The Lagrangean/surrogate relaxation and the column generation: new bounds and new columns

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Abstract
The column generation and the Dantzig-Wolfe decomposition methods are well known as efficient methods for treatment of linear programming problems with huge number of variables. One restrict master problem is identified and new columns are generated by a subproblem. It is also well known that these methods presents stabilizing issues. In order to ease these problems, the norm of the dual variables is kept under control, in order to avoid great variations. The Lagrangean/surrogate relaxation was proposed recently for stabilizing subgradient methods. This work proposes the combination of the column generation method and Lagrangean/surrogate relaxation as a stabilization method. Some computational results are showed for the *p*-median problem and several applications are suggested. Some opening questions are arising for future investigation.

1. Introduction

The recent computer science advances, with the construction of faster and more reliable equipments, provide robust systems for Mathematical Programming [3], allowing the resolution of problems with several constraints and/or variables. These tools allow that inherently complex problems can also be solved in acceptable computational time, by usage of combined techniques as, for example, the Column Generation Method applied to Integer Programming problems. Based on Dantzig and Wolfe [5], the first practical application of this technique was the determination of one-dimensional cutting patterns (Gilmore and Gomory [14, 15]) and, since then, its usage diffuses in an intensive way [2, 4, 6, 7, 8, 21, 26, 29, 31, 33, 43, 35].

The column generation technique can be employed for linear problems with huge dimensions, when all the columns are not known *a priori*, or when it is intended solve a problem using Dantzig-Wolfe decomposition, where the columns correspond to the extreme points of the convex set of feasible solutions of the problem. In this case, the resolution algorithm interchanges between a subproblem and a restrict master problem.

Approaches based on the column generation technique appears in a large number of recent works, as an alternative to nonlinear methods based in Lagrangean relaxation (Bundle and subgradient methods) to solve huge integer problems [1]. A search for articles in "*Web of Science*" on November 27th, 2001, with the subject "column generation", turned into 220 works, 93 only in the last three years.

The straight application of the column generation method usually produces a large number of counterproductive columns, which difficults the convergence to the solution of the problem. In this case, the dual variables oscillates around the optimal dual solution, then methods that avoid this performance can accelerate the resolution of the problem. Among these are the Boxstep Method [23], that restricts the searching of the dual solutions to a limited region that contains the dual solution as center; the Analytic Center Cutting Plane Method [9], that uses the analytic center of a region of dual function as solution, instead of the optimal solution, not permitting strong changes between two dual solutions in two consecutive iterations; the Bundle Method [23], that combines trust regions and penalizations, so the dual solutions do not vary so much from one iteration to another. Others methods are described in Neame [26].

This work intends to study the equivalence between the column generation method, arising from the Dantzig-Wolfe Decomposition of a problem, and the Cutting Plane Method (Kelley [18]) applied to the related Lagrangean problem. Our aim is to discuss and suggest research themes based on the application of the Lagrangean/surrogate relaxation described in Narciso and Lorena [25] as a stabilizing method for the column generation process, obtaining dual solutions with improved quality, accelerating the resolution of the original problem.

The work is organized as follows. Section 2 introduces the column generation process, the Dantzig-Wolfe decomposition and the Lagrangean relaxation, applied to an integer linear program. In Section 3 is described the combined use of the Lagrangean/surrogate relaxation and the column generation method. Some possible applications are proposed in Section 4, and Section 5 presents some opening questions about the theme.

2. Column generation

Integer Linear Problems deals with the optimization of a objective function over a feasible set, so that some or all variables must assume integer values. In particular, consider that the objective function and the equations that form the feasible set of the problem are linear.

Consider the following problem of Integer Programming:

$$\begin{aligned} z_{PI} = \; & \max_x cx \\ \text{subject to: } & \bar{A}x = \bar{b} \\ & x \in Z^n, \end{aligned} \qquad (2.1)$$

where x is the n-dimensional vector of integer variables of problem.

The constraint set $\bar{A}x = \bar{b}$ can be partitioned in two constraint sets, where one of them presents some structure that can be explored in advantageous way. Hence, after partitioning, the problem can be stated as:

$$\begin{aligned} \boldsymbol{z_{PI}} = \; & \max_x cx \\ \text{subject to: } & Ax = b \\ & A'x = b' \\ & x \in Z^n. \end{aligned}$$

Defining $W = Z^n \cap \{x : A'x = b'\}$, the problem will be formulated as:

$$\begin{aligned} z_{PI} = \; & \max_x cx \\ \text{subject to: } & Ax = b \\ & x \in W. \end{aligned} \qquad (2.2)$$

Suppose that the set W is finite, that is, the polyhedron $\{x : A'x = b'\}$ is limited, and $\{x^k : k \in K\}$ is the set of extreme points of the convex hull of W, **conv**(W), then any point $x \in W$ can be written as the convex linear combination of a finite number of extreme points of W:

$$x = \sum_{k \in K} \alpha_k x^k, \qquad \alpha_k \geq 0, \forall k \in K \text{ and } \sum_{k \in K} \alpha_k = 1. \qquad (2.3)$$

2.1 The Lagrangean Relaxation

Let λ be the vector of dual restrictions related to restrictions $Ax = b$, then the Lagrangean relaxation of (2.2) is as follows:

$$z_{RL}(\lambda) = \max_{x} \{cx + \lambda(b - Ax)\} \tag{2.4}$$

subject to: $x \in W$.

Considering the set of all extreme points of **conv**(W), and noting that optimizing $z_{RL}(\lambda)$ is equivalent to maximizing a linear function over W, then exists a extreme point of **conv**(W) that corresponds to the maximum value. Hence, (2.4) can be written as:

$$z_{RL}(\lambda) = \max_{k \in K} \{cx^k + \lambda(b - Ax^k)\}$$

that is a linear piecewise convex function in λ.

The best value of a upper bound for $z_{LR}(\lambda)$ is obtained solving the dual Lagrangean problem:

$$z_{DL} = \min_{\lambda} \max_{k \in K} \{cx^k + \lambda(b - Ax^k)\},$$

that can be written as:

$$z_{DL} = \min_{\lambda}\{\lambda b + \max_{k \in K}(c - \lambda A)x^k\},$$

or as the following Linear Program Problem:

$$z_{DL} = \min_{\lambda,\mu} \lambda b + \mu$$

subject to: $\mu \geq cx^k - \lambda Ax^k, \ \forall k \in K.$ (2.5)

In problems with a great number of variables (columns), where the set K of the index of extreme points of **conv**(W) is not known *a priori*, Kelley [18] proposes the use of the optimal dual solutions $\hat{\lambda}_l$ e $\hat{\mu}_l$ of (2.5), obtained in a given iteration l for some $K^l \subseteq K$, in the following subproblem:

$$z_{SP}(\hat{\lambda}) = \max_{x} \ cx - \hat{\lambda}Ax$$

subject to: $x \in W$, (2.6)

obtaining the optimal solution $x^{\hat{k}}$. The difference $z_{SP}(\hat{\lambda}) - \hat{\mu}$ is called reduced cost, and if $z_{SP}(\hat{\lambda}) \leq \hat{\mu}$, then the set K^l contains an optimal basis to the original problem; otherwise, the restriction (or cut) $cx^{\hat{k}} - \hat{\lambda}Ax^{\hat{k}}$ is added as a new line to the problem (2.5), set $K^{l+1} = K^l \cup \{\hat{k}\}$ and the problem is re-optimized. As a proposal to accelerate the solution of the original problem, any column x^p that satisfies $\hat{\mu} \leq cx^p - \hat{\lambda}Ax^p$ can be added to the set K^l.

This method is equivalent to the column generation process that will be presented in the next subsection, and results from the dual of problem (2.5) being equivalent to the Master Problem obtained from the linear relaxation of the Dantzig-Wolfe decomposition of the original problem.

2.2 The Dantzig-Wolfe decomposition

Another proposal to solve (2.1) to optimality consists in using the representation of solutions as a linear combination of extreme points of the optimization space in the formulation of the problem. The substitution of the expression (2.3) in formulation (2.2) leads to the follow Dantzig-Wolfe decomposition:

$$z_{DW} = \max_{\alpha_k : k \in K} c\left(\sum_{k \in K} \alpha_k x^k\right)$$

subject to:
$$A\left(\sum_{k \in K} \alpha_k x^k\right) = b$$
$$\sum_{k \in K} \alpha_k x^k \in W$$

Assuming the linear relaxation of the problem above, that is, allowing that $x \in$ conv(W), the following Master Problem is defined:

$$z_{PM} = \max_{\alpha_k : k \in K} c\left(\sum_{k \in K} \alpha_k x^k\right)$$

subject to:
$$A\left(\sum_{k \in K} \alpha_k x^k\right) = b \qquad (2.7)$$
$$\sum_{k \in K} \alpha_k = 1$$
$$\alpha_k \geq 0, \ \forall k \in K.$$

In any given iteration l of the column generation method, let $K^l \subseteq K$ be a subset of the extreme points of **conv**(W), defining the following Restrict Master Problem:

$$z_{PMR} = \max_{\alpha_k : k \in K^l} c\left(\sum_{k \in K^l} \alpha_k x^k\right)$$

subject to:
$$A\left(\sum_{k \in K^l} \alpha_k x^k\right) = b \qquad (2.8)$$
$$\sum_{k \in K^l} \alpha_k = 1$$
$$\alpha_k \geq 0, \ \forall k \in K^l.$$

Being $\hat{\lambda}$ and $\hat{\mu}$ the optimal dual solutions of (2.8), new columns can be computed as solution of the subproblem (2.6). The columns x^p that satisfy:

$$cx^p - \hat{\lambda} A x^p - \hat{\mu} \geq 0$$

(positive reduced costs) are included in (2.8) and the Restricted Master Problem is solved again, until no columns that improves the optimal value of (2.7) can be obtained.

The following general algorithm can be adapted to solve problems by the column generation method:

<u>Step 0:</u> Determine a initial set K^0 of columns for the Master Problem. Let $l = 0$;
<u>Step 1:</u> Solve the linear relaxation of the Restricted Master Problem;
<u>Step 2</u>: Use the optimal dual solution obtained in Step 1 and solve a subproblem to obtain new columns $x^{\hat{k}}$ in W;
<u>Step 3</u>: Add the columns $x^{\hat{k}}$ with positive reduced costs to the Restricted Master Problem. If such columns do not exist, then STOP $\rightarrow$ optimal solution.
<u>Step 4</u>: Otherwise, set $K^l \leftarrow K \cup \{\hat{k}\}$, $l \leftarrow l + 1$ and go to Step 1.

3. The Lagrangean/surrogate relaxation and column generation

3.1. The Lagrangean/surrogate relaxation

The Lagrangean/surrogate relaxation combines the Lagrangean and surrogate relaxations for a problem in an efficiency way. The Lagrangean relaxation is applied over the surrogate relaxation on a chosen set of constraints. It leads to a dual local Lagrangean problem in the one-dimensional variable, and this local optimization tends to correct the norm of subgradients vector, avoiding strong oscillations in the Lagrangean dual optimization methods that use the subgradients as search directions.

The Lagrangean/surrogate relaxation was applied with success in several problems of combinatory nature [20, 21, 22, 25, 27, 30]. The local optimization (local Lagrangean dual) does not need to be exact and a one-dimensional dichotomic search is employed. This optimization proved to be not necessary in every steps of subgradient methods, being enough to find the multiplier value in some initial iterations.

Being λ the surrogate multiplier related to the restrictions $Ax = b$, and $t \geq 0$ the Lagrangean multiplier related to the surrogate restrictions $\lambda Ax = \lambda b$, the Lagrangean/surrogate relaxation of (2.2) is obtained by:

$$z_{RL}(\lambda)_t = \max_{x} \{cx + t\lambda(b - Ax)\} \tag{3.1}$$

subject to: $\quad x \in W.$

It is immediate that for $t = 1$, $z_{RL}(\lambda)_1$ is the usual Lagrangean relaxation (2.4). The multiplier t is known as Lagrangean/surrogate multiplier, and its best value t^* is obtained fixing λ and solving the local dual problem using dichotomic search:

$$\min_{t\geq 0} \; z_{RL}(\lambda)_t \tag{3.2}$$

3.2. New bounds to column generation

The column generation process is generally unstable [10]. The selected columns can improve marginally the objective function value of the master problem, or its value can remain unaltered during several iterations. In some cases it is not possible to determine if the process is still converging or has stopped at some point. The calculation of upper bounds can indicate convergence of the column generation method. This section shows how Lagrangean/surrogate bounds can be directly obtained by the column generation process, using the multidimensional dual variable obtained as the optimal solution of the Restricted Master Problem. Also, the Lagrangean limit and the limit known as Farley Limit [11] are derived directly of Lagrangean/surrogate limit.

The calculation of the reduced cost CR using the Lagrangean/surrogate multiplier is given by:

$$CR_t = \max_{x\in W} \{cx - t.\lambda Ax\} - \mu,$$

or equivalently:

$$CR_t \geq \{cx - t.\lambda Ax\} - \mu, \; \forall x \in W, \text{ and } \{cx - t\lambda Ax\} \leq CR_t + \mu, \; \forall x \in W,$$

which indicate that $(t\lambda, CR_t + \mu)$ is a feasible solution to the problem (2.5). Hence, as (2.8) and (2.5) are primal and dual problems, respectively:

$$z_{PMR} \leq t.\lambda b + CR_t + \mu \quad , \tag{3.3}$$

which indicates that $t\lambda b + CR_t + \mu$ is an upper limit to the Restricted Master Problem.

Rewriting $t\lambda b + CR_t + \mu$, it leads to $\max_{x \in W} \{cx + t.\lambda(b - Ax)\}$, which is the Lagrangean/surrogate relaxation with multipliers λ and t.

The following particular cases can happen:

- for $t = 1$:
  it leads to $z_{\text{PMR}} \le \lambda b + CR_1 + \mu = \max_{x \in W} \{cx + \lambda(b - Ax)\}$, that is the traditional Lagrangean limit (2.4);
- for $t = t^*$, solution of (3.2):
  it leads to $z_{\text{PMR}} \le t^*\lambda b + CR_{t^*} + \mu = \max_{x \in W} \{cx + t^*\lambda(b - Ax)\}$, that is the best Lagrangean/surrogate limit (3.1); and
- for t_0 such that $\max_{x \in W} \{cx - t_0\lambda Ax\} = 0$:
  it leads to $z_{\text{PMR}} \le t_0\lambda b$, that is known as the Farley Limit for the particular case where $c \ge 0$ and $x \ge 0$.

It is immediate that $\max_{x \in W} \{cx + t^*\lambda(b - Ax)\} \le \max_{x \in W} \{cx + \lambda(b - Ax)\}$, and that $\max_{x \in W} \{cx + t^*\lambda(b - Ax)\} \le t_0\lambda b$, because $1 \ne t_0 \ne t^*$ (generally). Therefore the best Lagrangean/surrogate limit dominates the Lagrangean and the Farley limits (when the latter exists).

Figure 1 shows the typical behavior of the Lagrangean and Lagrangean/surrogate limits when mixed with the column generation process, for a p-median problem with 900 vertices and 300 medians [31].

3.3 Generating new columns

The column generation subproblem (2.6) can be modified by the Lagrangean/surrogate multiplier computed in (3.2), obtaining the new subproblem

$$z_{SP}(\lambda)_t = \max_{x \in W} \{cx - t\lambda Ax\}. \tag{3.4}$$

For $t \ne 1$, the problems (2.6) and (3.4) can produce different columns. If the columns x^p obtained in (3.4) satisfy $cx^p - \lambda Ax^p - \mu \ge 0$, then they become new columns to the Restricted Master Problem.

Senne and Lorena [31] formulated the p-median problem as a set partitioning problem. During the application of the column generation technique, the Lagrangean/surrogate relaxation showed to be an excellent alternative for the method stabilization, providing more productive columns than the traditional column generation method, accelerating the resolution of the problem.

Table 1 shows the results for an instance of a *p*-median problem with 200 vertices and 5 medians (showed in [31]). The maximal number of columns of the Restricted Master Problem was determied according a given value for the reduced cost. The numbers in brackets refers to the traditional column generation method while the other results refers to the columns obtained in (3.4) for $t = t^*$, solution of (3.2).

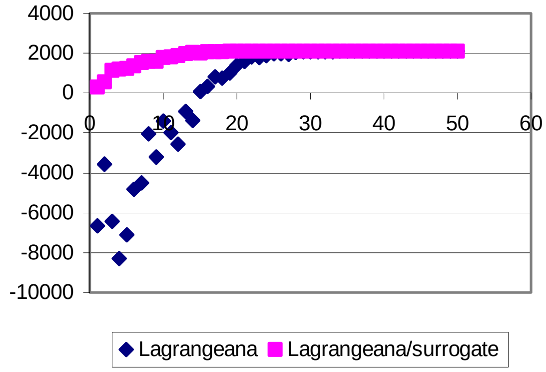

Figure 1: Usual behavior of Lagrangean and Lagrangean/surrogate limits.

Iterations	*Generated Columns*	*Used Columns*	*Primal Gap*	*Dual Gap*	*Total Time*
403 [487]	18493 [47634]	7543 [7364]	– [–]	– [–]	619.63 [971.59]
414 [1000]	20395 [167247]	6627 [3270]	– [0.631]	– [4.635]	613.79 [1370.99]
400 [1000]	23521 [186267]	3886 [421]	–0.276 [11.171]	2.010 [65.181]	532.27 [905.67]

Table 1: Number of generated and used columns.

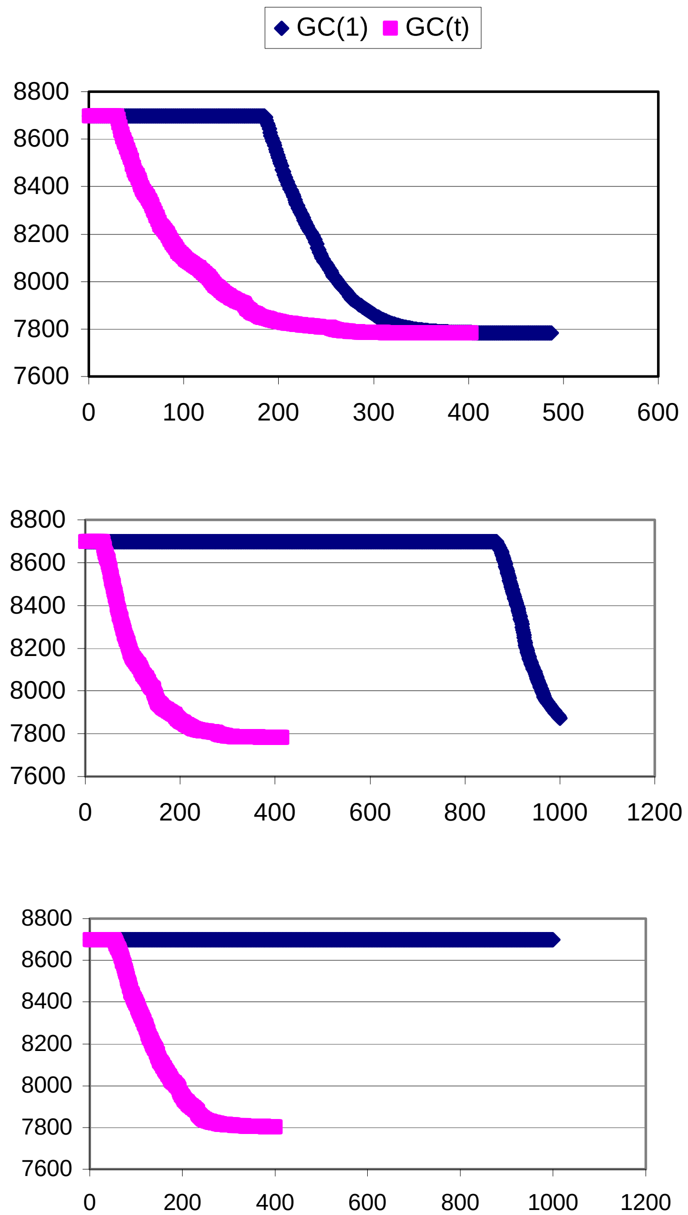

Figure 2: GC(1) = Usual Column Generation, GC(t) = Column Generation with columns of (3.4) (see table 1).

Figure 2 illustrates the results of Table 1. Note that even when a little number of columns is admitted to the restricted master problem, they produce better results when computed by expression

(3.4). The bounds obtained by the usual column generation method stands without improvement for several iterations.

4. Some applications

The Lagrangean/surrogate relaxation was applied at several Combinatorial Optimization problems. The usual subgradient method was used in the optimization of the dual problem. This section describes some possible applications of mixing columns generation and the Lagrangean/surrogate relaxation. Some of these applications are under development at *LAC/INPE*.

The column generation problem relating to (3.4) for each application will be formulated and examined. A complete formulation of problems can be referred in other works [4, 10, 12, 21, 24, 26, 29, 31, 32, 33, 35, 36]. In all of the following formulations the dual variables λ are obtained as solution to the Restricted Master Problem, and the associated Lagrangean/surrogate multiplier t is obtained as solution of problem (3.2)

4.1. The *p*-median problem

The p-median problem concerns the installation of p centers in a graph composed by arcs and vertices, minimizing the sum of all distances of each vertex to the closest center. If a demand is associated to each vertex, restrictions on the attending capacity of the centers may occur (capacitated p-median problem) [21, 31].

For the p-median problem, the Dantzig-Wolfe decomposition leads to the special case of unlimited set of extreme points, where the convexity constraints will not be present in the Restricted Master Problem formulation. Therefore, for clustering problems like this, one additional restriction will appear on the restricted master problem. This restriction is of type:

$$\sum_{k \in K^l} \alpha_k = p \,. \tag{4.1}$$

This restriction imposes that the number of medians (clusters) must be respected.

In this case, the column generation subproblem is:

$$z_{SP}(\lambda)_t = \min_{i \in N} \left[\min_{y_j \in \{0,1\}} \sum_{j \in N} \left(d_{ij} - t.\lambda_j \right) y_j \right], \tag{4.2}$$

where $[d_{ij}]_{n \times n}$ is the symmetric matrix that represent the costs (distances) between vertices i and j. Note that $d_{ii} = 0$, $\forall i \in N = \{1, ..., n\}$. The subproblem (4.2) is solved by inspection, verifying the sign of coefficients $(d_{ij} - t\lambda_j)$ of y_j.

For the capacitated case, the column generation subproblem is:

$$z_{SP}(\lambda)_t = \min_{i \in N} \left[\begin{array}{ll} \min & \sum_{j \in N} \left(d_{ij} - t.\lambda_j \right) y_j \\ \text{s.t.} & \sum_{j \in N} q_j y_j \le Q_i \,;\, y_j \in \{0,1\} \end{array} \right], \tag{4.3}$$

where Q_i is the capacity of the center allocated in vertex i and q_j is the demand of vertex j. This problem is the well-known Knapsack Problem, of class *NP-hard*, but well solved for large instances by branch-and-bound based algorithms.

Let γ be the dual variable of the Restricted Master Problem corresponding to constraint (4.1). If the columns $\left[\frac{y_j}{1} \right]$ obtained in (4.2) and (4.3) satisfies $\sum_{j \in N} \left(d_{ij} - \pi_j \right) y_j < |\gamma|$, these can be new columns to the Restricted Master Problem.

The solution of problems (4.2) and (4.3) can be significantly altered for different values on t. In Lorena and Senne [21] and Senne and Lorena [31], it was computationally verified that when t^* (the best Lagrangean/surrogate multiplier) is used in (4.2) and (4.3), its value is restricted to the interval (0,1] (tending to 1 as the process converges) and a simple analysis on these values on the formulas (4.2) and (4.3) showed that columns with small reduced costs are selected in the Restricted Master Problem as a consequence of the small number of allocated centers of clusters. Studies verifying this behavior in other problems are being conducted.

4.2. Generalized assignment problem

The generalized assignment problem consists in finding the most advantageous way of assigning n jobs to m machines so that each job is attributed to a single machine with limited capacity.

Such as the p-median problem, this is also a clustering problem with an additional constraint in the Restricted Master Problem:

$$\sum_{k \in K^i} \alpha_k^i \leq 1. \tag{4.4}$$

After the solution of the related Restricted Master Problem, the search for new columns can be computed solving the following knapsack subproblems (for $i = 1, ..., m$) [29]:

$$z_{SP}(\lambda)_t^i = \max \sum_{j=1}^{n} \left(p_{ij} - t\lambda_j\right) y_j^i \tag{4.5}$$

$$\text{subject to:} \quad \sum_{j=1}^{n} w_{ij} y_j^i \leq c_i$$

$$y_j^i \in \{0,1\}, \quad j \in \{1, 2, ..., n\},$$

where:

- w_{ij} is an integer positive number that represents the time that the machine i takes to do the job j, when j is assigned to i;
- c_i is an integer positive number that represents the available total time of machine i;
- p_{ij} is a positive number that represents the produced profit when the job j is assigned to machine i.

Let v_i the optimal dual cost related to constraint (4.4) corresponding to the agent i in the Restricted Master Problem. If for some i, $\sum_{j=1}^{n} \left(p_{ij} - \lambda_j\right) y_j^i - v_i > 0$, so the column $\left[\frac{y_j^i}{e_i}\right]$ can be added to the Restricted Master Problem.

The solutions of problems (4.5) can be significantly altered by different values of t. It is immediate that values for t in (0,1] favor the choice of columns of better quality.

4.3. Vehicle Routing Problem with Time Windows

Let V the vehicles set (identical) and $C = \{1, ..., n\}$ the customer set to be attended, associated to a depot by a directed graph [2, 4, 8, 16, 17]. Using the dual multipliers λ_i, $i \in C$, the coefficients that will be used at objective function of each subproblem $k \in V$ will be given by:

$$\hat{c}_{ij} = c_{ij} - t\lambda_i, \qquad \forall i, j \in C, i \neq j,$$

The subproblem $k \in V$ will be the following shortest path problem with time and capacity constraints[4]:

$$z_{SP}^{k}(\lambda)_t = \min \sum_{k\in V}\sum_{i\in N}\sum_{j\in N} \hat{c}_{ij} y_{ijk}$$

subject to:

$$\sum_{i\in C} d_i \sum_{j\in N} y_{ijk} \le q, \qquad \forall k \in V \tag{4.6}$$

$$\sum_{j\in N} y_{0jk} = 1, \qquad \forall k \in V \tag{4.7}$$

$$\sum_{i\in N} y_{ihk} - \sum_{j\in N} y_{hjk} = 0, \qquad \forall h \in C, \forall k \in V \tag{4.8}$$

$$\sum_{i\in N} y_{i,n+1,k} = 1, \qquad \forall k \in V \tag{4.9}$$

$$s_{ik} + t_{ij} - M(1 - y_{ijk}) \le s_{jk}, \qquad \forall I \in N,\ \forall j \in N, \forall\, k \in V \tag{4.10}$$

$$a_i \le s_{ik} \le b_i, \qquad \forall i \in N, \forall k \in V \tag{4.11}$$

$$y_{ijk} \in \{0, 1\}, \qquad \forall i \in N, \forall j \in N, \forall k \in V \tag{4.12}$$

Constraints (4.6) set up that the total demand of served customers by each vehicle cannot exceed its capacity. The three following constraints are flow constraints: constraints (4.7) set up that each vehicle must leave the depot only once; (4.8) specifies that each visited customer is left and (4.9) guarantee that all the vehicles must return to the depot only once. Precedence constraints (4.10) determine that a vehicle, starting from customer i, must not get to the customer j before instant $s_{ik} + t_{ij}$, where M is a suitable big value. Constraints (4.11) guarantees the start time of attendance of each customer to be within the specified time window. The binary nature of variables y_{ijk} is given by the constraint set (4.12).

The dual multiplier vector $\lambda = (\lambda_1, ..., \lambda_{|C|})$ reflects the attendance cost of each customer $i \in C$. By definition, λ_i is irrestricted and $t > 0$. The costs of the arcs to the subproblem are computed as in (4.6). If $t\lambda_i > c_{ij}$ prevails, the subproblem will tend to produce longer routes. In the case of prevailing $t\lambda_i < c_{ij}$, there will be preference for shorter routes. Kohl [19] notes that the difficult of resolution of the subproblem is directly proportional to the norm of the involved multipliers.

If t tends to 0, the norm of vector λ is always modified so that shorter routes are produced more often (more columns are generated). As long as t tends to 1, the values of λ_i, $i \in C$, contributes more effectively to the determination of the length of the best routes.

4.4. Symmetric traveling salesman problem

Consider a Traveling Salesman Problem defined in a graph $G = (V, E)$, $V = \{1, ..., n\}$, and let the binary variable y_{ij} equal to 1 if the arc $(i,j) \in E$ is used on the salesman optimal path. Define the matrix of costs (or distances) $C = [c_{ij}]$, where $c_{ij} = c_{ji}$ for all $i, j \in V$, that is associated to E [28].

The subproblem will be the following [36]:

$$z_{SP}(\lambda)_t = \min_{y} \left\{ \sum_{i<j} c_{ij} y_{ij} - t.\sum_{k\in V} \lambda_k \left(\sum_{i<k} y_{ik} + \sum_{j>k} y_{kj} \right) \right\}, \tag{4.13}$$

where y is a feasible solution to the minimum weight spanning 1-tree problem, that can be obtained considering the minimal cost spanning tree with vertices in $V\backslash\{1\}$ and two distinct arcs of minimal cost that links this tree to the vertex 1.

The cost matrix of (4.13) is $c_{ij} - t\lambda_k$ that represents the modified costs of the arcs of the original graph. It is immediate that some value of $t \neq 1$ can modify the solution of problem of the minimum weight spanning 1-tree. In particular, if the values of t* are restricted to the interval (0,1], columns with lower costs will be selected to the Restricted Master problem as a consequence of the cost reduction of the 1-trees, leading to solutions sequences that resembles salesman's tours.

4.5. Binary cutting problem

The binary cutting problem consists in determining the minimum number of rolls of length L necessary to attend a demand of rolls of smaller length a_i, $i = 1, ..., n$ [34, 35].

In this case, the columns of the Restricted Master Problem represents feasible cutting patterns to pieces of length L. So, the new patterns (columns) are generated solving the following 0-1 knapsack subproblem:

$$\max \sum_{i=1}^{n} \lambda_i y_i \tag{4.14}$$

$$\text{subject to:} \quad \sum_{i=1}^{n} a_i y_i \leq L \tag{4.15}$$

$$y_i \in \{0,1\} \tag{4.16}$$

where $y_i = 1$ if the item i is present at new column, or $y_i = 0$, otherwise.

It is known that the Lagrangean/surrogate limit must be a good dual limit for this problem. Let y_v be a feasible solution to the problem (4.14)-(4.16). The best value to the Lagrangean/surrogate multiplier can favor the determination of such solutions, producing different columns than the ones obtained by the traditional approach ($t = 1$). If the reduced cost, given by $(1 - \lambda y_v)$, is negative, then the correspondent column is a candidate to be added to the Restricted Master Problem.

5. Open questions

The formalization of the results of the following issues can perform the complementation of the existent theory:

a) Limits: It is known that the Lagrangean/surrogate relaxation provide better quality limits than the usual Lagrangean relaxation [25]. It is intended to establish relations between the obtained limits by these relaxations and others obtained to this class of problems, for example the Farley's Limit [11], specific for the column generation implementation;
b) Stabilization methods: Evaluate the influence of Lagrangean/surrogate relaxation and establish relations with other stabilization methods for the column generation technique (subgradients, bundle, Boxstep, Analytic Center Cutting Plane methods). Verify if such methods can be improved when mixed with the Lagrangean/surrogate relaxation;
c) Complexity of column generation subproblem: The Lagrangean/surrogate relaxation, when applied to problems with additional restrictions of capacities (knapsack) was adapted to permit the computation of the Lagrangean/surrogate multiplier (shortening the search process of t). Studies will be carried to evaluate how this search will be affected when combined with column generation;
d) Subproblem resolution: It is known that the column generation subproblem does not need to be solved to optimallity to generate new columns to the Restrict Master Problem. We intend to explore the influence of the Lagrangean/surrogate multiplier in the column generation process, considering it in the elaboration of new heuristics to the column generation subproblem;
e) Management of the number of columns: We intend to study the management of the number of columns to be considered on the Restricted Master Problem. Avail the relation quantity × quality of the columns obtained by the subproblems (in Senne and Lorena [31] the Lagrangeana/surrogate relaxation allowed to solve p-medians problem with less columns);
f) Lagrangean/surrogate relaxation and *the branch-and-price*: the Branch-and-Price method [1] uses the column generation technique on each node of a branch-and-bound search tree to obtain new non-basic variables to the problem. It will be explored the possibility of adapting the Lagrangean/surrogate relaxation and the column generation to branch-and-price and to verify if there is advantage in this adaptation.

6. Conclusions

This paper emphasizes the use of Lagrangean/surrogate relaxation as a stabilization method for the column generation problem. Some computational experience was produced to the p-median problem and other problems were suggested to test the stabilization algorithm.

The Lagrangean/surrogate relaxation and column generation method combination is done by the Dantzig-Wolfe decomposition theory and the Kelley method. The dual multiplier obtained as solution of a Restricted Master Problem is used to determine the Lagrangean/surrogate limit. The exploration of these limits is still open, as well as there are several other questions related to the efficient use of the lagrangean/surrogate multiplier at column generation process.

The suggested applications are being studied at LAC/INPE, but several other applications can be tested, including scheduling problems and new clustering problems.

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