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A GEOMETRIC METHOD FOR LOCATION OF GRAVITATIONAL WAVE SOURCES

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ABSTRACT

We show that the interaction of a gravitational wave with a spherical resonant-mass antenna changes the antenna's shape to that of an ellipsoid. These changes in shape always determine the direction of the incoming wave and may provide information on the wave's polarization. We present a new approach for determining the position of astrophysical sources of gravitational waves which involves fewer calculations than in earlier methods. We also show how the measured quantities relate to the energy density of the wave.

Subject headings: gravitation — instrumentation: detectors — relativity — waves

1. INTRODUCTION

The detection of gravitational waves is of interest to physicists and astrophysicists because of its broad implications: in testing the general theory of relativity, verifying the existence of black holes, finding the masses and abundance of neutron stars and black holes, "probing" the densities and viscosities of neutron stars, allowing new frontiers for astronomy and astrophysics, discovering new physics, and developing new technologies (cryogenics, SQUIDs, etc). Detectors are now in operation, and others are being built or projected (Blair 1991; Coccia, Pizzela, & Ronga 1995).

Since the pioneering work of Joseph Weber in the 1960s the detectors based on resonant antennas have improved significantly. Recently a new geometry for this kind of antenna was proposed: the truncated icosahedron (Johnson & Merkowitz 1993). It is expected to be the best spheroidal resonant-mass detector, its shape allowing for omnidirectionality. This antenna is designed to detect frequencies higher than those to which interferometric detectors are most sensitive. Together, spheroidal and interferometric detectors are expected to cover a wide range of interesting astrophysical sources of gravitational waves.

The interaction of a gravitational wave with the spherical antenna can be nicely visualized by inspecting the spacetime metric in the presence of the wave: the spherical shape changes to that of an ellipsoid. By observing changes in the ellipsoidal shape of the antenna one can obtain information about the wave's polarization. Because the spherical detector is expected to measure the five independent components of the $\boldsymbol{h}$ matrix, the shape of the ellipsoid could be completely determined.

The spherical detector can also provide information about the position in space of the astrophysical source. This has already been calculated (Dhurandhar & Tinto 1988; Magalhães et al. 1995), but we present here an alternative approach expected to involve fewer calculations than the previous one. Also, we present an explicit expression for the energy density of the gravitational wave calculated from the measured quantities which also imposes limits on $h_\times$ and h_+.

In § 2 we introduce the picture of the spatial distortion due to a gravitational wave as an ellipsoid. Physical implications of this approach are presented in § 3. In § 4 we determine the propagation direction of the gravitational wave, and in § 5 we analyze the Poynting vector for this wave and one more possible link between the linearized theory and electromagnetism. The wave's polarization is discussed in § 6, and further comments and extensions of the work are presented in § 7.

2. THE TIDAL ELLIPSOID

A gravitational wave far away from its source can be considered to be the result of a very weak disturbance $h_{\mu\nu}$ in the Minkowskian metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \quad \mu, \nu = 0, 1, 2, 3\ , \quad |h_{\mu\nu}| \ll 1\ , \qquad (1)$$

where

$$\eta = [\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

For a gravitational wave propagating in the z' direction of a convenient "wave frame" (with axis $x'y'z'$) and using the TT gauge (Misner, Thorne, & Wheeler 1973, § 35.4), the symmetric matrix $\boldsymbol{h}$ has the form

$$\boldsymbol{h}^{\mathrm{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \qquad (2)$$

In principle this disturbance varies with space and time $[h_{\mu\nu} = h_{\mu\nu}(t, \boldsymbol{x})]$, but we will omit this explicit dependence for the sake of simplicity.

The metric tensor $g_{\mu\nu}$ is known to relate the covariant and contravariant components of any vector $\boldsymbol{d}$. In particular, the squared modulus of $\boldsymbol{d}$ (which is also a proper distance from the spacetime origin) is given by

$$|\boldsymbol{d}|^2 = \sum_{\mu=0}^{3} x_\mu x^\mu = \sum_{\mu,\nu=0}^{3} x^\mu g_{\mu\nu} x^\nu\ . \qquad (3)$$

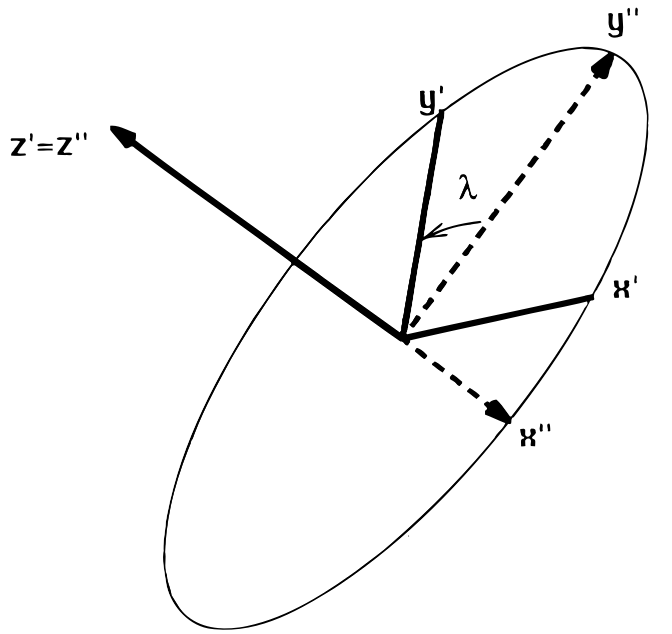

FIG. 1.—The relative positions between the wave proper frame ($x'y'z'$) and the diagonal frame ($x''y''z''$).

Notice that in the Minkowskian space $|\boldsymbol{d}|^2 = -t^2 + \boldsymbol{x}^2$. This means that for a fixed instant $t = t_0$ point particles located at the same distance $r^2 = |\boldsymbol{d}|^2 + t_0^2$ from another at the origin will describe a sphere in that space.

In order to visualize the effect of the gravitational wave on the relative positions of these particles we will change our point of view, rotating the x' and y' axes relative to the z' axis according to an angle λ such that, for a certain instant

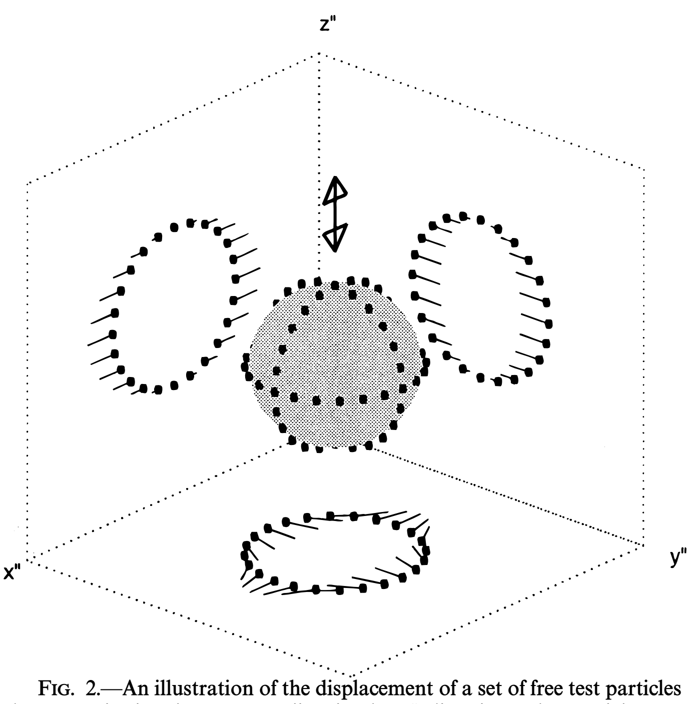

FIG. 2.—An illustration of the displacement of a set of free test particles by a gravitational wave traveling in the z'' direction. The particles are initially arranged in three orthogonal concentric rings. Each undeformed ring of particles is projected onto a plane. Their subsequent gravitational displacements are shown by the lines, which all lie in a plane normal to the direction of propagation.

$t = t_0$,

$$\tan \lambda = \sqrt{\frac{h_t - h_+(t_0)}{h_t + h_+(t_0)}} ,$$

with

$$h_t \equiv \sqrt{h_+^2(t_0) + h_\times^2(t_0)} . \quad (4)$$

We will call this rotated reference frame the instantaneous *diagonal* frame and label its axes by $x''y''z''$ (see Fig. 1). The reason for the name diagonal is simple: under this rotation g is diagonal since $\boldsymbol{h}$ is given by

$$\boldsymbol{h}'' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_t & 0 & 0 \\ 0 & 0 & -h_t & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(with frame $x'y'z'$) and

$$|\boldsymbol{d}|^2 = -t''^2 + (1 + h_t)x''^2 + (1 + h_t)y''^2 + z''^2 . \quad (5)$$

Because $h_t \ll 1$ we recognize that for a fixed instant $t'' = t_0$ the above equation describes an ellipsoid in the $x''y''z''$ space; it implies that particles located at the same distance r^2 from each other at the origin will describe an ellipsoid, in contrast with the sphere they described before the wave's arrival. We can call it the tidal ellipsoid, since gravitational waves produce tidal accelerations between particles. Tidal ellipsoids are also known in the Newtonian context (Ohanian & Ruffini 1994 , § 1.9): a very small drop of water with little surface tension would take the shape of an ellipsoid in the presence of a classic gravity field; this classic tidal ellipsoid could have its shape changed in all directions. On the other hand, from equation (5) we conclude that the relativistic tidal ellipsoid does not change in the z'' direction, the direction of propagation of the gravitational wave; this is a natural consequence of the transversality of the TT gauge.

The orientation of the ellipsoid in the lab frame is easily found from the transformation that takes $\boldsymbol{h}$ into its diagonal form, $\boldsymbol{h}''$. The eigenvectors of $\boldsymbol{h}$ are parallel to the three principal axes of the ellipsoid, and also the three orthogonal axes of the diagonal frame. One of the three eigenvalues of $\boldsymbol{h}$ is zero, and its eigenvector is parallel to the wave's propagation direction, as can be seen by inspection of equation (5). The other two eigenvalues have the same moduli and opposite signs, clearly distinguishing them from each other.

3. OTHER PHYSICAL EFFECTS DUE TO THE TIDAL ELLIPSOID

The force exerted by a gravitational wave on a particle with mass M can be given by the expression

$$F^i = \frac{M}{4} \nabla^i \sum_{j,k=1}^{3} x^j \ddot{h}_{jk} x^k , \quad i = 1, 2, 3 ,$$

as long as the relevant distances involved are much smaller than the gravitational wave wavelength (see Misner et al. 1973, § 37.2). From this equation we can find the work done by a gravitational wave to move this particle from the origin to a certain point P:

$$\int_0^P F\, dx = \frac{M}{4} \frac{d^2}{dt^2} \sum_{j,k=1}^{3} x_P^j g_{jk} x_P^k ,$$

where g_{jk} is given by equation (1). Because we assume that the variables x'' are independent, we conclude that the work done by the wave is related to how the shape of the tidal ellipsoid varies with time. This can be better understood by visualizing the spherical distribution of free particles in space. When the gravitational wave passes, it "works" on the distribution by changing its shape to that of an ellipsoid (Fig. 2).

A more complicated but similar effect happens in a solid sphere that resonates with the gravitational wave. For example, when a gravitational wave with the "+" polarization ($h_{x'x'} = h_t$, $h_{x'y'} = 0$) and traveling in the $\hat{z}$ direction of a frame fixed in the antenna (the lab frame) reaches the solid sphere, the displacement of a point on the sphere surface relative to the origin of this frame (assumed in the sphere's center of mass) is proportional to $\boldsymbol{d} = h_t(|x| - |y|)$, where x and y are the coordinates of the point in the lab frame. Notice that there is no oscillation in the $\hat{z}$ direction, as expected. The wave distorts the sphere such that the solid assumes the shape of an ellipsoid, similarly to what happens to the free particles of Figure 2. This distortion is, however, extremely small because h_t is also small.

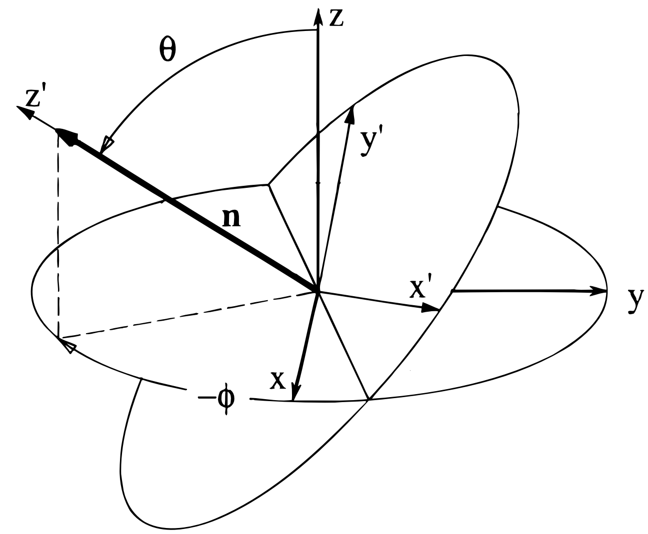

FIG. 3.—Position of the vector $\boldsymbol{n}$ and the wave proper frame ($x'y'z'$) relative to the lab frame (xyz).

4. DETERMINATION OF THE WAVE'S DIRECTION

In the absence of noise (Magalhães et al. 1995), or even in the presence of some noise (Merkowitz & Johnson 1995), the wave amplitudes h_{ij}, $i, j = 1, 2, 3$, are expected to be measured by a spherical detector. We may then use the eigenvalue equation (Dhurandhar & Tinto 1988)

$$\boldsymbol{h}\boldsymbol{n} = 0\boldsymbol{n}$$

to determine the wave's propagation direction, assumed to be the same as that of $\boldsymbol{n}$, the Cartesian unit base vector that points toward the direction of the z'' axis of the diagonal frame; again, the zero eigenvalue assures that there is no perturbation in the direction $\boldsymbol{n}$. This equation is valid in the diagonal frame and can be easily proved valid in any reference frame rotated from this frame according to a certain rotation matrix $\boldsymbol{M}$.

In the lab frame the vector $\boldsymbol{n}$ is given by (see Fig. 3)[1]

$$\boldsymbol{n} \equiv \begin{pmatrix} n^x \\ n^y \\ n^z \end{pmatrix} .$$

Five of the components of the wave tensor (h_{xx}, h_{xy}, h_{xz}, h_{yy}, and h_{yz}) can be obtained from the coefficients of the sphere's five normal modes (Wagoner & Paik 1977). The component h_{zz} can be obtained from the equation $h_{zz} = -h_{xx} - h_{yy}$, valid in the TT gauge.

The eigenvalue equation implies the three following equations:

$$h_{xx} n^x + h_{xy} n^y + h_{xz} n^z = 0 , \tag{6}$$

$$h_{yx} n^x + h_{yy} n^y + h_{yz} n^z = 0 , \tag{7}$$

$$h_{zx} n^x + h_{zy} n^y + h_{zz} n^z = 0 , \tag{8}$$

Because the matrix $\boldsymbol{h}$ has null determinant, the system of equations (6)–(8) is undetermined: one of the equations depends on the other two. But two independent equations suffice to determine the gravitational wave propagation direction, given by the angles θ and ϕ of the spherical coordinates of the lab frame,

$$\sin \phi = \frac{n^y}{\sqrt{(n^x)^2 + (n^y)^2}} ,$$

$$\cos \phi = \frac{n^x}{\sqrt{(n^x)^2 + (n^y)^2}} ,$$

$$\tan \phi = \frac{n^y}{n^x} , \tag{9}$$

$$\tan \theta = \frac{\sqrt{(n^x)^2 + (n^y)^2}}{n^z} = \frac{n^y}{n_z} \frac{1}{\sin \phi} . \tag{10}$$

For instance, suppose we choose the two equations

$$\sum_{i=1}^{3} h_{ki} n^i = 0 \quad \text{and} \quad \sum_{i=1}^{3} h_{li} n^i = 0 ,$$

where k and l may be either x, y, or z. By combining them we find that the components of $\boldsymbol{n}$ in the lab frame obey the relations

$$n^y = \left(\frac{h_{lx} h_{kz} - h_{lz} h_{kx}}{h_{ly} h_{kx} - h_{lx} h_{ky}} \right) n^z ,$$

$$n^x = -\frac{1}{h_{kx}} \left[h_{kz} + h_{ky} \left(\frac{h_{lx} h_{kz} - h_{lz} h_{kx}}{h_{ly} h_{kx} - h_{lx} h_{ky}} \right) \right] n^z .$$

Using the above equations in equations (9) and (10), we have

$$\tan \phi = \frac{h_{kz} h_{lx} - h_{kx} h_{lz}}{h_{ky} h_{lz} - h_{kz} h_{ly}}$$

and

$$\tan \theta = \left(\frac{h_{lx} h_{kz} - h_{lz} h_{kx}}{h_{ly} h_{kx} - h_{lx} h_{ky}} \right) \frac{1}{\sin \phi} .$$

[1] Because we are dealing only with instantaneous observations, from now on we will work only with vectors and matrices in three dimensions instead the usual four dimensions of metric theories of gravitation.

A similar but incomplete result was found previously by Dhurandhar & Tinto (1988).[2] In their work they used only the first two of the equations (6)–(8) to find the wave's direction of propagation. But these two equations alone do not lead to a well-determined result when the wave is propagating, for instance, in the x direction, since they yield $\tan\phi = \tan\theta = 0/0$ in this case; instead, equations (7) and (8) would be more appropriate, resulting in $\tan\phi = 0$ and $\tan\theta = \infty$, as it should be.

In order to illustrate the above method, suppose the following $\boldsymbol{h}$ matrix was obtained in the lab frame:

$$\boldsymbol{h} = \begin{bmatrix} 2.37 & 0.75 & -1.14 \\ 0.75 & -1.87 & -1.85 \\ -1.14 & -1.85 & -0.5 \end{bmatrix} . \qquad (11)$$

By choosing, for instance, the first and third rows of this matrix we find $\tan\phi \sim -1$ and $\tan\theta \sim \pm 0.99$, which imply $(\theta, \phi) \sim (\pi/4, 7\pi/4)$ or $(\theta, \phi) \sim (3\pi/4, 3\pi/4)$, as it should be.

[2] Notice that the angle ϕ in their eq. (34a) is actually the angle ϕ we use in this paper plus $\pi/2$.

Instead of solving the eigenvalue equation analytically we may obtain the direction of an astrophysical source from the direction of the eigenvector $\boldsymbol{n}$ with null eigenvalue when $\boldsymbol{h}$ is in its diagonal form. Using the values given in equation (11) as an example, the diagonal form of $\boldsymbol{h}$ is

$$\boldsymbol{h}'' = \begin{bmatrix} 3.16 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3.16 \end{bmatrix} .$$

The eigenvector that corresponds to the zero eigenvalue of $\boldsymbol{h}''$ is found to be

$$\boldsymbol{n} = \begin{bmatrix} -0.5 \\ 0.5 \\ -0.7 \end{bmatrix} ,$$

thus agreeing with the previous result (see Fig. 4). This procedure could have followed not only from the eigenvalue equation approach but also from the tidal ellipsoid picture, since the principal axes of the ellipsoid become evident in the diagonal frame.

Although both ways presented above may be used to determine the direction of the astrophysical source, they

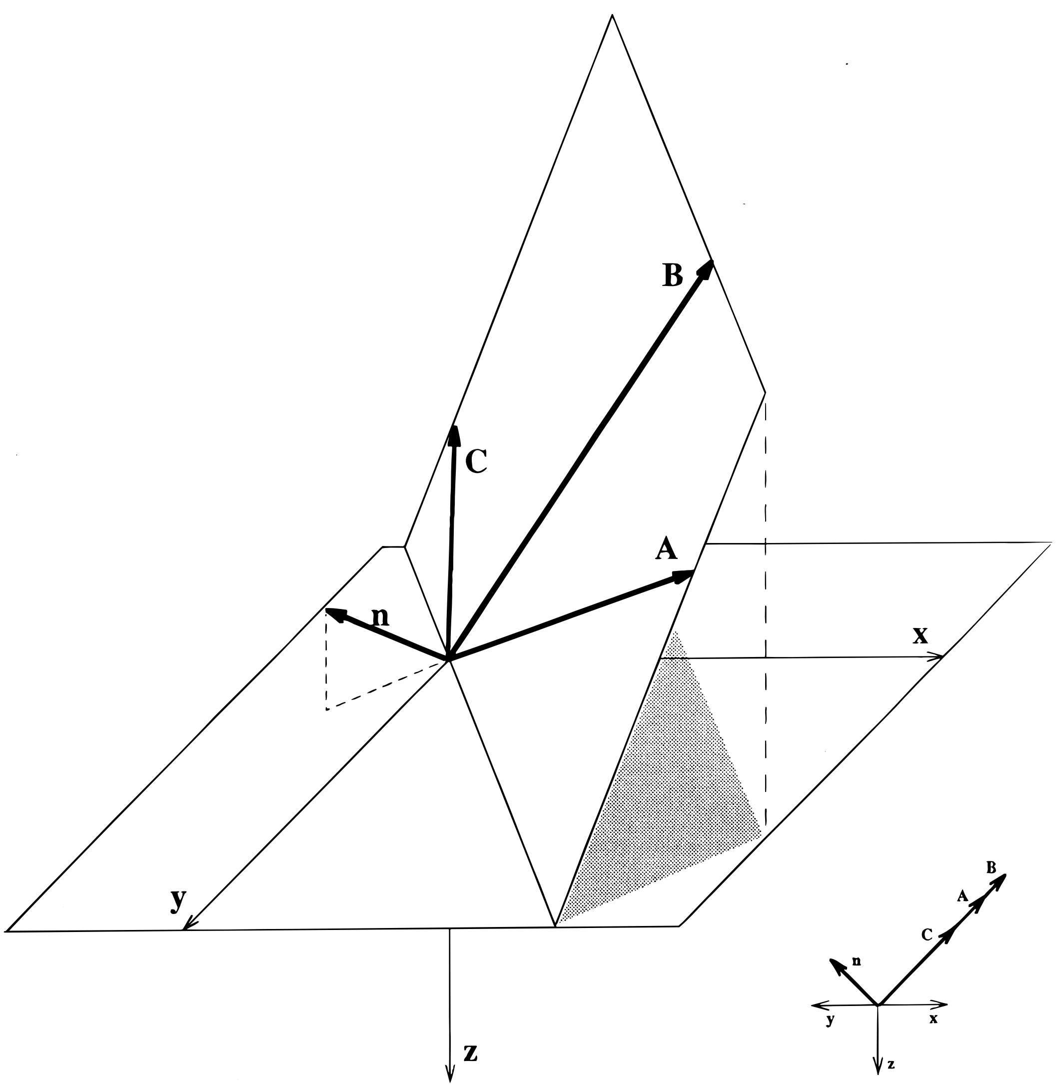

FIG. 4.—The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are coplanar and perpendicular to the wave's propagation direction, which is parallel to $\boldsymbol{n}''$

normally involve a considerable amount of algebra. We have devised a different approach to obtain the wave's direction of propagation. Notice that if we define the vectors

$$\boldsymbol{A} \equiv (h_{xx}, h_{xy}, h_{xz}) = \sum_{j=1}^{3} h_{xj} \hat{e}^j , \quad (12)$$

$$\boldsymbol{B} \equiv (h_{yx}, h_{yy}, h_{yz}) = \sum_{j=1}^{3} h_{yj} \hat{e}^j , \quad (13)$$

$$\boldsymbol{C} \equiv (h_{zx}, h_{zy}, h_{zz}) = \sum_{j=1}^{3} h_{zj} \hat{e}^j , \quad (14)$$

where $\hat{e}^j$ are the contravariant unit vectors that describe the lab frame when the wave is present, equations (6)–(8) can be rewritten as

$$\boldsymbol{A} \cdot \boldsymbol{n} = \boldsymbol{B} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0 . \quad (15)$$

The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are thus coplanar and perpendicular to the gravitational wave's propagation direction. Notice that these vectors are defined in terms of the components of the $\boldsymbol{h}$ tensor in a certain reference frame, so their magnitudes will change if the frame is changed. Equations (15) also implies that the vectors

$$\boldsymbol{I} \equiv \boldsymbol{A} \times \boldsymbol{B} , \quad \boldsymbol{J} \equiv \boldsymbol{B} \times \boldsymbol{C} , \quad \boldsymbol{K} \equiv \boldsymbol{C} \times \boldsymbol{A} , \quad (16)$$

will be parellel to $\boldsymbol{n}$.

The vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ provide a very practical and pictorial way to determine the wave's direction of propagation. For instance, the matrix given by equation (11) implies

$$\boldsymbol{A} = [2.37 \quad 0.75 \quad -1.14] ,$$
$$\boldsymbol{B} = [0.75 \quad -1.87 \quad -1.85] ,$$
$$\boldsymbol{C} = [-1.14 \quad -1.85 \quad -0.5] .$$

By drawing the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, equation (15) can be confirmed pictorially, as we see in Figure 4. Therefore, the use of these three vectors only, with no extra calculations, provides a very straightforward method of determining the direction of the source.

In fact, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are related to the differences between the covariant base vectors of the curved space, $\hat{e}_j$ (which describe the lab frame after the wave arrives) and the covariant unit Cartesian base vectors of the Euclidean space, $\boldsymbol{i}_j = \sum_{k=1}^{3} \eta_{jk} \hat{e}^k$ (which describe the lab frame before the wave arrives). For instance, the vector $\boldsymbol{A}$ is related to changes in the covariant base vector in the x-direction:

$$\hat{e}_x = \sum_{j=1}^{3} g_{xj} \hat{e}^j = \sum_{j=1}^{3} (\eta_{xj} + h_{xj}) \hat{e}^j = \boldsymbol{i}_x + \boldsymbol{A} \Rightarrow \boldsymbol{A} = \hat{e}_x - \boldsymbol{i}_x .$$

Similarly, $\boldsymbol{B}$ is related to changes in the y-direction, and $\boldsymbol{C}$ to those in the z-direction.

The tensor $\boldsymbol{h}$ is commonly referred to as the gravitational wave "strain," and the above relation makes clear the similarity between this tensor and the strain tensor of elestic mechanics: the components of $\boldsymbol{h}$ give the amount of the distortion of the axes of the lab frame relative to their positions when the space is Minkowskian, as we see in Figure 5.

The abstract idea of $\boldsymbol{h}$ as a strain, related to the distortion of reference frames, gains reality when we observe the changes in proper distances between free particles that are initially in a spherical distribution when the space is Minkowskian: the shape of this distribution changes to an ellipsoid in the presence of a gravitational wave. Because the three principal axes of this new ellipsoidal distribution are respectively smaller, bigger, and equal to the radius of the initial spherical configuration, we conclude that the gravitational wave stretches/shrinks the distances between the particles. Mathematically this result can be shown by rewriting equation (5) as ($t_0 \equiv 0$ for simplicity)

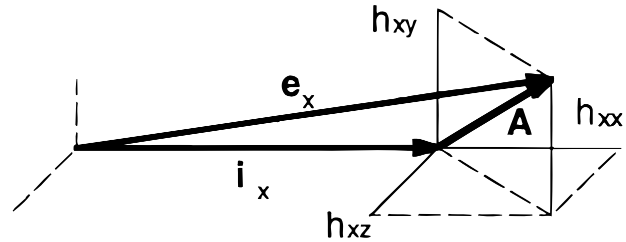

FIG. 5.—The Minkowskian axes $\boldsymbol{i}_x$ of the lab frame seem to be stretched by the gravitational wave, represented here by the vector $\boldsymbol{A}$. The components of $\boldsymbol{A}$ are given by a row of the "strain" tensor $\boldsymbol{h}$ that characterizes the wave.

$$\begin{aligned} |\boldsymbol{d}|^2 &= (x''^2 + y''^2 + z''^2) + h_t x''^2 - h_t y''^2 \\ &= |\boldsymbol{d}|^2_{\mathrm{Mink}} + h_{x''x''} x''^2 + h_{y''y''} y''^2 + h_{z''z''} z''^2 . \end{aligned}$$

Clearly, the components $h_{x''x''}$, $h_{y''y''}$, and $h_{z''z''}$ of the diagonal form of $\boldsymbol{h}$ display the stretching or shrinking (depending on their signs) of the distances in the x'', y'', and z'' directions, respectively, relative to the original Minkowskian distance $|\boldsymbol{d}|^2_{\mathrm{Mink}} = x''^2 + y''^2 + z''^2$. Therefore, by visualizing the tidal ellipsoid we are able to understand in three dimensions how the gravitational wave is changing the distances between particles.

5. THE WAVE'S POYNTING VECTOR

The intensity with which the gravitational wave distorts the lab frame can be characterized by the determinant of the metric tensor, Det $\boldsymbol{g} \equiv g$, since the volume of the space changes according to $g^{1/2}$ (Renton 1987). For the gravitational wave this invariant quantity is equal to

$$g = 1 - h_t^2 , \quad (17)$$

a result easily obtained in the diagonal frame. In this frame we have $\boldsymbol{A}'' = (h_t, 0, 0)$, $\boldsymbol{B}'' = (0, -h_t, 0)$, and $\boldsymbol{C}'' = (0, 0, 0)$, resulting in

$$\boldsymbol{I}'' = \boldsymbol{A}'' \times \boldsymbol{B}'' = -h_t^2 \boldsymbol{n} , \quad \boldsymbol{J}'' = \boldsymbol{K}'' = 0 . \quad (18)$$

The vector $\boldsymbol{I}''$ carries two important parameters of the wave: its propagation direction ($\boldsymbol{n}$) and the intensity with which it changes the volume of the initially Minkowskian space, $-h_t^2$ (see eq. [17]). These characteristics suggest that $\boldsymbol{I}''$ could be related to some kind of Poynting vector for the gravitational wave. In the lab frame its modulus can be calculated using equation (17), and we find

$$\begin{aligned} I'' &= -h_t^2 = h_{xx} h_{yy} - h_{xy}^2 + h_{xx} h_{zz} - h_{xz}^2 + h_{zz} h_{yy} - h_{zy}^2 \\ &= \boldsymbol{I} \cdot \hat{e}_z + \boldsymbol{K} \cdot \hat{e}_y + \boldsymbol{J} \cdot \hat{e}_x . \end{aligned} \quad (19)$$

The relationship between $\boldsymbol{I}''$ and the Poynting vector of a gravitational wave, $\boldsymbol{S}_g$, becomes clear in the monochromatic

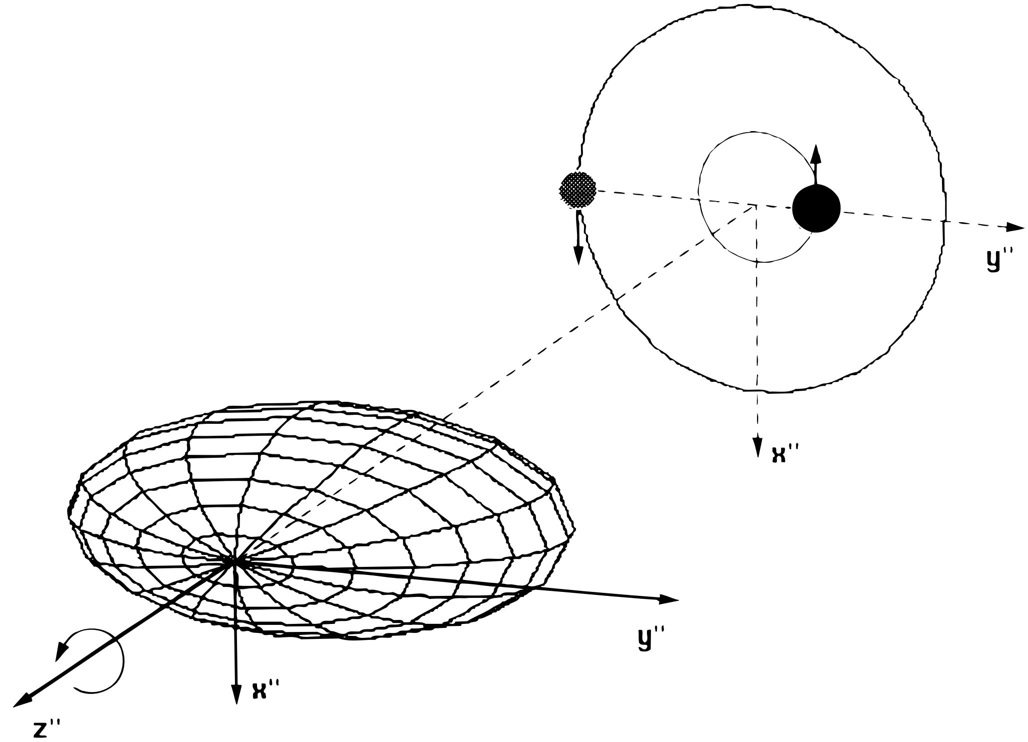

FIG. 6a

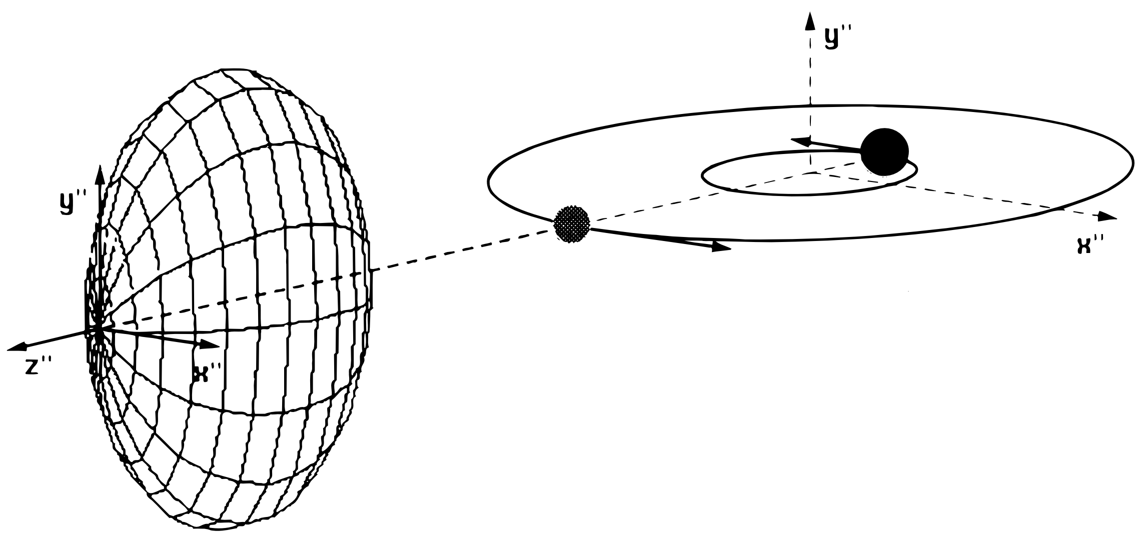

FIG. 6b

FIG. 6.—Changes in the shape of the spherical antenna when a gravitational wave emitted by a binary system arrives. (*a*) When the direction of propagation is perpendicular to the plane of the orbit, the antenna's ellipsoidal shape rotates in time with an angular velocity equal to the orbital angular velocity; (*b*) when the plane of the orbit is parallel to the wave's propagation direction, the ellipsoidal shape changes according to a linearly polarized wave.

case. In fact, in a nearly inertial frame of the linearized theory the energy density of a gravitational wave is given by

$$T_{00} = \frac{c^2}{32\pi G} \sum_{i,\, j=1}^{3} \langle \dot{h}^{ij} \dot{h}^{ij} \rangle \ ,$$

(see Misner et al. 1973, § 35.7), where c is the speed of light, G is the gravitational constant, the angle brackets denote an average over several wavelengths, and the dot implies a time derivative. In the diagonal frame the gravitational Poynting vector will thus be

$$\boldsymbol{S}_g = c T_{00} \boldsymbol{n} = \frac{c^3}{16\pi G} \langle \dot{h}_t^2 \rangle \boldsymbol{n} = \frac{c^3}{32\pi G} \langle \dot{\boldsymbol{B}}'' \times \dot{\boldsymbol{A}}'' \rangle \ .$$

This result is analogous to the one found in the electromagnetic theory (see Misner et al. 1973, § 5.6.)

By measuring $\boldsymbol{h}$ during several wavelengths and using equation (19) $\boldsymbol{S}_g$ could be easily found for any kind of gravitational wave. In particular, for a monochromatic signal of the type $h_{t''} = \mathscr{A} \cos \omega(t'' - z''/c)$, where $\mathscr{A}$ is a constant, we have

$$\boldsymbol{S}_g = - \frac{c^3\omega^2}{16\pi G} \langle \boldsymbol{I}'' \rangle .$$

6. COMMENTS ON THE WAVE'S POLARIZATION

The gravitational wave polarization is important from an astrophysical viewpoint. For example, a binary system of two stars in circular orbit about one another is expected to emit circularly polarized waves in the direction perpendicular to the plane of the orbit and linearly polarized waves in the direction of the plane of the orbit (Forward 1971; Schutz 1993, § 9.3). Assuming that in the wave frame a right-handed circularly polarized gravitational wave has amplitude ($\mathscr{A}$ is a constant)

$$\boldsymbol{h}' = \Re\left(\frac{\mathscr{A} e^{-i\omega(t-z/c)}}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right),$$

it is easy to show that for this wave $\dot{\lambda} = \omega/2$, implying that the diagonal frame (and, consequently, the tidal ellipsoid) is spinning about the z'' axis with angular velocity equal to $\omega/2$.

Therefore, from the behavior of the tidal ellipsoid in time we are able to obtain information about the total wave's polarization. For the binary system cited above the corresponding changes in the shape of the tidal ellipsoid are shown in Figure 6.

Although only continuous monitoring of the gravitational wave will provide a more precise determination of the values of h_+ and $h_\times$, the value of the modulus of $\boldsymbol{I}''$ calculated from experiment (see eq. [19]) will impose limits on the possible values of these polarization amplitudes by means of equation (4).

7. CONCLUSION

Gravitational waves far away from their sources can be interpreted as very weak disturbances in the Minkowskian space-time metric. We showed that a massive sphere is distorted into an ellipsoid with one principal axis unchanged, viz., the one that is parallel to the direction of propagation of the wave.

Since the spherical detector is expected to measure five independent components of the $\boldsymbol{h}$ matrix, changes in the antenna's shape can be determined. From them we can determine the wave's polarization and direction.

We have also devised a method for locating astrophysical sources that involves fewer calculations than in earlier methods. The geometric approach used yielded an expression for the Poynting vector of a gravitational wave that is analogous to expression for the Poynting vector in electromagnetic theory.

Our investigation suggests an analogy between the tensor $\boldsymbol{h}$ and the strain tensor of elastic mechanics. Further studies are in progress on this issue, which are expected to be presented in a forthcoming work.

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