

THE LEVI-CIVITA SPACETIME AS A LIMITING CASE OF THE γ SPACETIME

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Abstract

It is shown that the Levi-Civita metric can be obtained from a family of the Weyl metric, the γ metric, by taking the limit when the length of its Newtonian image source tends to infinity. In this process a relationship appears between two fundamental parameters of both metrics.

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1 Introduction

One of the most interesting metrics of the family of Weyl solutions [1] is the so called γ metric [2, 3]. This metric, which is also known as Zipoy-Voorhees metric [4], is continuously linked to the Schwarzschild spacetime through one of its parameters and corresponds to a solution of the Laplace equation in cylindrical coordinates. Its Newtonian image source [5] is given by a finite rod of matter. For a particular value of the mass density of the rod, the metric becomes spherically symmetric (Schwarzschild metric).

In this article we show that extending the length of the rod to infinity we obtain the Levi-Civita spacetime. At the same time a link is established between the parameter $\gamma/2$, measuring the mass density of the rod in the γ metric, and the parameter σ , which is thought to be related to the linear energy density of the source of the Levi-Civita spacetime [5]. Since σ is the real source, not the Newtonian image source and $\gamma/2$ measures the line mass density of the Newtonian image source, not of the real source, our result illustrates further the difficulties appearing in the interpretation of the Levi-Civita metric as representing an infinite line mass of density σ [6].

In the next section we describe the γ metric. In section 3 we show that it has a limit on the Levi-Civita spacetime. In section 4 some other limits are studied in order to build a limiting diagram for the γ metric. Finally section 5 presents our conclusions.

2 The γ metric

In cylindrical coordinates, static axisymmetric solutions to Einstein's equations are given by the Weyl metric [1]

$$ds^2 = e^{2\lambda} dt^2 - e^{-2\lambda} [e^{2\mu} (d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (2.1)$$

with

$$\lambda_{,\rho\rho} + \rho^{-1} \lambda_{,\rho} + \lambda_{,zz} = 0, \quad (2.2)$$

and

$$\mu_{,\rho} = \rho(\lambda_{,\rho}^2 - \lambda_{,z}^2), \quad (2.3)$$

$$\mu_{,z} = 2\rho\lambda_{,\rho}\lambda_{,z}, \quad (2.4)$$

where a comma denotes partial derivation. Observe the most amazing fact, as Synge writes [1], that (2.2) is just the Laplace equation for λ in Euclidean space. The γ metric is defined by [2]

$$e^{2\lambda} = \left[\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \right]^\gamma, \quad (2.5)$$

$$e^{2\mu} = \left[\frac{(R_1 + R_2 + 2m)(R_1 + R_2 - 2m)}{4R_1 R_2} \right]^{\gamma^2}, \quad (2.6)$$

where

$$R_1^2 = \rho^2 + (z - m)^2, \quad R_2^2 = \rho^2 + (z + m)^2. \quad (2.7)$$

It is worth noticing that λ as given by (2.5) corresponds to the Newtonian potential of a line segment of mass density $\gamma/2$ and length $2m$, symmetrically distributed along the z -axis. The particular case $\gamma = 1$ corresponds to the Schwarzschild metric. This is more easily seen using Erez-Rosen coordinates [4], given by

$$\rho^2 = (r^2 - 2mr) \sin^2 \theta, \quad z = (r - m) \cos \theta, \quad (2.8)$$

which yields the line element [2]

$$ds^2 = Fdt^2 - F^{-1}[Gdr^2 + Hd\theta^2 + (r^2 - 2mr)\sin^2\theta d\phi^2], \quad (2.9)$$

where

$$F = \left(1 - \frac{2m}{r}\right)^\gamma, \quad (2.10)$$

$$G = \left(\frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2\theta}\right)^{\gamma^2-1}, \quad (2.11)$$

$$H = \frac{(r^2 - 2mr)^{\gamma^2}}{(r^2 - 2mr + m^2 \sin^2\theta)^{\gamma^2-1}}. \quad (2.12)$$

Now, it is easy to check that $\gamma = 1$ corresponds to the Schwarzschild metric. The total mass of the source is $M = \gamma m$ [2, 3], and its quadrupole moment Q is given by

$$Q = \frac{\gamma}{3}M^3(1 - \gamma^2). \quad (2.13)$$

So that $\gamma > 1$ ($\gamma < 1$) corresponds to an oblate (prolate) spheroid. We shall now show that *elongating* the Newtonian image source to infinity we obtain the Levi-Civita spacetime. To achieve that, use will be made of the Cartan scalars. In the next section these scalars are obtained for the γ metric, and are compared to the corresponding quantities of the Levi-Civita metric in the limit $m \rightarrow \infty$.

3 The Levi-Civita limit

Since the limit $m \rightarrow \infty$ taken on the γ metric in the form (2.1) diverges, we use the Cartan scalar approach to obtain a finite limit [7, 8].

It is known [9] that the so called 14 algebraic invariants (and even all the polynomial invariants of any order) are not sufficient for locally characterizing a spacetime, in the sense that two metrics may have the same set of invariants and be not equivalent. As an example, all these invariants vanish for both Minkowski

and plane-wave [9] spacetimes and they are not the same. A complete local characterization of spacetimes may be done by the Cartan scalars. Briefly, the Cartan scalars are the components of the Riemann tensor and its covariant derivatives (up to possibly the 10th order) calculated in a constant frame [10, 9, 11, 12].

Therefore it is possible to establish unambiguously the local equivalence between two given metrics by comparing their respective Cartan scalars, in other words: Two metrics are equivalent if and only if there exist coordinate and Lorentz transformations which transform the Cartan scalars of one of the metrics into the Cartan scalars of the other. It should be stressed that, although the Cartan scalars provide a local characterization of the spacetime, global properties such as topological defects do not probably appear in them.

In practice, the Cartan scalars are calculated using the spinorial formalism. For the purpose here, the relevant quantities are the Weyl spinor Ψ_A , and its first covariant symmetrized derivative $\nabla\Psi_{AB'}$, which represent the Weyl tensor and its covariant derivative. Due to the amount of calculations, the computer algebra systems SHEEP/CLASSI [10, 9] and MAPLE were used throughout this section.

In order to calculate the Cartan scalars for the γ metric, we take the line element in spherical coordinates (eq. (2.9)) written in the same tetrad basis used by [2]. In the 0th order we find that the Ricci spinor and curvature scalar vanish and the Weyl spinor satisfies the relation: $\Psi_0 = \Psi_4$, $\Psi_1 = -\Psi_3$, $\Psi_2 \neq 0$. It can be easily shown that this corresponds to a Petrov type I metric, which therefore has no isotropies. For putting these Cartan scalars in a canonical form, two tetrad transformations are done, which in the spinorial formalism are given by:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & 1/A \end{bmatrix} \quad (3.1)$$

The first transformation puts the 0th order Cartan scalars in the form: $\Psi'_0 \neq 0$,

$\Psi'_1 = 0, \Psi'_2 \neq 0, \Psi'_3 = 0, \Psi'_4 \neq 0$. The second transformation with $A = (\Psi'_4/\Psi'_0)^{1/8}$ gives finally: $\Psi''_0 = \Psi''_4, \Psi''_1 = 0, \Psi''_2 \neq 0, \Psi''_3 = 0$, which is the canonical form for Petrov type I metrics.

We come out with two independent functions of the coordinates r and θ (eqs. (3.2) and (3.3)). So, up to 0^{th} order, the isometry group is of dimension $4 - 2 = 2$ (where 4 is the dimension of the spacetime). Since the metric is independent of the coordinates t and ϕ , its isometry group is of dimension 2. Therefore, the 1^{st} order Cartan scalars will present no new information about isometries and the Karlhede algorithm will end in the 1^{st} order.

Instead of calculating the 1^{st} order Cartan scalars in the new basis, for computational reasons they were calculated in the initial basis and afterwards transformed to the new basis. Finally, to have the Cartan scalars in the cylindrical coordinate system one has to invert the coordinate transformation from cylindrical to spherical given by eq. (2.8)¹ and apply it to the Cartan scalars, remembering that they transform like scalars.

In this new basis, in cylindrical coordinates, the 0^{th} order Cartan scalars of the γ metric (eqs. (2.1) and (2.5)–(2.7)) are (dropping the primes):

$$\Psi_2 = \frac{e^{2\lambda}}{e^{2\mu}} \frac{m\gamma(R_1 + R_2 - 2\gamma m)}{(R_1 + R_2 + 2m)(R_1 + R_2 - 2m)R_1 R_2} \quad (3.2)$$

$$\Psi_0 = \Psi_4 = -\Psi_2 \frac{\sqrt{f^2 + g^2}}{2R_1 R_2 (R_1 + R_2 - 2\gamma m)} \quad (3.3)$$

where

$$\begin{aligned} f^2 = & \{[(R_1 - R_2 - 2m)(R_1 - R_2 + 2m)\gamma^2 \\ & - (R_1 + R_2 + 2m)(R_1 + R_2 - 2m)](R_1 + R_2) \\ & - 2(R_1 + R_2 - 6\gamma m)R_1 R_2\}^2 \end{aligned} \quad (3.4)$$

¹This leads to $r = \frac{\sqrt{(z-m)^2 + \rho^2} + \sqrt{(z+m)^2 + \rho^2} + 2m}{2}$ and $\cos \theta = \frac{\sqrt{(z+m)^2 + \rho^2} - \sqrt{(z-m)^2 + \rho^2}}{2m}$.

$$g^2 = (\gamma^2 - 1)^2 (R_1 - R_2)^2 (R_1 + R_2 + 2m)(R_1 + R_2 - 2m) \\ (R_1 - R_2 + 2m)(R_1 - R_2 - 2m) \quad (3.5)$$

and R_1 and R_2 are given by eq. (2.7). The 1st order Cartan scalars are too long and will not be shown.

Although, at first sight, the Cartan scalars seem more complicated than the line element, a closer investigation shows that they are simpler. In fact they depend only on the coordinates ρ and z , while the line element depends on these coordinates and the differentials of the four coordinates. In other words, under coordinate transformations, the Cartan scalars transform like scalars while the metric components transforms like tensor components.

Due to this fact, it is easier to investigate limits using the Cartan scalars rather than using the metric. Besides and even more important is the fact that the metric usually has features that are due to the non-essential coordinates (like the singularity on the Schwarzschild horizon). On the other hand, since only the essential coordinates appear on the Cartan scalars, they do not present such problems. So, in principle, a coordinate system can be found which provides a well behaved limit for the Cartan scalars while the metric still diverges. Firstly, let us investigate the limits using Ψ_2 , later we shall investigate whether the other Cartan scalars share the same limits.

After a lengthy but straightforward calculation we may write:

$$\lim_{m \rightarrow \infty} \Psi_2 = 2^{-(\gamma-1)} m^{2(\gamma^2-\gamma)} \rho^{-2(\gamma^2-\gamma+1)} \gamma(1-\gamma) \quad (3.6)$$

which is either divergent or finite depending on the value of γ . Nevertheless, this expression suggests that a finite limit may arise for all values of γ if we define a new radial coordinate $\bar{\rho}$ by $m^{2(\gamma^2-\gamma)} \rho^{-2(\gamma^2-\gamma+1)} = \bar{\rho}^{-2(\gamma^2-\gamma+1)}$ that is,

$$\rho = 2^\beta m^\alpha \bar{\rho} \quad (3.7)$$

where

$$\alpha = \frac{\gamma^2 - \gamma}{\gamma^2 - \gamma + 1} \quad (3.8)$$

and

$$\beta = \frac{-\gamma}{\gamma^2 - \gamma + 1} \quad (3.9)$$

The constant β was introduced to provide the correct power of 2 in the limiting Cartan scalar.

Indeed, noting that $-\frac{1}{3} \leq \alpha < 1$ and using eq. (3.7) into eq. (3.2), a lengthy but straightforward calculation shows that in this new coordinate system one has:

$$\lim_{m \rightarrow \infty} \Psi_2 = \frac{1}{2} \bar{\rho}^{-2(\gamma^2 - \gamma + 1)} \gamma (1 - \gamma) \quad (3.10)$$

which is finite. Similarly, one finds that all Cartan scalars have a finite limit in this new coordinate system. The question now is to find out to which metric this set of Cartan scalars belongs. This is not a straightforward task, but fortunately, calling

$$\gamma = 2\sigma \quad (3.11)$$

and $\bar{\rho} = r$ we are led to following set of Cartan scalars:

$$\psi_2 = (1 - 2\sigma)\sigma r^{-2(4\sigma^2 - 2\sigma + 1)} \quad (3.12)$$

$$\psi_0 = \psi_4 = (4\sigma - 1)\psi_2 \quad (3.13)$$

$$\nabla\psi_{01'} = \nabla\psi_{50'} = \sqrt{2}(8\sigma^2 - 4\sigma + 1)(4\sigma - 1)(2\sigma - 1)\sigma r^{-3(4\sigma^2 - 2\sigma + 1)} \quad (3.14)$$

$$\nabla\psi_{10'} = \nabla\psi_{41'} = \sqrt{2}(4\sigma - 1)(2\sigma - 1)\sigma r^{-3(4\sigma^2 - 2\sigma + 1)} \quad (3.15)$$

$$\nabla\psi_{21'} = \nabla\psi_{30'} = \sqrt{2}(4\sigma^2 - 2\sigma + 1)(2\sigma - 1)\sigma r^{-3(4\sigma^2 - 2\sigma + 1)} \quad (3.16)$$

which are the Cartan scalars of the Levi-Civita spacetime [13]

$$ds^2 = r^{4\sigma} dt^2 - r^{8\sigma^2 - 4\sigma} (dr^2 + dz^2) - \frac{r^{2-4\sigma}}{a} d\phi^2 \quad (3.17)$$

This shows that in this new coordinate system, the limit of the γ metric as $m \rightarrow \infty$ is locally the Levi-Civita metric, provided the radial coordinates $\bar{\rho}$ and r are identified and the parameter γ divided by 2 of the γ metric is identified with the density parameter σ of the Levi-Civita metric, i.e., eq. (3.11) holds.

We use the word *locally* since the Cartan scalars provide a local characterization of the metric. Furthermore, there is a parameter a in the Levi-Civita metric which does not appear in its Cartan scalars since it is a topological defect and can be eliminated by a coordinate transformation. For studying the global properties of the limit one has to investigate the metric (or the line element) directly. In fact one may ask whether, using this new coordinate system, the Levi-Civita limit can be obtained directly from the line element of the γ metric.

In this new coordinate system, the γ metric may be written as

$$ds^2 = e^{2\lambda} dt^2 - e^{-2\lambda} e^{2\mu} 2^{2\beta} m^{2\alpha} d\bar{\rho}^2 - e^{-2\lambda} e^{2\mu} dz^2 - e^{-2\lambda} 2^{2\beta} m^{2\alpha} \bar{\rho}^2 d\phi^2 \quad (3.18)$$

where λ and μ are expressed in the new coordinates. The limit of the component $g_{\bar{\rho}\bar{\rho}}$ is precisely the g_{rr} of the Levi-Civita metric but the other metric components diverge. Now, the divergences can be easily removed by similar transformations on the coordinates t , z and ϕ , given by:

$$t = 2^{-\beta(\gamma^2+1)} m^{-\beta} \bar{t} \quad (3.19)$$

$$z = 2^\beta m^\alpha \bar{z} \quad (3.20)$$

$$\phi = 2^{\beta\gamma^2} m^{\beta\gamma} \frac{1}{\sqrt{a}} \bar{\phi} \quad (3.21)$$

In this new coordinate system, the γ metric becomes:

$$ds^2 = e^{2\lambda} 2^{-2(\gamma^2+1)\beta} m^{-2\beta} d\bar{t}^2 - e^{-2\lambda} e^{2\mu} 2^{2\beta} m^{2\alpha} d\bar{\rho}^2 - e^{-2\lambda} e^{2\mu} 2^{-2\beta} m^{-2\alpha} d\bar{z}^2 - e^{-2\lambda} 2^{2(\beta+\gamma^2\beta)} m^{2(\alpha+\gamma\beta)} \bar{\rho}^2 \frac{1}{a} d\bar{\phi}^2 \quad (3.22)$$

and its limit is precisely the Levi-Civita metric. The only drawback of this limit is the introduction of an infinite topological defect. In other words, the limit of the γ metric in this new coordinate system is the Levi-Civita metric only locally. So we have reproduced the result we found previously with the Cartan scalars. Whether a coordinate system for the γ metric exists which provides a global limit into the Levi-Civita metric, i.e., with a finite topological defect, is still an open question.

4 A limiting diagram for the γ metric

In the previous section we have shown that in the coordinate system defined by eqs. (3.7) and (3.19)–(3.21) the limit $m \rightarrow \infty$ of the γ metric is locally the Levi-Civita spacetime. We shall now study this limit, find other limits in the coordinate systems of the previous section and discuss some limits known in the literature in order to build the limiting diagram for the γ metric shown in figure 1.

4.1 Limits in the Schwarzschild coordinates

In the usual Schwarzschild coordinates, in the limit $m \rightarrow 0$, the Schwarzschild line element tends to Minkowski. The limit $m \rightarrow \infty$ diverges. This can be easily checked by hand or from the Cartan scalars [7].

4.2 The Geroch Limits

In 1969 Geroch [14] showed that in the coordinate system (Geroch coordinates) defined by

$$x = r + m^{4/3}, \quad \rho = m^{4/3}\theta, \quad t' = t, \quad \varphi' = \varphi \quad (4.1)$$

the limit of the Schwarzschild metric as $m \rightarrow \infty$ is the Minkowski spacetime. He also presented a coordinate system where the limit is a Kasner spacetime. These

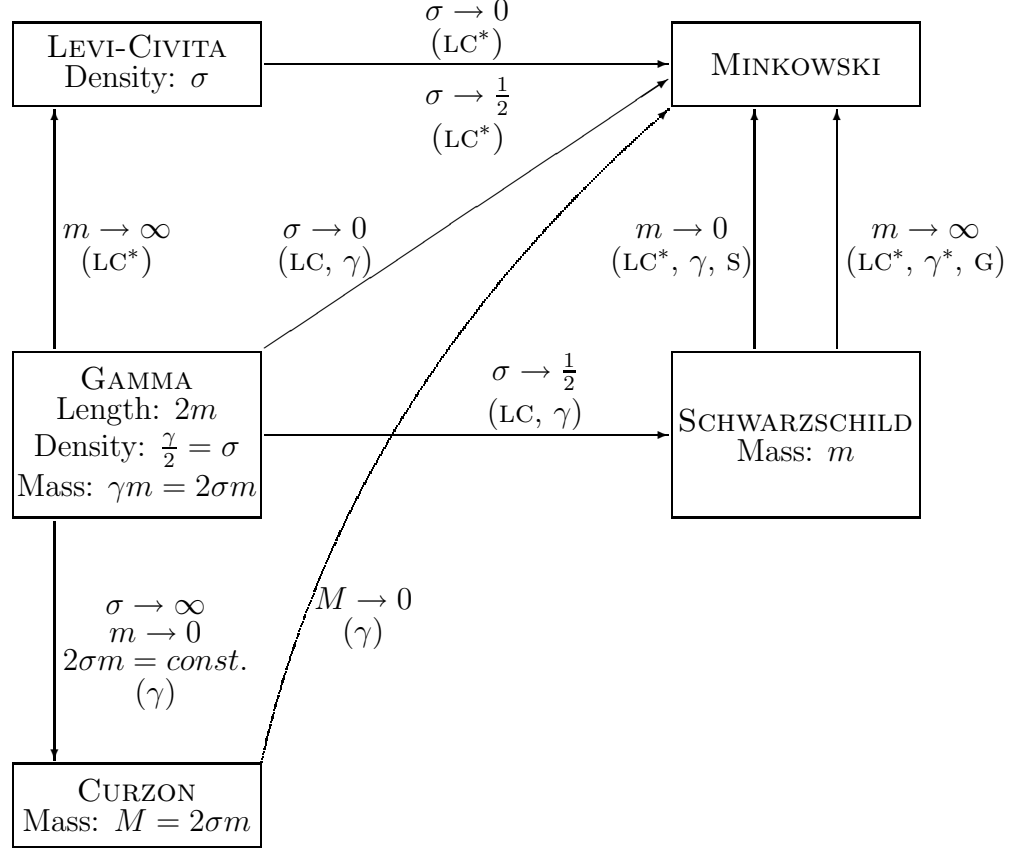


Figure 1: Limiting diagram for the γ metric. In brackets are the coordinate systems where each limit works. γ means the original cylindrical coordinate system for the γ metric; LC means the Levi-Civita coordinate system, i.e., the γ coordinate system plus the coordinate transformation given by eqs. (3.7) and (3.19)–(3.21); S means the usual Schwarzschild coordinates and G the Geroch coordinates first used to find the Minkowskian limit of Schwarzschild. The * means that, the limit is local, i.e., it was taken with the Cartan scalars and/or on the line element but a topological defect was found.

results show that the limit of a spacetime as some parameter goes to infinity is a coordinate dependent process.

Later, [7] re-obtained these limits by using the Cartan scalar technique, and extended the results presenting new limits of the Schwarzschild metric and developing an approach to find all limits of a given spacetime (see also [8, 15, 16])

4.3 Limits in the γ -coordinates

We shall call γ -coordinates the original cylindrical coordinates used for the γ metric (eqs. (2.1) and (2.5)–(2.7)). It is known [17] that in this coordinate system the limit $\gamma \rightarrow \infty$, $m \rightarrow 0$ with $\gamma m = \text{const.}$ leads to the Curzon metric and, as shown in section 2, the limit $\gamma \rightarrow 1$ leads to Schwarzschild. Besides one can easily see that as $\gamma \rightarrow 0$ the γ metric tends to Minkowski. The coordinate systems in which the Curzon and Schwarzschild metrics are expressed when obtained as limit of the γ metric will also be called γ -coordinates.

In the γ -coordinate system, the line elements of Curzon (see [17]) and Schwarzschild tend to Minkowski as $m \rightarrow 0$ (this arises directly from the line element). Although the Schwarzschild line element in γ -coordinates diverges as $m \rightarrow \infty$ it can be shown that its Cartan scalars (those of the γ metric with $\gamma = 1$) tend to zero, which is locally Minkowski.

4.4 Limits in the LC-coordinates

In the previous section, starting from the γ -coordinates we defined new coordinates by scaling ρ with a coordinate transformation which depends on m and γ (eq. (3.7)). As we have then shown, the Cartan scalars of the γ metric in this new coordinate system tend to the Cartan scalars of the Levi-Civita spacetime if the radial coordinates of both metrics are identified ($\bar{\rho} = r$) and $\gamma = 2\sigma$. Since the Cartan scalars give a complete local characterization of each metric, therefore,

the metric corresponding to this limit is locally the Levi-Civita metric. Rescaling also the coordinates t , z and ϕ (eqs. (3.19)–(3.21)) the Levi-Civita limit was obtained directly from the line element but with a topological defect.

This new coordinate system will therefore be called Levi-Civita-coordinates or LC-coordinates for short, for both the γ metric and the Levi-Civita metric (although the metric equivalence is only local). Coincidentally, this is the usual coordinate system for the Levi-Civita metric.

As $\gamma \rightarrow 0$ or $\gamma \rightarrow 1$, the γ metric in LC-coordinates tends to Minkowski or Schwarzschild, as can be seen directly from the line element (3.22). The last one giving Schwarzschild in LC-coordinates.

The limits of the Levi-Civita metric as $\sigma \rightarrow 0$ and $\sigma \rightarrow 1/2$ giving locally Minkowski can be directly found from the line-element in LC-coordinates.

As $m \rightarrow \infty$, the Schwarzschild line element in LC-coordinates diverge but its Cartan scalars tend to 0, i.e., locally Minkowski.

Finally, the LC-coordinate system turns out to be a new coordinate system (Geroch coordinates is the old one) where the Schwarzschild line element tends to Minkowski as $m \rightarrow \infty$. The equivalence is local since an infinite topological defect appears. This limit can be also done with the Cartan scalars.

5 Conclusion

We have seen so far that extending the length of the Newtonian image source of the γ metric to infinity, we arrive at the Levi-Civita spacetime. The amazing fact is that the finite rod does not represent the *real* source of the γ metric (it is just its Newtonian image source), whereas the infinite line singularity is thought to be the real source of the LC spacetime. The link between the parameters γ and σ ($\gamma = 2\sigma$) appearing in the limiting process is quite consistent with previous

results [14, 7], in the sense that the Schwarzschild metric ($\gamma = 1$) leads (locally) to Minkowski spacetime as $m \rightarrow \infty$ and the Levi-Civita metric ($m \rightarrow \infty$) leads (locally) also to Minkowski if $\sigma = 1/2$. It should be interesting to find out if restrictions on σ , based on the existence of timelike circular geodesics [18] in LC ($\sigma < 1/4$) do appear in the γ metric.

We shall now proceed to the interpretation of the limiting diagram of the γ metric (figure 1). In order to build this diagram, we introduced a new coordinate system for this metric (the LC-coordinates) and found two new limits as $m \rightarrow \infty$: Schwarzschild \rightarrow Minkowski in γ -coordinates and LC-coordinates and γ metric \rightarrow Levi-Civita in LC-coordinates.

One notices, that as it is presented, the diagram is quite consistent. It supports the current interpretations of σ as being the density in the Levi-Civita metric; $\gamma/2$ and $2m$, respectively, as the density and the length in the γ metric; m as the mass in the Schwarzschild solution and M as the mass in the Curzon solution.

Note that from the γ metric one can reach Minkowski either through Levi-Civita making $m \rightarrow \infty$ and then $\sigma \rightarrow 1/2$ or through Schwarzschild by making $\sigma \rightarrow 1/2$ and then $m \rightarrow \infty$. The limit $\sigma \rightarrow 0$ is similar; the difference being that, since the mass in the γ metric is $2\sigma m$, the limit $\sigma \rightarrow 0$ leads to Schwarzschild with zero mass, which is Minkowski.

This work would not be complete if we did not show the known weakness of the diagram. The first one is that specific coordinate systems were used. We do not know which other limits could arise if we explored more coordinate possibilities. One away to solve this problem would be the use of the Cartan scalar techniques developed in [7, 8] to find all limits of the concerning metric. The main difficulty is computational, since in the present case too many Cartan scalars are different from zero and they depend on more than one coordinate and parameter.

The second weakness are two other known limits of the Schwarzschild solution found in [14] and [7], namely, a Kasner solution and plane wave solutions, respectively. How do they match in this diagram is still an open problem.

Another extension of this work is finding a single coordinate system which provides all the limits on the diagram and does not present an infinite topological defect. This would help the understanding of the topological defects in the Levi-Civita metric.

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