'w$, w, and w' are values of w at three consecutive iterations of algorithm S, $w' \leftarrow w + t\, g_w/\|g_w\|^2$, $t > 0$, and $x*$ is an optimal solution to (SR_w) (idem to w' and $'w$).

PROPOSITION 2: (i) If $g_w g_{w'} \geq 0$ then $v(SR_w) \geq v(SR_{w'})$;

(ii) If $g_w g_{w'} < 0$ and $t \leq |wg_{w'}| \cdot \|g_w\|^2 / |g_w g_{w'}|$ then $v(SR_w) \geq v(SR_{w'})$.

Proof: Of $(SR_{w'})$ solution, $w' g_{w'} \leq 0$, or

$$wg_{w'} + t\, g_w g_{w'}/\|g_w\|^2 \leq 0.$$

(i) Like $t > 0$ and $g_w g_{w'} \geq 0$, $wg_{w'} \leq 0$, that is $x*'$ is feasible to (SR_w) and then $v(SR_w) \geq v(SR_{w'})$;

(ii) If $g_w g_{w'} < 0$ and $t \leq |wg_{w'}| \cdot \|g_w\|^2 / |g_w g_{w'}|$, a similar analysis shows that $v(SR_w) \geq v(SR_{w'})$.|||

COROLLARY 1: (i) If $v(SRw) < v(SR_{w'})$ then $g_w g_{w'} < 0$;

(ii) If $t/\|g_w\|^2 \leq |wg_{w'}|/|g_w g_{w'}|$ then $v(SR_w) \geq v(SR_{w'})$.

Proof: immediate.|||

The multiplier w' is a function of t, and naturally it is not known *a priori* if $g_w g_{w'}$ is greater, less or equal 0. But if t is "small enough" then $v(SR_w)$ may be greater than $v(SR_{w'})$.

The following result shows that t can't be "much small" to avoid that $v(SR_w) = v(SR_{w'})$.

PROPOSITION 3: (i) If $t \le |wg_w|$ then $v(SR_w) \le v(SR_{w'})$;

(ii) If $t \le |wg_w|$ and $g_w g_{w'} \ge 0$ then $v(SR_w) = v(SR_{w'})$;

(iii) If $t < |wg_w|$, $g_w g_{w'} < 0$ and $t/\|g_w\|^2 \le |wg_{w'}|/|g_w g_{w'}|$ then $v(SR_w) = v(SR_{w'})$.

Proof: (i) $w'g_w = wg_w + t$ and if $t \le |wg_w|$ then $w'g_w \le 0$, that is, $x*$ is feasible to $(SR_{w'})$;

(ii) immediate from proposition 2(i) and (i);

(iii) immediate of proposition 2(ii) and (i).|||

COROLLARY 2: If $v(SR_w) > v(SR_{w'})$ then $t > |wg_w|$.

Suppose now that $J_w = J_{w'}$.

PROPOSITION 4: If $J_w = J_{w'}$ then

(i) $t \le |wg_w|$;

(ii) $0 \le |w'g_w| < |wg_w|$.

Proof: If $J_w = J_{w'}$ then $g_w = g_{w'}$ and $w'g_w = wg_w + t \le 0$. Then (i) $t \le |wg_w|$, because $wg_w \le 0$ and $t > 0$;

(ii) immediate from (i).|||

Then if $J_w = J_{w'}$ the two hypothesis of proposition 3 (ii) are fulfilled and $v(SR_w) = v(SR_{w'})$.

COROLLARY 3: (i) If $t > |wg_w|$ then $J_w \neq J_{w'}$;

(ii) If $|wg_w| < t < |wg_{w'}| \cdot \|g_w\|^2 / |g_w g_{w'}|$ then $J_w \neq J_{w'}$ and $v(SR_w) \geq v(SR_{w'})$.

Proof: (i) immediate from proposition 4;

(ii) immediate from (i) and proposition 2.|||

Some consequences and conditions for the scalar product $g_w g_{w'}$ sign are presented in the next proposition.

PROPOSITION 5: (i) If $|wg_{w'}| \leq |w'g_{w'}|$ then $g_w g_{w'} \leq 0$;

(ii) $g_w g_{w'} > 0$ if and only if $|wg_{w'}| > |w'g_{w'}|$ and $wg_{w'} \leq 0$.

Proof: $w'g_{w'} = wg_{w'} + t\, g_w g_{w'} / \|g_w\|^2 \leq 0$, $t > 0$. Then

(i) if $|wg_{w'}| \leq |w'g_{w'}|$ then $g_w g_{w'} \leq 0$;

(ii) if $g_w g_{w'} > 0$ then $|wg_{w'}| > |w'g_{w'}|$ and $wg_{w'} \leq 0$.|||

If $wg_{w'} > 0$ the solution $x*'$ is not feasible to (SR_w) and then $v(SR_{w'})$ may be greater than $v(SR_w)$. In the following we examine the consequence.

PROPOSITION 6: If $wg_{w'} > 0$ then $g_w g_{w'} < 0$ and

$$wg_{w'} \leq t\, |g_w g_{w'}| / \|g_w\|^2.$$

Proof: $w'g_{w'} = wg_{w'} + t\, g_w g_{w'} / \|g_w\|^2$. Because $w'g_{w'} \leq 0$, $t > 0$ and $wg_{w'} > 0$ then $wg_{w'} \leq t\, |g_w g_{w'}| / \|g_w\|^2$ and $g_w g_{w'} < 0$.|||

We can now examine a suggestion value for t to search the monotony in the sequence $\{v(SR_{w'})\}$.

The use of $t = |'wg_w|$ may be considered a natural derivation of proposition 2 (ii) upper limit in the worst case, and produces good computational results. Then,

$$t = |'wg_w|,$$
$$w' \leftarrow w + |'wg_w|\ g_w/\|g_w\|^2, \text{ and}$$
$$w'g_{w'} = wg_{w'} + |'wg_w|\ g_w g_{w'}/\|g_w\|^2.$$

PROPOSITION 7: If $t = |'wg_w|$ and $J_w = J_{w'}$ then $g_{'w}g_w \le 0$.
Proof: Of proposition 4 (i) if $J_w = J_{w'}$ then $|'wg_w| = t \le |wg_w|$. Then, of proposition 5 (i), $g_{'w}g_w \le 0$.||||

COROLLARY 4: If $t = |'wg_w|$ and $J_w = J_{w'}$ then $J_{'w} \ne J_w = J_{w'}$.
Proof: immediate.||||

With this very interesting result, two is the greater number of iterations with the same optimal solution for the relaxation ($SR_{w'}$). In almost all the cases this result remain valid for the optimal value.

PROPOSITION 8: (i)If $t = |'wg_w|$ and $g_{'w}g_w > 0$ then $J_w \ne J_{w'}$.
(ii) If $t = |'wg_w|$, $g_{'w}g_w > 0$ and $g_w g_{w'} \ge 0$ then $v(SR_w) \ge v(SR_{w'})$ and $J_w \ne J_{w'}$.

Proof: (i) By proposition 5 (ii), if $g_{'w}g_w > 0$ then $t = |'wg_w| > |wg_w|$ and by corollary 1 (i) $J_w \ne J_{w'}$;
(ii) immediate from propositions 2 and 5.||||

It is impossible that $t \le |wg_w|$ for two consecutive iterations of algorithm S. If $g_{'w}g_w > 0$ there is a sufficient condition ($t > |wg_w|$) for $v(SR_w) > v(SR_{w'})$.

It is important to note that some results can be changed when the projection of w' in the non-negative orthant of R^m is made, because for an $i \in \{1,\ldots,m\}$, if $w_i \neq 0$ and $w'_i < 0$, when $w'_i \leftarrow 0$, and maybe

$$w' g_w \neq w g_w + t$$

$$w' g_{w'} \neq w g_{w'} + t . g_w g_{w'} / \|g_w\|^2 \text{ (idem for 'w and w).}$$

Algorithm S can be modified introducing

if $w_i \leftarrow 0$ **then** $(g_w)_i \leftarrow 0$, for $i \in \{1,\ldots,m\}$.

This simple modification is necessary at implementation phase to preserve the main results of this section.

4. LAGRANGEAN VERSION

Let **Algorithm** L be a derivation of algorithm G for such that relaxation $(REL_w) = (SRC_w)$. From (SRC_w) solution we have: $i*$, $\lambda_w = c_{i*}/wA^{i*}$, g_w, and J_w. Suppose that w and w' are two consecutive iterations of algorithm L, and from $(SRC_{w'})$ solution: $i*'$, $\lambda_{w'} = c_{i*'}/w'A^{i*'}$, $g_{w'}$ and $J_{w'}$.

The result of proposition 1, $v(SRC_w) = v(LRw_c)$, for $w_c = \lambda_w . w$, will be used in algorithm L, and at each iteration, problem (SRC_w) may be solved by the expected linear time complexity algorithm **NKR** of Fayard and Plateau (Fayard D. and Plateau G. [3]). This will produce a sequence of Lagrangean values $v(LRw_{c'})$.

The solution of (SRC_w) have an integer part, the $x_j = 1$, $j \in J_{w,}$ and the real part $x_{i*} = |wg_w|/wA^{i*}$, $|wg_w| < wA^{i*}$. For the integer part a similar analysis than the one made for the (SR_w) problem can be repeated here, that is, the search for t such that $wg_{w'} \le 0$, and all propositions and corollaries of section 3 are applied to (SRC_w). For the (SRC_w) case the following proposition is a stronger result than all those for (SR_w), and this is a consequence of the function sc characteristics (see section 2).

PROPOSITION 9: If $wg_{w'} \le 0$ and $|wg_{w'}| > wA^{i*'}$ then $v(SRC_w) > v(SRC_{w'})$.

Proof: If $wg_{w'} \le 0$ and $|wg_{w'}| > wA^{i*'}$ then $w(g_{w'} + A^{i*'}) \le 0$ and $v(SRC_{w'}) = \sum_{j\in J_{w'}} c_j - c_{i*'} w'g_{w'}/w'A^{i*'} \le \sum_{j\in J_{w'}} c_j + c_{i*'} <$ $v(SRC_w)$.||||

Suppose now that $wg_{w'} \le 0$ and $|wg_w| \le wA^{i*'}$.

PROPOSITION 10: If $wg_{w'} \le 0$ and $|wg_{w'}| \le wA^{i*'}$ then there exists a $\Omega \ge 0$ such that

$$\sum_{j\in J_{w'}} c_j - \lambda'_w wg_{w'} + \Omega = v(SRC_w), \ \Omega \ge 0,$$

where $\lambda'_w = c_{i*'}/wA^{i*'}$.

If $i* = i^{*'}$ then $\Omega = \sum_{j\in K} |c_j - \lambda_w wA^j|$,

where $K = (J_w \cup J_{w'}) - (J_w \cap J_{w'})$.

Proof: If $wg_{w'} \le 0$ and $|wg_{w'}| \le wA^{i*'}$,
the solution

$$\begin{cases} y_j = 1, \text{ for all } j \in J_{w'}, \\ y_{i*'} = |wg_{w'}|/wA^{i*'}, \\ y_j = 0, \text{ otherwise,} \end{cases}$$

is feasible for (SRC_w), then there is a $\Omega \ge 0$ such that

$$\sum_{j\in J_{w'}} c_j - \lambda'_w w g_{w'} + \Omega = v(SRC_w).$$

Suppose now that $i* = i*'$, then $\lambda'_w = \lambda_w$ and for $\Omega = \sum_{j\in K} |c_j - \lambda_w wA^j|$,

$$\sum_{j\in J_{w'}} c_j - \lambda_w wg_{w'} + \sum_{j\in K} |c_j - \lambda_w wA^j| =$$

$$= \sum_{j\in J_{w'}} (c_j - \lambda_w wA^j) + \lambda_w wb - \sum_{j\in K_{J'}} (c_j - \lambda_w wA^j) +$$

$$\sum_{j\in K_J} (c_j - \lambda_w wA^j) = \sum_{j\in J_w} c_j - \lambda_w wg_w = v(SRC_w),$$

where $K_{J'} = J_{w'} - (J_w \cap J_{w'})$,

$$K_J = J_w - (J_w \cap J_{w'}).$$ ||||

Using proposition 10, conditions can be examined for $v(SR_w) \ge v(SR_{w'})$

PROPOSITION 11: If $wg_{w'} \le 0$ and $|wg_{w'}| \le wA^{i*'}$ and

(i) if $|wg_{w'}|/wA^{i*'} \ge |w'g_{w'}|/w'A^{i*'}$, or

(ii) if $|wg_{w'}|/wA^{i*'} < |w'g_{w'}|/w'A^{i*'}$ and

$$c_{i*}|w'g_{w'}|/w'A^{i*'} - c_{i*}|wg_{w'}|/wA^{i*'} \le \Omega \text{ then}$$

$v(SRC_w) \ge v(SRC_{w'})$.

Proof: Using proposition 10 and of $(SRC_{w'})$ solution,

$v(SRC_w) - v(SRC_{w'}) = c_{i*'}\, w'g_{w'}/w'A^{i*'} - c_{i*'}\, wg_{w'}/wA^{i*'} + \Omega$

Then, because $wg_{w'}$ and $w'g_{w'}$ are both ≤ 0 and $\Omega \ge 0$, (i) and (ii) are immediate.||||

Remark 1: As $v(SRC_w)$ and $v(SRC_{w'}) \in R$, the equality in (i) and (ii) has a small probability, and this implies strictly decreasing surrogate values.

CORROLARY 5: If $J_w = J_{w'}$ and $g_wA^{i*'} \ge 0$ then $v(SRC_w) > v(SRC_{w'})$ and $0 < v(SRC_w) - v(SRC_{w'}) \le c_{i*'}(w'g_w/w'A^{i*'} - wg_w/wA^{i*'}) + \Omega$.

proof: If $J_w = J_{w'}$, $g_w = g_{w'}$ and by proposition 4 (i) $t \le |wg_w|$. Then for $g_wA^{i*'} \ge 0$,

$$|wg_w|/wA^{i*'} > |wg_w + t|/(wA^{i*'} + t\, g_wA^{i*'}/\|g_w\|^2),$$

that is, by proposition 11 (i), $v(SRC_w) > v(SRC_{w'})$ and $v(SRC_w) - v(SRC_{w'}) = c_{i*'}(w'g_w/w'A^{i8'} - wg_w/wA^{i*'}) + \Omega$.||||

Remark 2: If $J_w = J_{w'}$ and $i^* = i^{*'}$ then $\Omega = 0$.

The following result gives an upper bound to t at each iteration of algorithm L.

PROPOSITION 12: $t < \max_{j\in\{1,\ldots,n\}} \{ wA^j\|g_w\|^2/|g_wA^j| \}$;

Proof: From the $(SRC_{w'})$ solution $w'A^{i*'} > 0$, or

$$wA^{i*'} + t\, g_wA^{i*'}/\|g_w\|^2 > 0. \text{ And then}$$

$$t < wA^{i*'}\|g_w\|^2/|g_wA^{i*'}| \text{ or } t < \max_{j\in\{1,...,n\}} \{ wA^j\|g_w\|^2/|g_wA^j| \}$$

(if $g_wA^{i*} = 0$, $0 \le t < \infty$).||||

The step size t may be:

$$t = [v(SRC_w) - v_h]/(\lambda_w \cdot p)$$

where v_h is a lower bound on $v(P)$ obtained by any heuristics, and p is a positive constant defined at the first iteration of the algorithm.

A nice characteristic of this definition is that with a judicious choice of p we can estimate of when the property $t \le |wg_w|$ (sufficient condition for decreasing in the surrogate case) will be fulfilled.

PROPOSITION 13: If $t \le |wg_w|$ then

$$[(SRC_w) - v_h] \le c_{i*}\cdot p$$

Proof: $t = [(SRC_w)-v_h].wA^{i*}/c_{i*}.p$, then like $|wg_w| \le wA^{i*}$ the result is immediate.||||

COROLLARY 5: If $[(SRC_w) - v_h] > c_{i*}p$ then $t > |wg_w|$.

proof: immediate.||||

5. COMPUTATIONAL TESTS:

In this section we present computational tests using algorithms S and L with 13 problems of the literature.

A reduction phase for algorithm L are also presented. This makes possible to fix variables at their optimal values and/or eliminate redundant constraints. The main features of these tests are reported in tables 1, 2 and 3 of the Appendix.

The test problems of literature are:

W1-W8 due to Weingartner and Ness [17],

F due to Fleisher [4],

P6 and P7 due to Petersen [12], and

ST1 and ST2 due to Senju and Toyoda [16].

The initial multiplier w used in both algorithm is

$$w_i = \left(\sum_{j=1}^{n} a_{ij} - b_i \right) / \sum_{j=1}^{n} a_{ij}, \quad i = 1, \ldots, m$$

For algorithm S, t = $|{'}wg_w|$ is used at all iterations. The stopping criteria in algorithm G is changed to a more flexive one, in order to search for an improved solution, i.e., a counter is introduced to stop algorithm S after a fixed number of incresing values of $v(SR_w)$. But, when $v(SR_{w^{\cdot}})$ increases for the first time ($v(SR_w{}^{K}{}_{c})$ is reached), it is near the dual surrogate optimal solution and very few iterations were necessary (see table 1). The nice characteristic of corollary 3, i.e., the $v(SRC_w)$ values and solutions can not be repeated at the third consecutive algorithm iteration is preserved.

We also have made tests making $t \leftarrow t/2$ when $v(SR_w) < v(SR_{w'})$, and the bounds $v(SR_w{}^{K*})$ are comparable with that ones of table 1, but with an increasing at the number of iterations.

For algorithm L, $t = [v(SRC_w)-v_h]/p.\lambda_w$ at all iterations and none counter are introduced at implementation (see table 2). The parameter p is fixed at the first iteration to estimate when $t \le |wg_w|$, and for problems W1-W8, P6 e P7, e F , p = 10 is a good choice, but for ST1 and ST2, p must be small, equal to 0.5. This is a directly consequence of the problems data. It is interesting to note that this adequate values for p are all equivalent to make $10^{-4} \le t/\|g_w\|^2 \le 10^{-3}$ at the initial iteration of algorithm L for each test problem. The computational tests show that with the choice of the interval $[10^{-4}, 10^{-3}]$ for $t/\|g_w\|^2$, corollary 1 is valid in a great number of consecutive iterations.

The lower bound of v(P), v_h , is obtained by the following greedy heuristic :

HEURISTIC: (i) sort the ratios c_j/wA^j, $j=1,\ldots,n$, in decreasing order (this was already obtained by the solution of (SRC_w);

(ii) fix variables at 1 according this order while each constraint of (P) is feasible. When one constraint is violated, set the correspondent $x_j = 0$ and continue.

Because the sequence $\{v(SRC_{w^{\cdot}})\}$ is monotone decreasing, the results of table 2 are obtained in very few iterations (k*) comparing with traditional subgradient algorithms. The bounds $v(SRC_w{}^{k*})$ are very good comparing with the (PL) bound (the solution of the (P) dual). The values of v_h are also very good (less than 1% error of v(P)). Lorena and Plateau [11] presents other successful tests with diferent values of $t/\|g_w\|^2$ in the interval $[10^{-4}, 10^{-3}]$.

The monotony characteristic of sequence $\{v(SRC_{w^{\cdot}})$ is attractive to apply reduction tests of type described in Fréville and Plateau [5], while the algorithm L is in course. At the end we have fixed some variables and/or eliminated some redundant constraints.

Table 3 shows for each test problem the number of variables fixed at iteration 5, 10, 15, 20, and at the final iteration of algorithm L, using the following simple test in wich all we need is completly given at each algorithm iteration :

Reduction test 1: Given $\partial \in \{0,1\}$, $w \in R_+{}^m$ and v_h, the following is a sufficient condition to reduce the size of (P) fixing variables to their optimal value

If there is an index $j \in \{1,...,n\}$ such that

$$\left\lfloor v(SRC_w) - | d^{\partial}_j | \right\rfloor \le v_h$$

where $d^0{}_j = c_j - \lambda_w.wA^j$ (resp. $d^1{}_j$) if $c_j - \lambda_w.wA^j$ is positive (resp. negative),

then the variable x_j must be fixed at value $1 - \partial$.

The following elimination test of redundant constraints can be made at some iterations of algorithm L:

Reduction test 2: Let the problem

$$(P^k{}_w) \qquad \begin{array}{ll} \max & A_k x \\ \text{subj. to} & wAx \le wb \\ & x \in [0,1]^n, \end{array}$$

then, the following condition is sufficient for elimination of constraint k

If there is an index $k \in \{1,...,m\}$ such that

$$\left\lfloor v(P^k{}_w) \right\rfloor \le v_h$$

then constraint $A_K \leq b_K$ can be eliminated. This test can be made only for constraints k such that w'_K is near 0 and $w_K \neq 0$, for some $K \in \{1,\ldots,m\}$.

The main advantage of this approach is the automatic reduction induced by the algorithm, but the results are not comparable with the ones of Fréville and Plateau [5], because here we have used a less elaborated heuristic to obtain v_h.

6. CONCLUSION:

The good results of tables 1, 2 and 3 can be extended to other 0-1 problems for that problem (SRC_w) is a continuous 0-1 knapsack problem, because in this case all propositions remain valid.

The current work concerns an improvement at the reduction phase for algorithm L using more elaborated tests and heuristics, and for both algorithms the validation of the suggested values for t in other randomly generated problems of great size. The first experiments leads to efficient and strong results.

APPENDIX

Table 1

Problem	k_c	$v(SR_w{}^{k_c})$	k*	cont	$v(SR_w{}^{k*})$	$v(SR_w{}^0)$
P 6	13	10667	27	5	10659	11081
P 7	22	16613	28	2	16596	17236
ST 1	33	7853	44	3	7850	8341
ST 2	34	8774	50	3	8771	9110
F	13	2230	25	3	2219	2410
WN 1	8	141548	8	1	141548	146260
WN 2	8	130883	8	1	130883	136708
WN 3	6	98416	15	4	97906	101507
WN 4	7	121087	21	5	120647	124504
WN 5	11	98796	11	1	98796	122363
WN 6	10	130773	10	1	130773	140447
WN 7	17	1095491	22	3	1095491	1101533
WN 8	6	627442	12	3	627442	636916

Where:

k_c = iteration number of when $v(SR_{w'}) > v(SR_w)$;

k* = number of iterations;

cont = counter of times in wich $v(SR_{w'}) > v(SR_w)$;

$v(SR_w{}^0)$ = initial optimal value of (SR_w);

$v(SRw^{k}{}_c)$ = optimal value of (SR_w) for $w = w^{k}{}_c$; and

$v(SR^{k*})$ = final optimal value of (SR_w).

Table 2

Prob	v(P)	(PL)	$v(SRC_w^0)$	k*	$v(SRC_w^{k*})$	$t/\|g_w\|^2$
W1	141278	142019	148363.8	22	142019	4.2×10^{-4}
W2	130883	131637.5	137840.2	36	131638.4	4.7×10^{-4}
W3	95677	99647	102400.3	66	99650	4.5×10^{-4}
W4	119337	122505.2	125896	10	122507.9	4.3×10^{-4}
W5	98796	100433.1	123271.1	34	100433.1	2.1×10^{-4}
W6	130623	131335	141157.1	24	131335	5.4×10^{-4}
W7	1095445	1095721.2	1101848	5	1095722	3.5×10^{-4}
W8	624319	628773.7	637939.9	14	628775.2	3.5×10^{-4}
F	2139	2221.8	2447.8	20	2229.6	3.3×10^{-4}
P6	10618	10672.3	11091.6	53	10675.9	5.0×10^{-4}
P7	16537	16612.8	17248.1	40	16613.9	4.5×10^{-4}
ST1	7772	7839	8356.5	48	7853.4	2.5×10^{-4}
ST2	8722	8773	9123.3	21	8774.6	7.0×10^{-4}

Where:

v(P) = the optimal solution value of (P);

(PL) = the optimal solution value of the linear programming relaxation of (P);

k* = number of iterations;

$v(SRC_w^0)$ = initial optimal value of (SRC_w); and

$v(SRC_w^{k*})$ = final optimal value of (SRC_w).

Table 3

Probl.	size	v_h	# of fixed variables at iteration 5	10	15	20	final
WN 1	2x28	140618	7	8	10	10	10
WN 2	2x28	130723	10	11	15		15
WN 3	2x28	95627	9	9	9	9	10
WN 4	2x28	119337	10	11	14	14	14
WN 5	2x28	98796	2	4	5	6	16
WN 6	2x28	130233	5	10	10	14	14
WN 7	2x105	1095352	38	81	89	90	90
WN 8	2x105	620060	42	47			47
P 6	5x39	10547	1	1	1	1	8
P 7	5x50	16499	4	9	9	9	18
ST 1	30x60	7761	11	15			15
ST 2	30x60	8722	14	16	16	16	16

REFERENCES:

[1] Dantzig G.B., "Discrete-variable extremum problems", Management Science 5(1957),266-277.

[2] Dyer M. E., "Calculating surrogate constraints", Mathematical Programming 19(1980):255-278.

[3] Fayard D. and Plateau G., "Contribution a la résolution des programmes mathématiques en nombres entiers", dissertation, 1979.

[4] Fleisher J., Sigmap Newsletter 20, february 1976.

[5] Fréville A. and Plateau G., "Heuristics and reduction methods for multiple constraints 0-1 linear programming problems", European Journal of Operational Research 24(1986):206-215.

[6] Gavish B. and Pirkul H., Efficient algorithms for solving multiconstrained zero-one knapsack problems to optimality", Mathematical Programming 31(1985):78-105.

[7] Glover F., "Surrogate constraint duality in mathematical programming", Operations Research 23(1975):434-453.

[8] Greenberg H. J., "The generalized penalty-function surrogate model", Operations Research 21(1973):162-178.

[9] Greenberg H. J. and Pierskalla W. P., "A review of quasi-convex functions" Operations Research 19(1971):1553-1570.

[10] Karwan M. H. and Rardin R. L., "Some relationships between lagrangian and surrogate duality in integer programming" Mathematical Programming 18(1979):320-334.

[11] Lorena L. A. N. and Plateau G., "A monotone decreasing algorithm for the 0-1 multiknapsack dual problem" Technical Report, Université de Paris-Nord (1988).

[12] Petersen C. C., "Computational experience with variant of the Balas algorithm applied to the selection of R&D projects", Management Science 13(9)(1967):736-750.

[13] Rockafellar R. T., "Convex Analysis", Princenton University Press, Princenton, N. J. (1970).

[14] Sarin S., Karwan M.H. and Rardin R.L., "A new surrogate dual multiplier search procedure", Naval Research Logistics 34(1987):431-450.

[15] Sarin S., Karwan M.H. and Rardin R.L., "Surrogate duality in branch-and-bound procedure for integer programming", European Journal of Operational Research 33(1988):326-333.

[16] Senju S. and Toyoda Y., "An approach to linear programming with 0-1 variables", Management Science 15(4)(1968):196-207.

[17] Weingartner H.M. and Ness D.N., "Methods for the solution of the multidimensional 0-1 knapsack problem", Operations Research 15(1)(1967):83-103.